On some properties of $p$-nuclear and $p$-integral operators

by

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Introduction. In a recent paper [5] A. Pietsch and the present author investigated the general properties of so called $p$-nuclear and $p$-integral operators. These are natural generalizations to arbitrary $1 \leq p < \infty$ of the classes of nuclear and integral operators, respectively, which were introduced and studied by Grothendieck [4]. Among other things Grothendieck proved that an integral operator between two Banach spaces $E$ and $F$ is nuclear provided that one of the following four conditions is satisfied:

(i) $E$ is reflexive;
(ii) $E$ has a separable dual;
(iii) $F$ is reflexive;
(iv) $F$ is separable and isomorphic to the dual of a Banach space.

Neither of conditions (iii) and (iv) is sufficient for the corresponding statement for $p$-integral and $p$-nuclear operators to hold true. This is shown for example by the natural injection of $l^p$ into $l^p$, which is 2-integral ([5], §10) but not 2-nuclear; in fact, it is not even compact. The principal aim of the present paper is to show that either of conditions (i) and (ii) is sufficient in order that a $p$-integral mapping of $E$ into $F$ be $p$-nuclear. The first case is obtained as a corollary of the more general statement (due to Grothendieck [4]) in the case $p = 1$ that the composition of a weakly compact linear operator and a $p$-integral operator is $p$-nuclear. The proof of this fact depends on a formulation for arbitrary $p$ (cf. also Chevet [2]) of another result of Grothendieck [4], stating that the space of nuclear operators of $E$ into an $L^p$-space can be identified with the corresponding space of $E'$-valued integrable functions. This extension makes it also possible to give a complete characterization of a $p$-nuclear operator of an $L^p'$-space into an $L^p$-space, where $1/p + 1/p' = 1$. The situation turns out to be quite analogous to the case of Hilbert-Schmidt mappings between $L^p$-spaces.

Throughout the paper, $E$ and $F$ will denote complex Banach spaces with unit balls $U$ and $V$, respectively. The corresponding weakly compact
unit balls in the dual Banach spaces $E'$ and $F'$ will be denoted by $B^n$ and $V^m$. The letters $X$ and $Y$ are used for locally compact spaces, $\mu$ and $\gamma$ denote positive Radon measures on $X$ and $Y$, and the spaces $L^p(X)$ or $L^q(Y)$, $1 \leq p < \infty$, of equivalence classes of complex-valued $\mu$-measurable functions on $X$ are defined in the usual way. The Banach space of (equivalence classes of) $E$-valued $\mu$-measurable functions on $X$ such that
\[
\left( \int \|f(x)\|^p d\mu(x) \right)^{\frac{1}{p}} < \infty
\]
will be denoted by $L^p(X, E)$ or $L^p(E)$. We shall make no notational distinction between an element $f$ in an $L^p$-space and a representing function for $f$.

1. $p$-nuclear operators between $L^p$-spaces. As in [3], § 1, we denote by $N_p(E, F)$ the set of all linear mappings $T$ of $E$ into $F$ which can be written in the form
\[
T u = \sum_{n=1}^{\infty} \langle u, w_n \rangle v_n
\]
with
\[
\left( \sum_{n=1}^{\infty} |\langle u, w_n \rangle|^p \right)^{\frac{1}{p}} < \infty \quad \text{and} \quad \sup_{n \geq 1} \left( \sum_{n=1}^{\infty} |\langle u, v_n \rangle|^p \right)^{\frac{1}{p}} < \infty
\]
when $1 \leq p < \infty$ and with the additional requirement $|\langle u, w_n \rangle| \to 0$, $n \to \infty$, in the case $p = \infty$. It turns out that $N_p(E, F)$ is a linear space and in fact a Banach space when equipped with the norm
\[
N_p(T) = \inf \left( \sum_{n=1}^{\infty} |\langle u, w_n \rangle|^p \sup_{n \geq 1} \left( \sum_{n=1}^{\infty} |\langle u, v_n \rangle|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}},
\]
where the infimum is taken over all adequate representations (1) of $T$. The elements in $N_p(E, F)$ are called $p$-nuclear operators or operators of type $N_p$. Interchanging the roles of the sequences $(w_n)$ and $(v_n)$ one obtains in the same way another Banach space $N^p(E, F)$ of operators, the norm being given by
\[
N^p(T) = \inf \left( \sum_{n=1}^{\infty} |\langle u, w_n \rangle|^p \sup_{n \geq 1} \left( \sum_{n=1}^{\infty} |\langle u, v_n \rangle|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]

For $p = 1$ the two classes $N_1(E, F)$ and $N^1(E, F)$ are equal and coincide with the space of nuclear operators of $E$ into $F$. The set $L_1(E, F)$ of finite-dimensional linear mappings of $E$ into $F$ is dense in all the spaces $N_p(E, F)$ and $N^p(E, F)$. It is easy to see that an operator $T$ of type $N_p$ has a factorization of the form
\[
E \overset{T}{\longrightarrow} E' \overset{D}{\longrightarrow} L^p \overset{\Phi}{\longrightarrow} F
\]
where $||F|| \leq 1$, $||Q|| \leq 1$ and $D$ is multiplication with a sequence $(\lambda_n)$ satisfying
\[
\sum_{n=1}^{\infty} |\lambda_n|^p \leq N_p(T) + \varepsilon,
\]
$\varepsilon > 0$ being given in advance. Similarly, every $T \in N^p(E, F)$ can be written
\[
E \overset{P}{\longrightarrow} E' \overset{Q}{\longrightarrow} L^p \overset{\Phi}{\longrightarrow} F
\]
with $P, Q, D$ as above and
\[
\sum_{n=1}^{\infty} |\lambda_n|^p \leq N^p(T) + \varepsilon.
\]

This fact is used in the proof of the following lemma, which in the case of operators of type $N_p$ is contained in [5], § 6, Lemma 7, and which can be proved in a similar way in the case of $N^p$-operators.

**LEMMA 1.** Suppose $E$ or $F$ satisfies the metric approximation condition and that $T : E \to F$ has finite-dimensional range. Then, for every $\varepsilon > 0$, there exists a finite representation $T u = \sum_{n=1}^{M} \langle u, w_n \rangle v_n$ such that
\[
\sum_{n=1}^{M} |\langle u, w_n \rangle|^p \sup_{n \geq 1} \left( \sum_{n=1}^{M} |\langle u, v_n \rangle|^p \right)^{\frac{1}{p}} < N_p(T) + \varepsilon.
\]

Similarly, there exists a finite representation $T u = \sum_{n=1}^{N} \langle u, x_n \rangle y_n$ such that
\[
\sup_{n \geq 1} \left( \sum_{n=1}^{N} |\langle u, x_n \rangle|^p \right)^{\frac{1}{p}} \sum_{n=1}^{N} |\langle u, y_n \rangle|^p < N^p(T) + \varepsilon.
\]

If $\Phi \in L_1^0(X, E')$, then the formula
\[
(T u)(x) = \langle u, \Phi(x) \rangle \quad \text{a.e.}
\]
determines a bounded linear mapping $T$ of $E$ into $L_1^0(X)$. Grothendieck [4] proved that this correspondence is an isometry between $L_1^0(E')$ and $N_1(E, L_1^0)$. We shall prove the following extension of this result to arbitrary $1 \leq p < \infty$ (a similar generalization has been announced by Chevet [3]):

**THEOREM 1.** For $1 \leq p < \infty$ we have natural embeddings
\[
N_p(E, L_1^0) \subset L_p(E') \subset N_p(E, L_p^0),
\]
each of norm $< 1$ and such that elements $T : E \to L_p^0$ and $\Phi \in L_1^0(E')$ corresponding to each other satisfy identity (2).
Proof. A linear operator $T$ of $E$ into $L_p^q$ with finite-dimensional range has a finite representation of the form

$$T = \sum_{t=1}^{M} \langle u_t, u'_t \rangle v_t,$$

and it is easy to see that the function

$$\Phi_T = \sum_{t=1}^{M} u'_t v_t$$

in $L_p^q(E')$ does not depend on the actual choice of representation of $T$. Hence the correspondence $T \rightarrow \Phi_T$ defines a linear mapping of $L_p(E, L_p^q)$ into $L_p^q(E')$, and since

$$\left( \int \|\Phi_T(x)\|^{p'} d\mu(x) \right)^{1/p'} = \left( \int \sup_{\|x\| \leq 1} \left( \sum_{t=1}^{M} \langle u_t, u'_t \rangle \tau v_t (x) \right)^{p'} d\mu(x) \right)^{1/p'},$$

for all $g \in L_p^q$. According to the definition of the norm in $N_p(E, L_p^q)$ this shows the desired inequality (3) and thus completes the proof of the theorem.

In the same way one proves

**Theorem 2.** For $1 \leq p < \infty$ we have natural embeddings

$$N_p(L_p^q, E) \subset L_p^q(E', F) \subset N_p(L_p^q, F),$$

each of norm $\leq 1$ and such that elements $T: L_p^q \rightarrow F$ and $\Phi \in L_p^q(E)$ corresponding to each other satisfy

$$Tu = \int \Phi(y) u(y) d\mu(y).$$

Before stating the last theorem of this section we recall that a linear mapping $T: E \rightarrow F$ is called absolutely $p$-summing (cf. [1]) if there exists a constant $C$ such that for every finite sequence $(u_n)_{n=1}^M$ satisfying

$$\sum_{t=1}^{M} \langle u_n, u'_n \rangle \leq ||u_n||_p, \quad u' \in E',$$

one has

$$\left( \sum_{t=1}^{M} ||Tu_n||_q \right)^{1/q} \leq C.$$
Theorem 3. For linear operators $T : L^p_c(Y) \to L^p_c(X)$, $1 < p < \infty$, the following conditions are equivalent:

(a) $T$ is $p$-nuclear;
(b) $T$ is absolutely $p$-summing;
(c) $T$ can be written in the form

$$\langle Tu(x), y \rangle = \int g(x, y) u(y) \, dv(y)$$

with $g \in L^p_{\text{ess}}(X \times Y)$;
(d) there exists a non-negative function $h \in L^p$, such that $\|T u(x)\| < h(x)$ a.e. for all $u \in L^p$ with $\|u\|_{L^p}^{p} \leq 1$.

Moreover

$$N_p(T) = A_p(T) = \left( \int \|y^n \|_{L^p}^p \, dx \right)^{\frac{1}{p}} \leq \left( \int h^p \, dx \right)^{\frac{1}{p}}.$$

Proof. The equivalence of (a) and (c) follows immediately from Theorem 1 and Theorem 2 by observing the familiar fact that $L^p_c(X, L^p_c(Y))$ is isometrically isomorphic to $L^p_{\text{ess}}(X \times Y)$ (cf. [3, ch-III, p. 198]). Condition (c) implies (b); if (c) is satisfied and $(u_n)_n$ is any finite sequence of functions in $L^p$, with

$$\sum_{n=1}^{N} |\langle u, u_n \rangle|^{p} \leq \int \|u\|_{L^p}^p \, dx,$$

then

$$\sum_{n=1}^{N} \|Tu_n\| = \left( \int \sum_{n=1}^{N} \left| \int g(x, y) u_n(y) \, dv(y) \right|^p \, dx \right)^{\frac{1}{p}} \leq \left( \int \|g\|_{L^p}^p \, dx \right)^{\frac{1}{p}} \leq \left( \int \|u\|_{L^p}^p \, dx \right)^{\frac{1}{p}} \leq \left( \int \|h\|_{L^p}^p \, dx \right)^{\frac{1}{p}}.$$

To prove that (b) implies (c), we use the fundamental fact (cf. [5], § 8) that the dual of $N_p(L^p, L^p)$ via $\langle S, T \rangle = \text{tr}(ST)$ can be identified with the space of absolutely $p'$-summing operators of $L^p$ into $L^p$. Therefore, if $T$ is a given element in $A_p(L^p, L^p)$, then for any $S \in N_p(L^p, L^p)$ with a representation $f(s, y)$ according to (c), we have the inequality

$$\text{tr}(ST) \leq A_p(T) N_p(S) \leq A_p(T) \left( \int \|f\|_{L^p}^p \, dx \right)^{\frac{1}{p}}.$$

This shows that there exists a function $g \in L^p_{\text{ess}}(X \times Y)$ such that

$$\text{tr}(TS) = \int \int f(x, y) g(y, x) \, dx \, dy.$$

Choosing in particular operators of the form $Su = \langle u, u' \rangle v$, $u' \in L^p$, $v \in L^p$, we obtain

$$\langle u'; T v \rangle = \text{tr}(TS) = \int \int u'(y) v(s(y, x)) \, dx \, dv(y)$$

$$= \int u'(y) \, dv(y) \int g(y, x) v(x) \, dx \, dv(x),$$

whence

$$(Tv)(y) = \int g(y, x) v(x) \, dx \, dv(x) \text{ a.e.}$$

The implication (b) $\Rightarrow$ (c) follows from this by interchanging $L^p_c$ and $L^p$. In the course of the proof we have also obtained the equality signs in formula (4).

It is obvious that (c) $\Rightarrow$ (d), and we complete the proof of the theorem by showing that (d) $\Rightarrow$ (b). Put

$$M = \{u \in L^p_c(X) : |(u(x)| \leq h(x) \text{ a.e.} \}$$

and let $L^p_M$ denote the linear hull of $M$. A Banach space structure is introduced on $L^p_M$ by considering $M$ as the unit ball. If $T$ satisfies (d), then it has a factorization of the form

$L^p_c \xrightarrow{T_1} L^p_M \xrightarrow{T_2} L^p_{\text{ess}}$.

where $T_1 = T$, $I$ is the identity and

$$(Tu)(x) = \begin{cases} u(x) & h(x) \neq 0, \\ h(x)/h(x) & h(x) = 0, \end{cases}$$

$$(Tv)(x) = h(x) v(x).$$

However, by [8] the mapping $f$ is absolutely $p$-summing with

$$A_p(I) \leq \left( \int h^p \, dx \right)^{\frac{1}{p}}.$$
with \( L_p(T) \subseteq N_p(T) \), and this inclusion is in general a proper one. However, the following theorems will tell us that in certain important situations we actually have equality in (5).

**Theorem 4.** If \( S : E \rightarrow F \) is weakly compact and \( T : F \rightarrow G \) is \( p \)-integral (\( 1 \leq p < \infty \)), then the composed mapping \( TS : E \rightarrow G \) is \( p \)-nuclear and \( \|N_p(TS)\| \leq \|I_p(T)\|^\epsilon \|G\| \).

**Proof.** Let
\[
\begin{align*}
E & \subseteq C(V^\nu) \subseteq L^p_r(V^\nu) \subseteq G \\
&\subseteq G
\end{align*}
\]
be a factorization of \( T \) with \( |P| \leq 1 \), \( |Q| \leq 1 \) and \( \mu(V^\nu)^{\frac{1}{p}} \leq \mu(T) + \epsilon \), where \( \epsilon > 0 \) is given in advance. We first consider the weakly compact mapping \( R = PS \) of \( E \) into \( C(V^\nu) \). If, for any \( x \in V^\nu \), the Dirac measure on \( V^\nu \) with support in \( x \) is denoted by \( \delta_x \), then the identity
\[
\langle Ru(x), \delta_y \rangle = \langle R\delta_u, \delta_y \rangle = \langle u, E\delta_y \rangle
\]
shows that the function \( \Phi(x) = E\delta_x \) is continuous in the weak topology \( \sigma(E', E) \) and generates a representation of the form (2) of \( R \). We shall prove that there exists a \( \mu \)-measurable function \( \Psi \) on \( V^\nu \) with values in \( E' \) such that
\[
\langle u, \Psi(x) \rangle = \langle u, \Phi(x) \rangle = \langle Ru, \delta_x \rangle \quad \forall u \in E.
\]
For this purpose we note that, since \( R \) is a weakly compact mapping, so is the transposed mapping \( E^* \) of \( E \) (cf. [3], ch. VI). In particular, \( \Phi \) takes its values in a \( c(E', E') \)-compact subset, and consequently the weak integral \( \int \Phi(f(x))d\mu(x) \) is in \( E' \) for all \( f \in L^p_r(V^\nu) \) (cf. [1], p. 1). Since
\[
\langle u, \int \Phi(f(x))d\mu(x) \rangle = \langle u, \Phi(f) \rangle = \langle Ru, f \rangle, \quad u \in E,
\]
the operator \( E^* \) defined by \( f \mapsto \int \Phi(f(x))d\mu(x) \) is the composition of the natural isometric embedding of \( L^p_r(V^\nu) \) into \( C(V^\nu) \) and the weakly compact mapping \( E \), and hence \( E^* \) is weakly compact. According to a sharp form of the Dunford-Pettis theorem (cf. e.g. [1], p. 215, Exercices) it therefore exists a strongly \( \mu \)-measurable function \( \Psi \) on \( V^\nu \) such that \( \|\Psi(x)\| \leq \|E^*\| \) everywhere in \( V^\nu \) and
\[
\int \Phi(f(x))d\mu(x) = \int \Psi(f(x))d\mu(x), \quad f \in L^p_r(V^\nu).
\]
Equality (6) follows immediately from this, and hence the construction of \( \Psi \) is complete.

Applying the second inclusion in Theorem 1 we now obtain that \( L_E : E \rightarrow L^p_r(V^\nu) \) is \( p \)-nuclear with
\[
N_p(L_E) \leq \left( \int \|\Psi(x)\|^p d\mu(x) \right)^{\frac{1}{p}} \leq \|V^\nu\|^{\frac{1}{p}} \|\mu\|^{\frac{1}{p}} \|\Psi(x)\|_{L^p_r(V^\nu)} \leq \|I_p(T) + \epsilon\|^\epsilon \|G\|.
\]

Hence, by an easily verified property of \( p \)-nuclear operators, \( T_{S+}N_p(E, G) \) and
\[
N_p(TS) \leq (I_p(T) + \epsilon)^{\epsilon} \|G\|.
\]
Since \( \epsilon > 0 \) is arbitrary, this concludes the proof of the theorem.

**Corollary 1.** If \( E \) is reflexive, then, for \( 1 \leq p < \infty \), \( N_p(E, F) = I_p(E, F) \) with equality of the corresponding norms.

**Theorem 5.** If \( E \) has a strongly separable dual \( E' \), then, for \( 1 \leq p < \infty \), \( N_p(E, F) = I_p(E, F) \) with equality of the corresponding norms.

**Proof.** If \( T \subseteq N_p(E, F) \), let \( E \subseteq C(V^\nu) \subseteq L^p_r(V^\nu) \subseteq F \) be a factorization of \( T \) with \( |P| \leq 1 \), \( |Q| \leq 1 \) and \( \mu(V^\nu)^{\frac{1}{p}} \leq \mu(T) + \epsilon \). We have
\[
\langle Pu(x), \delta_y \rangle = \langle Pu, \delta_y \rangle = \langle u, E'\delta_y \rangle, \quad u \in E^*,
\]
and as in the proof of the preceding theorem we are through if we can prove that \( \Phi(x) = E'\delta_x \) is strongly \( \mu \)-measurable and satisfies \( \|\Phi(x)\| \leq 1 \). However, since \( E' \) is strongly separable, for each \( u \) in the unit ball of \( E' \) we can find a sequence \( u_k \in U \) such that \( \langle u_k, u \rangle \rightarrow \langle u_k, u \rangle \rightarrow \langle u_k, u \rangle \) for all \( u \in E' \). Hence the sequence consisting of the continuous functions \( \langle u_k, \Phi(x) \rangle \) converges pointwise to \( \langle u, \Phi(x) \rangle \), proving that \( \langle u, \Phi(x) \rangle \) is \( \mu \)-measurable for all \( u \in E' \). Using once more the separability of \( E' \) we conclude that \( \Phi(x) \) is strongly \( \mu \)-measurable (cf. [1], p. 1). Since obviously
\[
\|\Phi(x)\| = \sup_{p \in I} \|Pu(x)\| \leq 1,
\]
the proof is finished.

We conclude the paper by indicating a couple of applications to the general theory of \( L \)-operators. It was proved in [5], p. 4, that the composition of two operators \( S \) and \( T \), one of which is \( p \)-nuclear and hence the other one absolutely \( q \)-summing, is a mapping of type \( L \), where
\[
1 \leq \frac{1}{r} = \min \left( \frac{1}{p} + \frac{1}{q}, 1 \right).
\]
Using Theorem 4 and the methods of proof in [5] one immediately finds that \( \Phi \) is in fact an \( r \)-nuclear operator provided that \( r \) is defined by (7) and \( 1 \leq p, q < \infty \). Similarly one proves using Corollary 1 and Theorem 5 that, if \( E \) is reflexive or \( E' \) strongly separable, then every absolutely \( p \)-summing operator of \( E \) into \( F \) is quasi-\( p \)-nuclear and hence in particular compact (cf. [5], § 4).

**References**

Ideals in group algebras

by

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Throughout this paper $G$ shall denote a locally compact abelian group. The ideal structure of $L^1(G)$ is still not fully known. For example, at a recent international symposium on functional analysis held at Sopot, Poland, the following questions were asked: (i) are there maximal non-closed ideals in $L^1(G)$ and (ii) what type of prime ideals are in $L^1(G)$? The purpose of this paper is to answer the above mentioned questions. The crux of the matter lies in the following theorem on Banach algebras:

**Theorem 1.** Let $R$ be a commutative Banach algebra with bounded approximate identity and continuous involution. Then every maximal ideal of $R$ is regular and, consequently, closed.

**Proof.** In as much as $R$ has non-regular maximal ideals if, and only if $R' \subsetneq R$ (cf. [3], p. 87-88), we shall assume $R' = M \subset R$, $M$ a non-regular maximal ideal. Hence, there exists $a \in R'$ such that $M + \{ae\} = R$ (cf. [3]). Define $f$ such that $f(m) = 0$, $m \in M$, and $f(\alpha e) = \alpha$. $f$ is a linear functional and, in fact, is positive since $f(\bar{a} \alpha e) = 0$, for $\alpha \in M$ for every $a$. Varopoulos [5] has proved that $f$ is continuous under the hypothesis on $R$. Hence, if $\{e_\alpha\}$ denotes the approximate identity, then $f(\alpha e) = \lim f(e, e) = 0$ since $e_\alpha x \in M$; i.e. $f = 0$. This contradiction establishes Theorem 1.

**Corollary 1.** Every maximal ideal of $L^1(G)$ is regular and, therefore, closed.

**Proof.** $L^1(G)$ has the properties described above for the ring $R$.

**Remark.** We could have proved the Corollary by using a result of Hewitts [2] which shows $(L^1(G))^f = L^1(G)$ and applying the results cited in [3] (cf. [3], p. 96).

We shall now study the prime ideals of $L^1(G)$. Corollary 1 allows us to use spectral synthesis methods in such a study.

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