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On some properties of p-nuclear and p-integral operators

by

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Introduction. In a recent paper [5] A. Pietsch and the present author investigated the general properties of so called p-nuclear and p-integral operators. These are natural generalizations to arbitrary $1 \leqslant p \leqslant \infty$ of the classes of nuclear and integral operators, respectively, which were introduced and studied by Grothendieck [4]. Among other things Grothendieck proved that an integral operator between two Banach spaces E and F is nuclear provided that one of the following four conditions is satisfied:

(i) E is reflexive;

(ii) E has a separable dual;

(iii) F is reflexive;

(iv) F is separable and isomorphic to the dual of a Banach space.

Neither of conditions (iii) and (iv) is sufficient for the corresponding statement for p-integral and p-nuclear operators to hold true. This is shown for example by the natural injection of l^1 into l^2, which is 2-integral ([5], § 10) but not 2-nuclear; in fact, it is not even compact. The principal aim of the present paper is to show that either of conditions (i) and (ii) is sufficient in order that a p-integral mapping of E into F be p-nuclear. The first case is obtained as a corollary of the more general statement (due to Grothendieck [4] in the case $p = 1$) that the composition of a weakly compact linear operator and a p-integral operator is p-nuclear. The proof of this fact depends on a formulation for arbitrary p (cf. also Chevet [2]) of another result of Grothendieck [4], stating that the space of nuclear operators of E into an L^1-space can be identified with the corresponding space of E'-valued integrable functions. This extension makes it also possible to give a complete characterization of a p-nuclear operator of an $L^{p'}$-space into an L^p-space, where $1/p+1/p' = 1$. The situation turns out to be quite analogues to the case of Hilbert-Schmidt mappings between L^2-spaces.

Throughout the paper, E and F will denote complex Banach spaces with unit balls U and V, respectively. The corresponding weakly compact unit balls in the dual Banach spaces E' and F' will be denoted by U^0 and V^0. The letters X and Y are used for locally compact spaces, μ and ν denote positive Radon measures on X and Y, and the spaces $L^p_\mu(X) = L^p_\mu$, $1 \leqslant p \leqslant \infty$, of equivalence classes of complex-valued μ-measurable functions on X are defined in the usual way. The Banach space of (equivalence classes of) E-valued μ-measurable functions on X such that

$$\Big(\int \|f(x)\|^p d\mu(x)\Big)^{1/p} < \infty$$

will be denoted by $L^p_\mu(X, E)$ or $L^p_\mu(E)$. We shall make no notational distinction between an element f in an L^p-space and a representing function for f.

1. p-nuclear operators between L^p-spaces. As in [5], § 1, we denote by $N_p(E, F)$ the set of all linear mappings T of E into F which can be written in the form

$$Tu = \sum_1^\infty \langle u, u_n'\rangle v_n \tag{1}$$

with

$$\Big(\sum_1^\infty \|u_n'\|^p\Big)^{1/p} < \infty \quad \text{and} \quad \sup_{\|v'\|\leqslant 1} \Big(\sum_1^\infty |\langle v_n, v'\rangle|^{p'}\Big)^{1/p'} < \infty$$

when $1 \leqslant p < \infty$ and with the additional requirement $\|u_n'\| \to 0$, $n \to \infty$, in the case $p = \infty$. It turns out that $N_p(E, F)$ is a linear space and in fact a Banach space when equipped with the norm

$$N_p(T) = \inf \Big(\sum_1^\infty \|u_n'\|^p\Big)^{1/p} \sup_{\|v'\|\leqslant 1} \Big(\sum_1^\infty |\langle v_n, v'\rangle|^{p'}\Big)^{1/p'},$$

where the infimum is taken over all adequate representations (1) of T. The elements in $N_p(E, F)$ are called *p-nuclear operators* or *operators of type N_p*. Interchanging the roles of the sequences $(u_n')_1^\infty$ and $(v_n)_1^\infty$ one obtains in the same way another Banach space $N^p(E, F)$ of operators, the norm being given by

$$N^p(T) = \inf \sup_{\|u\|\leqslant 1} \Big(\sum_1^\infty |\langle u, u_n'\rangle|^{p'}\Big)^{1/p'} \Big(\sum_1^\infty \|v_n\|^p\Big)^{1/p}.$$

For $p = 1$ the two classes $N_p(E, F)$ and $N^p(E, F)$ are equal and coincide with the space of nuclear operators of E into F. The set $L_0(E, F)$ of finite-dimensional linear mappings of E into F is dense in all the spaces $N_p(E, F)$ and $N^p(E, F)$. It is easy to see that an operator T of type N_p has a factorization of the form

$$E \xrightarrow{P} l^\infty \xrightarrow{D} L^p \xrightarrow{Q} F$$

where $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$ and D is multiplication with a sequence $(\lambda_n)_1^\infty$ satisfying

$$\Big(\sum_1^\infty |\lambda_n|^p\Big)^{1/p} \leqslant N_p(T) + \varepsilon,$$

$\varepsilon > 0$ being given in advance. Similarly, every $T \in N^p(E, F)$ can be written

$$E \xrightarrow{P} l^{p'} \xrightarrow{D} l^1 \xrightarrow{Q} F$$

with P, Q, D as above and

$$\Big(\sum_1^\infty |\lambda_n|^p\Big)^{1/p} \leqslant N^p(T) + \varepsilon.$$

This fact is used in the proof of the following lemma, which in the case of operators of type N_p is contained in [5], § 6, Lemma 7, and which can be proved in a similar way in the case of N^p-operators.

LEMMA 1. *Suppose E' or F satisfies the metric approximation condition and that $T: E \to F$ has finite-dimensional range. Then, for every $\varepsilon > 0$, there exists a finite representation $Tu = \sum_1^M \langle u, u_n'\rangle v_n$ such that*

$$\Big(\sum_1^M \|u_n'\|^p\Big)^{1/p} \sup_{\|v'\|\leqslant 1} \Big(\sum_1^M |\langle v_n, v'\rangle|^{p'}\Big)^{1/p'} < N_p(T) + \varepsilon.$$

Similarly, there exists a finite representation $Tu = \sum_1^N \langle u, x_n'\rangle y_n$ such that

$$\sup_{\|u\|\leqslant 1} \Big(\sum_1^N |\langle u, x_n'\rangle|^{p'}\Big)^{1/p'} \Big(\sum_1^N \|y_n\|^p\Big)^{1/p} < N^p(T) + \varepsilon.$$

If $\Phi \in L^p_\mu(X, E')$, then the formula

$$(Tu)(x) = \langle u, \Phi(x)\rangle \text{ a.e.} \tag{2}$$

determines a bounded linear mapping T of E into $L^p_\mu(X)$. Grothendieck [4] proved that this correspondence is an isometry between $L^1_\mu(E')$ and $N_1(E, L^1_\mu)$. We shall prove the following extension of this result to arbitrary $1 \leqslant p < \infty$ (a similar generalization has been announced by Chevet [2]):

THEOREM 1. *For $1 \leqslant p < \infty$ we have natural embeddings*

$$N^p(E, L^p_\mu) \subset L^p_\mu(E') \subset N_p(E, L^p_\mu),$$

each of norm $\leqslant 1$ and such that elements $T: E \to L^p_\mu$ and $\Phi \in L^p_\mu(E')$ corresponding to each other satisfy identity (2).

Proof. A linear operator T of E into L^p_μ with finite-dimensional range has a finite representation of the form

$$Tu = \sum_1^M \langle u, u_n'\rangle v_n,$$

and it is easy to see that the function

$$\Phi_T = \sum_1^M u_n' v_n$$

in $L^p_\mu(E')$ does not depend on the actual choice of representation of T. Hence the correspondence $T \to \Phi_T$ defines a linear mapping of $L_0(E, L^p_\mu)$ into $L^p_\mu(E')$, and since

$$\Big(\int \|\Phi_T(x)\|^p d\mu(x)\Big)^{1/p} = \Big(\int \sup_{\|u\|\leqslant 1} \Big|\sum_1^M \langle u, u_n'\rangle v_n(x)\Big|^p d\mu(x)\Big)^{1/p}$$
$$\leqslant \sup_{\|u\|\leqslant 1} \Big(\sum_1^M |\langle u, u_n'\rangle|^{p'}\Big)^{1/p'} \Big(\sum_1^M \int |v_n(x)|^p d\mu(x)\Big)^{1/p},$$

it follows from Lemma 1 that this mapping is of norm $\leqslant 1$ if $L_0(E, L^p_\mu)$ is considered as a subspace of $N^p(E, L^p_\mu)$. We can extend it by continuity to the whole of $N^p(E, L^p_\mu)$ without increasing its norm. Using the fact that a convergent sequence in an L^p-space contains a subsequence which converges almost everywhere, one finds that, for each $T \in N^p(E, L^p_\mu)$,

$$(Tu)(x) = \langle u, \Phi_T(x)\rangle \text{ a.e.}, \quad u \in E.$$

This concludes the proof of the first half of the theorem.

A density argument shows that for the second inclusion it is enough to prove that if Φ is a step-function

$$\Phi(x) = \sum_1^m u_n' \chi_n(x),$$

where χ_n are characteristic functions of disjoint integrable sets K_n, then (2) defines a mapping T with

$$N_p(T) \leqslant \Big(\int \|\Phi(x)\|^p d\mu(x)\Big)^{1/p} = \Big(\sum_1^m \|u_n'\|^p \mu(K_n)\Big)^{1/p}. \tag{3}$$

However, if we write

$$\Phi(x) = \sum_1^m \big(\mu(K_n)^{1/p} u_n'\big)\cdot\big(\mu(K_n)^{-1/p}\chi_n(x)\big) = \sum_1^m v_n' \psi_n(x),$$

then T has a representation of the form

$$Tu = \sum_1^m \langle u, v_n'\rangle \psi_n$$

with

$$\Big(\sum_1^m \|v_n'\|^p\Big)^{1/p} = \Big(\sum_1^m \|u_n'\|^p \mu(K_n)\Big)^{1/p} = \Big(\int \|\Phi(x)\|^p d\mu(x)\Big)^{1/p}$$

and

$$\Big(\sum_1^m |\langle \psi_n, g\rangle|^{p'}\Big)^{1/p'} = \Big(\sum_1^m \mu(K_n)^{-p'/p} \Big|\int_{K_n} g(x)\, d\mu(x)\Big|^{p'}\Big)^{1/p'}$$
$$\leqslant \Big(\sum_1^m \int_{K_n} |g|^{p'} d\mu\Big)^{1/p'} = \Big(\int |g|^{p'} d\mu\Big)^{1/p'}$$

for all $g \in L^{p'}_\mu$. According to the definition of the norm in $N_p(E, L^p_\mu)$ this shows the desired inequality (3) and thus completes the proof of the theorem.

In the same way one proves

THEOREM 2. *For $1 \leqslant p < \infty$ we have natural embeddings*

$$N_p(L^{p'}_\nu, F) \subset L^p_\nu(F) \subset N^p(L^{p'}_\nu, F),$$

each of norm $\leqslant 1$ and such that elements $T: L^{p'}_\nu \to F$ and $\Phi \in L^p_\nu(F)$ corresponding to each other satisfy

$$Tu = \int \Phi(y) u(y)\, d\nu(y).$$

Before stating the last theorem of this section we recall that a linear mapping $T: E \to F$ is called *absolutely p-summing* (cf. [6]) if there exists a constant C such that for every finite sequence $(u_n)_1^M$ satisfying

$$\Big(\sum_1^M |\langle u_n, u'\rangle|^p\Big)^{1/p} \leqslant \|u'\|, \quad u' \in E',$$

one has

$$\Big(\sum_1^M \|Tu_n\|^p\Big)^{1/p} \leqslant C.$$

The set of all such operators is a Banach space when provided with the norm

$$A_p(T) = \inf C.$$

THEOREM 3. *For linear operators* $T: L_\nu^{p'}(Y) \to L_\mu^p(X)$, $1 < p < \infty$, *the following conditions are equivalent*:

(a) T *is p-nuclear*;

(b) T *is absolutely p-summing*;

(c) T *can be written in the form*

$$(Tu)(x) = \int g(x, y) u(y) d\nu(y) \quad \text{with } g \in L^p_{\mu\times\nu}(X\times Y);$$

(d) *there exists a non-negative function* $h \in L_\mu^p$, *such that* $|(Tu)(x)| \leqslant h(x)$ *a.e. for all* $u \in L_\nu^{p'}$ *with* $\left(\int |u|^{p'} d\nu\right)^{1/p'} \leqslant 1$.

Moreover

(4) $$N_p(T) = A_p(T) = \left(\int\int |g|^p d\mu d\nu\right)^{1/p} \leqslant \left(\int h^p d\mu\right)^{1/p}.$$

Proof. The equivalence of (a) and (c) follows immediately from Theorem 1 and Theorem 2 by observing the familiar fact that $L_\mu^p(X, L_\nu^p(Y))$ is isometrically isomorphic to $L^p_{\mu\times\nu}(X\times Y)$ (for a special case see [3, ch-III, p. 198]). Condition (c) implies (b); for if (c) is satisfied and $(u_n)_1^M$ is any finite sequence of functions in $L_\nu^{p'}$ with

$$\sum_1^M |\langle v, u_n\rangle|^p \leqslant \int |v|^p d\nu,$$

then

$$\sum_1^M \|Tu_n\|^p = \int \left[\sum_1^M \left|\int g(x, y) u_n(y) d\nu(y)\right|^p\right] d\mu(x)$$
$$\leqslant \int \left(\int |g(x, y)|^p d\nu(y)\right) d\mu(x) = \int\int |g|^p d\mu d\nu.$$

To prove that (b) implies (c) we use the fundamental fact (cf. [5], § 8) that the dual of $N_p(L_\nu^{p'}, L_\mu^p)$ via $\langle S, T\rangle = \operatorname{tr}(ST)$ can be identified with the space of absolutely p'-summing operators of L_μ^p into $L_\nu^{p'}$. Therefore, if T is a given element in $A_{p'}(L_\mu^p, L_\nu^{p'})$, then, for any $S \in N_p(L_\nu^{p'}, L_\mu^p)$ with a representation $f(x, y)$ according to (c), we have the inequality

$$|\operatorname{tr}(TS)| \leqslant A_{p'}(T) N_p(S) \leqslant A_{p'}(T)\left(\int\int |f|^p d\mu d\nu\right)^{1/p}.$$

This shows that there exists a function $g \in L^{p'}_{\nu\times\mu}(Y\times X)$ such that

$$\operatorname{tr}(TS) = \int\int f(x, y) g(y, x) d\mu(x) d\nu(y).$$

Choosing in particular operators of the form $Su = \langle u, u'\rangle v$, $u' \in L_\nu^p$, $v \in L_\mu^p$, we obtain

$$\langle u', Tv\rangle = \operatorname{tr}(TS) = \int\int u'(y) v(x) g(y, x) d\mu(x) d\nu(y)$$
$$= \int u'(y) d\nu(y) \int g(y, x) v(x) d\mu(x),$$

whence

$$(Tv)(y) = \int g(y, x) v(x) d\mu(x) \text{ a.e.}$$

The implication (b) $\Rightarrow$ (c) follows from this by interchanging L_μ^p and $L_\nu^{p'}$. In the course of the proof we have also obtained the equality signs in formula (4).

It is obvious that (c) $\Rightarrow$ (d), and we complete the proof of the theorem by showing that (d) $\Rightarrow$ (b). Put

$$M = \{u \in L_\mu^p(X) : |u(x)| \leqslant h(x) \text{ a.e.}\}$$

and let L_M denote the linear hull of M. A Banach space structure is introduced on L_M by considering M as the unit ball. If T satisfies (d), then it has a factorization of the form

$$L_\nu^{p'} \xrightarrow{T_1} L_M \xrightarrow{T_2} L^\infty_{h^p\cdot\mu} \xrightarrow{I} L^p_{h^p\cdot\mu} \xrightarrow{T_3} L_\mu^p,$$

where $T_1 = T$, I is the identity and

$$(T_2u)(x) = \begin{cases} \dfrac{u(x)}{h(x)}, & h(x) \neq 0, \\ 0, & h(x) = 0, \end{cases}$$

$$(T_3v)(x) = h(x) v(x).$$

However, by [6] the mapping I is absolutely p-summing with

$$A_p(I) \leqslant \left(\int h^p d\mu\right)^{1/p},$$

and since clearly $\|T_i\| \leqslant 1$ $(i = 1, 2, 3)$ it follows that T is absolutely p-summing and that the inequality in (4) is satisfied. The proof is finished.

2. Connection between p-integral and p-nuclear operators. A p-integral operator $T: E \to F$ $(1 \leqslant p < \infty)$ is characterized by the fact that it has a factorization of the form

$$E \xrightarrow{P} C(U^0) \xrightarrow{I} L_\mu^p(U^0) \xrightarrow{Q} F,$$

where $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$ and I is the identity mapping (cf. [5], § 3). The set $I_p(E, F)$ of all p-integral mappings of E into F is a linear space and in fact a Banach space when provided with the norm

$$I_p(T) = \inf \mu(U^0)^{1/p},$$

where the infimum is taken over all adequate factorizations. One has

(5) $$N_p(E, F) \subset I_p(E, F)$$

with $I_p(T) \leqslant N_p(T)$, and this inclusion is in general a proper one. However, the following theorems will tell us that in certain important situations we actually have equality in (5).

THEOREM 4. *If $S: E \to F$ is weakly compact and $T: F \to G$ is p-integral $(1 \leqslant p < \infty)$, then the composed mapping $TS: E \to F \to G$ is p-nuclear and $N_p(TS) \leqslant I_p(T)\|S\|$.*

Proof. Let
$$F \xrightarrow{P} C(V^0) \xrightarrow{I} L^p_\mu(V^0) \xrightarrow{Q} G$$
be a factorization of T with $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$ and $\mu(V^0)^{1/p} < I_p(T)+\varepsilon$, where $\varepsilon > 0$ is given in advance. We first consider the weakly compact mapping $R = PS$ of E into $C(V^0)$. If, for any $x \in V^0$, the Dirac measure on V^0 with support in x is denoted by δ_x, then the identity
$$(Ru)(x) = \langle Ru, \delta_x\rangle = \langle u, R'\delta_x\rangle$$
shows that the function $\Phi(x) = R'\delta_x$ is continuous in the weak topology $\sigma(E', E)$ and generates a representation of the form (2) of R. We shall prove that there exists a μ-measurable function Ψ on V^0 with values in E' such that

$$\langle u, \Psi(x)\rangle = \langle u, \Phi(x)\rangle = (Ru)(x) \text{ a.e.} \tag{6}$$

for all $u \in E$. For this purpose we note that, since R is a weakly compact mapping, so is the transposed mapping R' of R (cf. [3], ch. VI). In particular, Φ takes its values in a $\sigma(E', E'')$-compact subset, and consequently the weak integral $\int \Phi(x)f(x)\,d\mu(x)$ is in E' for all $f \in L^1_\mu(V^0)$ (cf. [1], § 1). Since
$$\langle u, \int \Phi(x)f(x)\,d\mu(x)\rangle = \int \langle u, \Phi(x)\rangle f(x)\,d\mu(x) = \langle Ru, f\cdot\mu\rangle, \qquad u \in E,$$
the operator R^* defined by $f \to \int \Phi(x)f(x)\,d\mu(x)$ is the composition of the natural isometric embedding of $L^1_\mu(V^0)$ into $C(V^0)'$ and the weakly compact mapping R', and hence R^* is weakly compact. According to a sharp form of the Dunford-Pettis theorem (cf. e.g. [1], § 2, Exercises) it therefore exists a strongly μ-measurable function Ψ on V^0 with values in E' such that $\|\Psi(x)\| \leqslant \|R^*\|$ everywhere in V^0 and
$$\int \Phi(x)f(x)\,d\mu(x) = \int \Psi(x)f(x)\,d\mu(x), \qquad f \in L^1_\mu(V^0).$$

Equality (6) follows immediately from this, and hence the construction of Ψ is complete.

Applying the second inclusion in Theorem 1 we now obtain that $IR: E \to L^p_\mu(V^0)$ is p-nuclear with
$$N_p(IR) \leqslant \Big(\int \|\Psi(x)\|^p\,d\mu(x)\Big)^{1/p} \leqslant \mu(V^0)^{1/p}\sup_{x\in V^0}\|\Psi(x)\|$$
$$\leqslant \big(I_p(T)+\varepsilon\big)\|R^*\| \leqslant \big(I_p(T)+\varepsilon\big)\|S\|.$$

Hence, by an easily verified property of p-nuclear operators, $TS \in N_p(E, G)$ and
$$N_p(TS) \leqslant \big(I_p(T)+\varepsilon\big)\|S\|.$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof of the theorem.

COROLLARY 1. *If E is reflexive, then, for $1 \leqslant p < \infty$, $N_p(E, F) = I_p(E, F)$ with equality of the corresponding norms.*

THEOREM 5. *If E has a strongly separable dual E', then, for $1 \leqslant p < \infty$, $N_p(E, F) = I_p(E, F)$ with equality of the corresponding norms.*

Proof. If $T \in I_p(E, F)$, let $E \xrightarrow{P} C(U^0) \xrightarrow{I} L^p_\mu(U^0) \xrightarrow{Q} F$ be a factorization of T with $\|P\| \leqslant 1$, $\|Q\| \leqslant 1$ and $\mu(U^0)^{1/p} < I_p(T)+\varepsilon$. We have
$$(Pu)(x) = \langle Pu, \delta_x\rangle = \langle u, P'\delta_x\rangle, \qquad x \in U^0,$$
and as in the proof of the preceding theorem we are through if we can prove that $\Phi(x) = P'\delta_x$ is strongly μ-measurable and satisfies $\|\Phi(x)\| \leqslant 1$. However, since E' is strongly separable, for each u'' in the unit ball of E'' we can find a sequence $u_n \in U$ such that $\langle u_n, u'\rangle \to \langle u'', u'\rangle$ for all $u' \in E'$. Hence the sequence consisting of the continuous functions $\langle u_n, \Phi(x)\rangle$ converges pointwise to $\langle u'', \Phi(x)\rangle$, proving that $\langle u'', \Phi(x)\rangle$ is μ-measurable for all $u'' \in E''$. Using once more the separability of E' we conclude that $\Phi(x)$ is strongly μ-measurable (cf. [1], § 1). Since obviously
$$\|\Phi(x)\| = \sup_{\|u\|\leqslant 1}|(Pu)(x)| \leqslant 1,$$
the proof is finished.

We conclude the paper by indicating a couple of applications to the general theory of I_p-operators. It was proved in [5], § 7, that the composition of two operators S and T, one of which is p-integral and the other one absolutely q-summing, is a mapping of type I_r, where

$$\frac{1}{r} = \min\Big(\frac{1}{p}+\frac{1}{q}, 1\Big). \tag{7}$$

Using Theorem 4 and the methods of proof in [5] one immediately finds the sharper result that TS is in fact an r-nuclear operator provided that r is defined by (7) and $1 \leqslant p, q < \infty$. Similarly one proves using Corollary 1 and Theorem 5 that, if E is reflexive or E' strongly separable, then every absolutely p-summing operator of E into F is quasi-p-nuclear and hence in particular compact (cf. [5], § 4).

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Ideals in group algebras *

by

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Throughout this paper G shall denote a locally compact abelian group. The ideal structure of $L_1(G)$ is still not fully known. For example, at a recent international symposium on functional analysis held at Sopot, Poland, the following questions were asked: (i) are there maximal non-closed ideals in $L_1(G)$ and (ii) what type of prime ideals are in $L_1(G)$? The purpose of this paper is to answer the afore mentioned questions. The crux of the matter lies in the following theorem on Banach algebras:

THEOREM 1. *Let R be a commutative Banach algebra with bounded approximate identity and continuous involution. Then every maximal ideal of R is regular and, consequently, closed.*

Proof. In as much as R has non-regular maximal ideals if, and only if $R^2 \subsetneqq R$ (cf. [3], p. 87-88), we shall assume $R^2 \subset M \subset R$, M a non-regular maximal ideal. Hence, there exists $x \in R - R^2$ such that $M + \{ax\} = R$ (cf. [3]). Define f such that $f(m) = 0$, $m \in M$, and $f(ax) = a$. f is a linear functional and, in fact, is positive since $f(x^* x) = 0$, for $x^* x \in M$ for every x. Varopoulos [5] has proved that f is continuous under the hypothesis on R. Hence, if $\{e_a\}$ denotes the approximate identity, then $f(x) = \lim_a f(e_a x) = 0$ since $e_a x \in M$; i.e. $f \equiv 0$. This contradiction establishes Theorem 1.

COROLLARY 1. *Every maximal ideal of $L_1(G)$ is regular and, therefore, closed.*

Proof. $L_1(G)$ has the properties described above for the ring R.

Remark. We could have proved the Corollary by using a result of Hewitts [2] which shows $(L_1(G))^2 = L_1(G)$ and applying the results cited in [3] (cf. [3], p. 96).

We shall now study the prime ideals of $L_1(G)$. Corollary 1 allows us to use spectral synthesis methods in such a study.

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