$X = C[-a, a]$. The operator K is anticommutative with S. Indeed,

$$(SKx)(t) = \int_{-a}^{a} K(-t, s)x(s)\,ds,$$

$$(KSx)(t) = \int_{-a}^{a} K(t, s)x(-s)\,ds = -\int_{a}^{-a} K(t, -u)x(u)\,du$$
$$= \int_{-a}^{a} K(t, -u)x(u)\,du = -\int_{-a}^{a} K(-t, u)x(u)\,du = -(SKx)(t),$$

because $K(-t, s) = -K(t, -s)$.

Let us consider the equation $(a_0^2 - b_0^2 \neq 0 \neq a_1^2 - b_1^2)$

$$(4.2)\qquad a_0 x(t) + b_0 x(-t) + a_1 \int_{-a}^{a} K(t, s)x(s)\,ds + b_1 \int_{-a}^{a} K(-t, s)x(s)\,ds = y(t),$$
$$y \in C[-a, a].$$

According to Theorem 2.9, for solving (4.2) it is sufficient to know a solution of the equation $(K^2 - \lambda I)x = y$ and all the solutions of the equation $(K - \sqrt{\lambda} I)z_1 = 0$. But

$$(K^2 x)(t) = \int_{-a}^{a} K_1(t, s)x(s)\,ds, \quad \text{where } K_1(t, s) = \int_{-a}^{a} K(t, u)K(u, s)\,du.$$

This means that for solving (4.2) it is enough to solve the following equations:

$$-\lambda \tilde{x}(t) + \int_{-a}^{a} K_1(t, s)\tilde{x}(s)\,ds = y(t),$$

$$-\sqrt{\lambda} z_1(t) + \int_{-a}^{a} K(t, s)z_1(s)\,ds = 0.$$

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A uniform algebra with non-global peak points

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1. Introduction. By a *uniform algebra* A on a topological space X we mean an algebra of continuous complex-valued functions on X which contains the constants and is closed under uniform convergence on compact subsets of X. A point $p \in X$ is said to be a *local peak point* of A in X if there exists a neighborhood U of p in X and a function $a \in A$ such that $a(p) = 1$ and $|a(x)| < 1$ if $x \in U \setminus \{p\}$. If X is compact and the space $M(A)$ of non-zero continuous homomorphisms of A, with the w*-topology, is (homeomorphic to) X, it is known that every local peak point of A in X is a global peak point, i.e., U can be taken to be X [3]. It is the purpose of this paper to show that this result is not true for general uniform algebras. We exhibit a uniform algebra A on a σ-compact space M (which is set-wise just the complex numbers C) which has local peak points in $M = M(A)$ but has no global peak points. In fact, A contains no non-constant bounded functions.

2. The construction. We describe a sequence of subsets M_m of the plane which satisfy

(i) $M_m \subset M_{m+1}$;

(ii) $\bigcup_{m=1}^{\infty} M_m = C$;

(iii) each M_m is compact and non-separating;

(iv) for every z_0 on the positive y-axis, there is a sequence $\{z_m\}_{m=1}^{\infty}$, $z_m \in M_m - M_{m-1}$, such that $z_m \to z_0$;

(v) for every positive integer m and every z_0 on the positive y-axis with $|z_0| \leqslant m$ there is a sequence $\{z_n\}_{n=1}^{\infty} \subset M_m$ such that $z_n \to z_0$;

(vi) for every z not lying on the positive y-axis, z lies in the interior $\operatorname{int} M_m$ of M_m for some m.

Let B_m be the union of the lines $\{(1/n, y) : m/2n \leqslant y \leqslant m\}$ and the line segments joining the following pairs of points:

$$(0, -m) \text{ and } (m, -m); \quad (m, -m) \text{ and } (m, m);$$

$$(m, m) \text{ and } \left(\frac{1}{m}, m\right); \quad \left(\frac{1}{m}, m\right) \text{ and } \left(\frac{1}{m}, \frac{1}{2}\right);$$

$$\left(\frac{1}{m}, \frac{1}{2}\right) \text{ and } (0, 0); \quad (0, 0) \text{ and } (0, m).$$

Take M_m to be the closed and bounded subset of C bounded by B_m and its reflection about the y-axis.

For each m, let $A_m = (P|M_m)^-$ be the uniform closure on M_m of the algebra P of polynomials in z. By a well-known theorem of Mergelyan, A_m is the commutative Banach algebra of all continuous functions on M_m which are analytic on int M_m. Let $\pi_j^m\colon A_m \to A_j$, $m \geqslant j$, be the algebra isomorphism given by $\pi_j^m(f_m) = f_m|M_j$, $f_m \epsilon A_m$. The collection $\{A_m, \pi_j^m\}$ determines a dense inverse limit system [1]; Let A be the inverse limit of this system and write π_m for the canonical projection of A into A_m. Then A is a commutative F-algebra whose non-zero continuous homomorphism space M is σ-compact and can be identified set-wise with C. Further, A can be regarded as a uniform algebra on M with $M(A) = M$ by defining $f(z) = \pi_m(f)(z)$, $f \epsilon A$, $z \epsilon M_m$.

Birtel and Lindberg show in [2] that an algebra A defined in this way contains no bounded functions except constants, so of course no point of M can be a (global) peak point of A in M. However, if $q = iy$, $y > 1$, then q is a local peak point of A in M, as the following lemmas show.

LEMMA 1. *Let $q = iy_0$, $y_0 > 1$, and take an integer r with $y_0 < r-1$ and $|y_0 - 1| > 1/r$. Let*

$$K_m = \left\{(x, y): \frac{4m+7}{4(m+1)(m+2)} \leqslant |x| \leqslant \frac{4m+1}{4m(m+1)}, 1 \leqslant y \leqslant r\right\}$$

and let K be the closure in C of $\bigcup_{m=1}^{\infty} K_m$. Then there is a neighborhood U of q in M with $U \subset K$.

Proof. Set

$$Y_m = \left\{(x, y): \frac{4m+1}{4m(m+1)} \leqslant |x| \leqslant \frac{4m+3}{4m(m+1)}, \ 1 \leqslant y \leqslant m\right\}.$$

Then Y_m is compact, $Y_m \cap M_m = \emptyset$, and $Y_m \subset M_{m+1}$. We find for each m a polynomial p_m as follows: Let

$$f_1(z) = \begin{cases} 2, & z \epsilon Y_1, \\ 0, & z \epsilon M_1. \end{cases}$$

By Mergelyan, there is a polynomial p_1 with $\|p_1 - f_1\|_{M_1 \cup Y_1} < \frac{1}{4}$. Assume now that p_{m-1} has been chosen. Since

$$f_m(z) = \begin{cases} p_{m-1}(z), & z \epsilon M_m, \\ 2, & z \epsilon Y_m, \end{cases}$$

is continuous on $M_m \cup Y_m$ and analytic on the interior of this set, there is a polynomial p_m such that $\|p_m - f_m\|_{M_m \cup Y_m} < 1/2^{m+1}$. If $\varepsilon > 0$ and m is taken large enough so that $\sum_{k=m}^{\infty} \frac{1}{2^{k+2}} < \varepsilon$, then for $n > m$,

$$\|p_n - p_m\|_{M_m} \leqslant \sum_{k=m}^{n-1} \|p_{k+1} - p_k\|_{M_m} \leqslant \sum_{k=m}^{n-1} (\|p_{k+1} - f_{k+1}\|_{M_m} + \|f_{k+1} - p_k\|_{M_m})$$
$$< \sum_{k=m}^{n-1} \frac{1}{2^{k+2}} < \varepsilon,$$

so $\{p_n\}_{n=1}^{\infty}$ converges uniformly on each set M_m and thus determines an element $f \epsilon A$. But

$$|f(q)| \leqslant |p_1(q)| + \sum_{k=1}^{\infty} |p_{k+1}(q) - p_k(q)| < \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{2^{k+2}} < 1$$

while if $z \epsilon Y_m$ for some m, then

$$|f(z)| \geqslant |p_m(z)| - \sum_{k=1}^{\infty} |p_{k+1}(z) - p_k(z)| > \left(2 - \frac{1}{2^{m+1}}\right) - \sum_{k=m}^{\infty} \frac{1}{2^{k+2}} > 1.$$

Thus if $Y = \bigcup_{m=1}^{\infty} Y_m$, then $\{z \epsilon M : |f(z)| < 1\} \cap Y = \emptyset$, so the w*-open neighborhood $U = \{z : |f(z)| < 1, \ |z-q| < 1/r\}$ of q in M is contained in K since $K \cup Y \supset \{z : |z-q| < 1/r\}$.

LEMMA 2. *There is an element $a \epsilon A$ which peaks at q in K.*

Proof. Set $X_m = M_m \cup K$ and let $U_m = \{z : |z-q| < 1/m\}$. Note that X_m is a compact non-separating subset of C. We find by induction polynomials b_m which satisfy

$$\|b_{m+1} - b_m\|_{X_m} < \frac{1}{2^{m-1}}, \tag{1}$$

$$\|b_{m+1} - b_m\|_{K \setminus U_m} < \frac{1}{2^{m+1}}, \tag{2}$$

$$b_m(q) = 3\left(1 - \frac{1}{2^m}\right), \tag{3}$$

$$\|b_m\|_K \leqslant 3\left(1 - \frac{1}{2^{m+1}}\right). \tag{4}$$

By Mergelyan, there is a function $h_m \in (P|\bar{X_m})$ which peaks in X_m at q. Let φ_m be a conformal map of int M_m onto $\{z : |z| < \frac{1}{2}\}$ and φ_0 a conformal map of the right half-plane H onto $\{z : |z| < 1\}$ with continuous boundary values taken so that $\varphi_0(q) = 1$.

Write

$$h_m(z) = \begin{cases} \varphi_m(z), & z \in \operatorname{int} M_m, \\ \varphi_0(z), & z \in K_n \cap H,\ n > m, \\ \varphi_0\left(z + \dfrac{1}{n}\right), & z \in K_n \setminus H,\ n > m, \end{cases}$$

and extend h_m to a function continuous on X_m with $|h_m(z)| < 1$ for $z \in M_m \setminus \{q\}$.)

Let $a_1 \in P$ be such that $\|a_1 - f_1\|_{X_1} < \frac{1}{8}$ and set $b_1 = \frac{3}{2}(a_1/a_1(q))$. Then $\|b_1\|_K < \frac{3}{2}(1+\frac{1}{8})/(1-\frac{1}{8}) \leqslant 3(1-\frac{1}{4})$ and $b_1(q) = 3(1-\frac{1}{2})$. Suppose $b_1, \ldots, b_k$ have been picked satisfying (1)-(4). Since $b_k(q) = 3(1-1/2^k)$, there is some $j > k$ such that

$$|b_k(z)| < 3\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+2}} \quad \text{for } z \in U_j.$$

Choosing n_j large enough so that $\|h_j^{n_j}\|_{X_j \setminus U_j} < \frac{1}{16}$ and $a_j \in P$ with $\|a_j - h_j^{n_j}\|_{X_j} < \frac{1}{16}$, set $g_j = a_j - \frac{1}{8}$.

Then $\|g_j\|_{X_j} < 1$, $\|g_j\|_{X_j \setminus U_j} < \frac{1}{4}$, and $|g_j(q)| > \frac{3}{4}$. Let

$$b_{k+1} = b_k + \frac{3}{2^{k+1}} \frac{g_j}{g_j(q)}.$$

It is easily checked that b_{k+1} satisfies (1), (2), and (3). To see (4), note that if $z \in K \cap U_j$,

$$|b_{k+1}(z)| \leqslant 3\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+2}} + \frac{3}{2^{k+1}} \cdot \frac{4}{3} = 3\left(1 - \frac{1}{2^{k+2}}\right),$$

while if $z \in K \setminus U_j$,

$$|b_{k+1}(z)| \leqslant 3\left(1 - \frac{1}{2^{k+1}}\right) + \frac{3}{2^{k+1}} \cdot \frac{4}{3} \cdot \frac{1}{4} < 3\left(1 - \frac{1}{2^{k+2}}\right).$$

Thus we have by induction the desired polynomials b_m. By (1), $\{b_m\}_{m=1}^{\infty}$ converges uniformly on the sets M_m to $b \in A$. By (3), $b(q) = 3$. If $z \in K \setminus \{q\}$, then there exists m such that $z \in U_m$, so by (2) and (4)

$$|b(z)| \leqslant |b_m(z)| + \sum_{k=m}^{\infty} |b_{k+1}(z) - b_k(z)| \leqslant 3\left(1 - \frac{1}{2^{m+1}}\right) + \frac{1}{2^m} = 3 - \frac{1}{2^{m+1}} < 3.$$

Thus $a = b/3$ peaks at q in K and the lemma is proved.

That q is a local peak point of A in M (where $q = iy$, $y > 1$) now follows immediately since the w*-neighborhood U of q is contained in K.

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