$T\times K$ to K, then there are $\bar{x}$ in K and $\bar{\varphi}$ in Φ such that $\bar{x}(\bar{\varphi}(s)) = \bar{x}\big(\varphi(\bar{\varphi}(s))\big)$ for every φ in Φ and every s in S. If, moreover, given any s, s' in S there are φ, φ' in Φ such that $\varphi(s) = \varphi'(s')$, then $\bar{x}(\bar{\varphi})$ is a constant element of K.

More particularly, if S is compact regular and Φ is a semigroup of transformations in S such that Φ has a compact regular topology such that $(\varphi, s)\to\varphi(s)$ is continuous, and if F is a complete locally convex linear topological space and E is the space of continuous functions on S to F topologized by uniform convergence, let $O_x = \{x(\varphi)\colon \varphi \text{ in } \Phi\}$ for each x in E and $K_x =$ the closed convex cover of O_x. Then for each x in E there exists y_x in K_x and φ_x in Φ such that $y_x(\varphi_x(s)) = y_x\big(\varphi(\varphi_x(s))\big)$ for all φ and s; if, moreover, given s, s' there are φ, φ' such that $\varphi(s) = \varphi'(s')$, then for each x the function y_x is constant. For, the map $(\varphi, s)\to\varphi(s)$ is uniformly continuous and hence the map $(x, \varphi)\to x(\varphi)$ is continuous on $E\times\Phi$ to E. Clearly O_x is then compact for each x and so therefore is K_x since E is complete. Applying the preceding paragraph to K_x for each x yields the above conclusion [4].

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On equations with reflection

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If an equation contains together with the unknown function $x(t)$ the value $x(-t)$, then it will be called an *equation with reflection*. For example, the differential equation

(1) $$a_0x(t)+b_0x(-t)+a_1x'(t)+b_1x'(-t) = y(t)$$

is an equation with reflection.

Let us denote the reflection by S. Since $S^2 = I$, where I is identity operator, S is an involution. The differentiation operator D is anticommuting with S. Indeed,

$$(SDx)(t) = x'(-t),\quad (DSx)(t) = x(-t)' = -x'(-t) = (-SDx)(t).$$

Hence $SD+DS = 0$.

In this paper we shall consider a linear equation

(2) $$(a_0I+b_0S)x+(a_1I+b_1S)Dx = y,$$

where S is an involution on a linear space X, D is a linear operator acting in X and anticommuting with S, and a_0, b_0, a_1, b_1 are scalars.

As examples we shall consider equation (1) and an integral equation of form (2).

1. Let X be a linear space (over complex scalars). Let S be an involution: $S^2 = I$ on X. Let

$$P^+ = \frac{1}{2}(I+S),\quad P^- = \frac{1}{2}(I-S).$$

The following properties of an involution, shown in [1] (see also [2]) will be used further:

1° The operators P^+ and P^- are disjoint projectors giving a partition of unity:

(1.1) $$P^+P^- = P^-P^+ = 0,\quad (P^+)^2 = P^+,\quad (P^-)^2 = P^-,\quad P^++P^- = I$$

Moreover, $P^+-P^- = S$, $SP^+ = P^+$, $SP^- = -P^-$.

2° The eigenvalues of the operator S are $+1$, -1 and the respective eigenspaces are $X^+ = P^+X$, $X^- = P^-X$, i.e. if we write $x^+ = P^+x$, $x^- = P^-x$ for any $x \in X$, we have

$$Sx^+ = x^+, \quad Sx^- = -x^-. \tag{1.2}$$

3° The space X is a direct sum of spaces X^+ and X^-,

$$X = X^+ \oplus X^-, \tag{1.3}$$

which implies that any element $x \in X$ can be written in a unique manner as a sum

$$x = x^+ + x^-, \quad \text{where } x^+ \in X^+, x^- \in X^-, \tag{1.4}$$

and

$$Sx = x^+ - x^-. \tag{1.5}$$

Let S be an involution in the space X and let D be a linear operator acting in X and anticommutative with S, i.e.

$$SD + DS = 0.$$

Let us remark that the operator D^2 is commutative with S:

$$SD^2 - D^2S = 0. \tag{1.6}$$

Indeed,

$$SD^2 = (SD)D = -(DS)D = -D(SD) = -D(-DS) = D^2S.$$

The following property will play a very important role in further considerations:

PROPERTY 1.1.

$$P^+D = DP^-, \quad P^-D = DP^+. \tag{1.7}$$

In fact, since $SD = -DS$, we have

$$P^+D = \frac{1}{2}(I+S)D = \frac{1}{2}(D+SD) = \frac{1}{2}(D-DS) = \frac{1}{2}D(I-S) = DP^-,$$

$$P^-D = \frac{1}{2}(I-S)D = \frac{1}{2}(D-SD) = \frac{1}{2}(D+DS) = \frac{1}{2}D(I+S) = DP^+.$$

This implies that

$$Dx^+ = (Dx)^-, \quad Dx^- = (Dx)^+, \tag{1.8}$$

because

$$Dx^+ = DP^+x = P^-Dx = (Dx)^-, \quad Dx^- = DP^-x = P^+Dx = (Dx)^+.$$

Then the operator D changes the role of spaces X^+, X^-, the operator D^2, as commutative with S, maps each of these spaces into itself.

2. Suppose we are given in the space X an operator

$$A = (a_0I + b_0S) + (a_1I + b_1S)D,$$

where S is an involution in X, D is a linear operator acting in X and anticommutative with S, and a_0, b_0, a_1, b_1 are scalars.

Let us suppose that

$$a_0^2 - b_0^2 \neq 0, \quad a_1^2 - b_1^2 \neq 0. \tag{2.1}$$

The case where either $a_0^2 - b_0^2 = 0$ or $a_1^2 - b_1^2 = 0$ will be considered in the next section.

PROPOSITION 2.1. *Let*

$$B = (a_0I - b_0S) - (a_1I + b_1S)D, \quad R_A = -(a_1^2 - b_1^2)^{-1}B. \tag{2.2}$$

Then

$$AR_A = R_AA = D^2 - \lambda I, \tag{2.3}$$

where

$$\lambda = \frac{a_0^2 - b_0^2}{a_1^2 - b_1^2} \neq 0. \tag{2.4}$$

Proof. Let us remark that $SDS = -S^2D = -D$ and $SDSD = -DS^2D = -D^2$ because $DS = -SD$ and $S^2 = I$. Hence

$$\begin{aligned} BA &= (a_0I - b_0S)(a_0I + b_0S) - (a_1I + b_1S)D(a_0I + b_0S) + \\ &\quad + (a_0I - b_0S)(a_1I + b_1S)D - (a_1I + b_1S)D(a_1I + b_1S)D \\ &= (a_0^2 - b_0^2)I + (a_0a_1 - b_0b_1)D + (a_0b_1 - a_1b_0)SD - (a_0a_1 - b_0b_1)D - \\ &\quad - (a_0b_1 - a_1b_0)SD - (a_1^2 - b_1^2)D^2 = (a_0^2 - b_0^2)I - (a_1^2 - b_1^2)D^2. \end{aligned}$$

Similarly, we can show that

$$AB = (a_0^2 - b_0^2)I - (a_1^2 - b_1^2)D^2;$$

then, if we assume $R_A = -(a_1^2 - b_1^2)B$ and $\lambda = (a_0^2 - b_0^2)(a_1^2 - b_1^2)^{-1}$, we obtain the required formula (2.3).

Let us denote by D_T the domain of an operator T and by Z_T the kernel of T:

$$Z_T = \{x \in D_T : Tx = 0\}.$$

PROPOSITION 2.2. $Z_A \subset Z_{D^2 - \lambda I}$. (*Similarly*, $Z_{R_A} \subset Z_{D^2 - \lambda I}$).

Proof. Let $x \in Z$. Then $Ax = 0$ and $(D^2 - \lambda I)x = R_A(Ax) = 0$, which implies $x \in Z_{D^2 - \lambda I}$. Hence $Z_A \subset Z_{D^2 - \lambda I}$. There is similar proof for Z_{R_A}.

THEOREM 2.3. $Z_{D^2 - \lambda I} = \{z : z = z_1 + Sz_2,\ z_1, z_2 \in Z_{D - \sqrt{\lambda} I}\}$.

Proof. Let us suppose that z is of the form $z_1 + Sz_2$, where $z_1, z_2 \in Z_{D-\sqrt{\lambda}I}$. Then

$$(D^2-\lambda I)z = (D^2-\lambda I)z_1 + (D^2-\lambda I)Sz_2 = (D^2-\lambda I)z_1 + S(D^2-\lambda I)z_2$$
$$= (D+\sqrt{\lambda}I)(D-\sqrt{\lambda}I)z_1 + S(D+\sqrt{\lambda}I)(D-\sqrt{\lambda}I)z_2 = 0,$$

because $SD^2 = D^2S$. Hence $z \in Z_{D^2-\lambda I}$.

Conversely, let us suppose that $z \in Z_{D^2-\lambda I}$. The operator $\frac{1}{\sqrt{\lambda}} D$ is an involution on the space $Z_{D^2-\lambda I}$. In fact, since $\lambda \neq 0$, for any $z \in Z_{D^2-\lambda I}$

$$\left(\frac{1}{\sqrt{\lambda}} D\right)^2 z - z = 0.$$

This implies the decomposition of $Z_{D^2-\lambda I}$ onto a direct sum

$$Z_{D^2-\lambda I} = Z_{D+\sqrt{\lambda}I} \oplus Z_{D-\sqrt{\lambda}I}.$$

Hence $z = z_1 + z_2'$, where $z_1 \in Z_{D-\sqrt{\lambda}I}$, $z_2' \in Z_{D+\sqrt{\lambda}I}$ are linearly independent. We have to show that $z_2' = Sz_2$, where $z_2 \in Z_{D-\sqrt{\lambda}I}$. Since $z_2' \in Z_{D+\sqrt{\lambda}I}$, we find $Dz_2' = -\sqrt{\lambda}z_2'$. Hence $\sqrt{\lambda}Sz_2' = S(\sqrt{\lambda}z_2') = -SDz_2' = DSz_2'$, which mplies $(D-\sqrt{\lambda}I)Sz_2' = 0$ and $z_2 = Sz_2' \in Z_{D-\sqrt{\lambda}I}$. But $z_2' = S^2z_2' = S(Sz_2') = Sz_2$, which gives the required form of z.

THEOREM 2.4.

$$Z_A = \{z : z = \mu[(a_0 - a_1\sqrt{\lambda})I - (b_0 + b_1\sqrt{\lambda})S]z_1,\ z_1 \in Z_{D-\sqrt{\lambda}I},\ \mu \text{ being scalar}\}.$$

Proof. Proposition 2.2 implies that $Z_A \subset Z_{D^2-\lambda I}$. Theorem 2.5 implies that any element of $Z_{D^2-\lambda I}$ is of the form $z = z_1 + Sz_2$, $z_1, z_2 \in Z_{D-\sqrt{\lambda}I}$. We shall choose z_1 and z_2 in such a manner that $Az = 0$. Let $z_1, z_2 \in Z_{D-\sqrt{\lambda}I}$. Then $Sz_1, Sz_2 \in Z_{D+\sqrt{\lambda}I}$ (compare with the proof of Theorem 2.3). Hence

$$Az_1 = (a_0I + b_0S)z_1 + (a_1 + b_1S)Dz_1 = (a_0I + b_0S)z_1 + (a_1I + b_1S)\sqrt{\lambda}z_1$$
$$= [(a_0 + a_1\sqrt{\lambda})I + (b_0 + b_1\sqrt{\lambda})S]z_1,$$
$$ASz_2 = (a_0I + b_0S)Sz_2 + (a_1I + b_1S)DSz_2 = (a_0I + b_0S)Sz_2 +$$
$$+ (a_1I + b_1S)(-\sqrt{\lambda})Sz_2$$
$$= [(a_0 - a_1\sqrt{\lambda})I + (b_0 - b_1\sqrt{\lambda})S]Sz_2,$$
$$Az = A(z_1 + Sz_2) = [(a_0 + a_1\sqrt{\lambda})I + (b_0 + b_1\sqrt{\lambda})S]z_1 +$$
$$+ [(a_0 - a_1\sqrt{\lambda})I + (b_0 - b_1\sqrt{\lambda})S]Sz_2$$
$$= (a_0 + a_1\sqrt{\lambda})z_1 + (b_0 + b_1\sqrt{\lambda})Sz_1 + (a_0 - a_1\sqrt{\lambda})Sz_2 + (b_0 - b_1\sqrt{\lambda})z_2.$$

But the space $Z_{D^2-\lambda I}$ is a direct sum of spaces $Z_{D-\sqrt{\lambda}I}$ and $Z_{D+\sqrt{\lambda}I}$ (compare with the proof of Theorem 2.3) and $z_1, z_2 \in Z_{D-\sqrt{\lambda}I}$, $Sz_1, Sz_2 \in Z_{D+\sqrt{\lambda}I}$. Then the equality $Az = 0$ holds if and only if

$$(a_0 + a_1\sqrt{\lambda})z_1 + (b_0 - b_1\sqrt{\lambda})z_2 = 0, \quad (b_0 + b_1\sqrt{\lambda})Sz_1 + (a_0 - a_1\sqrt{\lambda})Sz_2 = 0.$$

Transforming the second equation by S and using the property $S^2 = I$ we obtain the following system of equations:

$$(2.5)\qquad \begin{aligned} (a_0 + a_1\sqrt{\lambda})z_1 + (b_0 - b_1\sqrt{\lambda})z_2 &= 0, \\ (b_0 + b_1\sqrt{\lambda})z_1 + (a_0 - a_1\sqrt{\lambda})z_2 &= 0. \end{aligned}$$

From these equations it follows that z_1 and z_2 are linearly dependent. Indeed, let us suppose that z_1 and z_2 are linearly independent. Then all coefficients in (2.5) are equal to zero:

$$a_0 + a_1\sqrt{\lambda} = 0, \quad a_0 - a_1\sqrt{\lambda} = 0, \quad b_0 + b_1\sqrt{\lambda} = 0, \quad b_0 - b_1\sqrt{\lambda} = 0.$$

Since $\lambda \neq 0$, we find $a_0 = a_1 = b_0 = b_1 = 0$, which is a contradiction. Hence there are scalars μ_1 and μ_2 such that

$$\mu_1 z_1 + \mu_2 z_2 = 0, \quad |\mu_1| + |\mu_2| > 0.$$

Using the last equality, we write (2.5) in the following form:

$$[-(a_0 + a_1\sqrt{\lambda})\mu_2 + (b_0 - b_1\sqrt{\lambda})\mu_1]z_2 = 0,$$
$$[-(b_0 + b_1\sqrt{\lambda})\mu_2 + (a_0 - a_1\sqrt{\lambda})\mu_1]z_2 = 0.$$

These equalities hold if and only if

$$(2.6)\qquad \begin{aligned} (a_0 - a_1\sqrt{\lambda})\mu_1 - (b_0 + b_1\sqrt{\lambda})\mu_2 &= 0, \\ (b_0 - b_1\sqrt{\lambda})\mu_1 - (a_0 + a_1\sqrt{\lambda})\mu_2 &= 0. \end{aligned}$$

The determinant of system (2.8) is

$$\Delta = -(a_0^2 - a_1^2\lambda) + (b_0^2 - b_1^2\lambda) = -[(a_0^2 - b_0^2 - \lambda(a_1^2 - b_1^2)]$$
$$= -\left[a_0^2 - b_0^2 - \frac{a_0^2 - b_0^2}{a_1^2 - b_1^2}(a_1^2 - b_1^2)\right] = 0.$$

Hence system (2.6) has a non-trivial solution:

$$\mu_1 = (b_0 + b_1\sqrt{\lambda}), \quad \mu_2 = (a_0 - a_1\sqrt{\lambda})$$

And we have (μ is an arbitrary scalar)

$$\mu(b_0 + b_1\sqrt{\lambda})z_1 + \mu(a_0 - a_1\sqrt{\lambda})z_2 = 0.$$

Taking the formula of solutions of system (2.6) from the second equation (2.6) we check that the first of the equations (2.5) is also satisfied.

This implies that

$$z = \mu[(a_0 - a_1\sqrt{\lambda})z_1 - (b_0 + b_1\sqrt{\lambda})Sz_1],$$

which was to be proved.

In a similar way we can determine the set

$$Z_{R_A} = \{z : z = \mu[(a_0 + a_1\sqrt{\lambda})I + (b_0 + b_1\sqrt{\lambda})S]z_1,\ z_1 \in Z_{D-\sqrt{\lambda}I},\ \mu \text{ being scalar}\}.$$

PROPOSITION 2.6. *If $\tilde{x}$ is a solution of the equation $(D^2 - \lambda I)\tilde{x} = y$, then $x = R_A\tilde{x}$ is a solution of the equation $Ax = y$.*

Proof. Let $\tilde{x}$ satisfy the equation $(D^2 - \lambda I)\tilde{x} = y$. Then

$$Ax = AR_A\tilde{x} = (D^2 - \lambda I)\tilde{x} = y.$$

Similarly, $u = A\tilde{x}$ is a solution of the equation $R_A u = y$.

Finally, we obtain the following theorem on the general form of the solution of the equations $Ax = y$ and $R_A u = y$:

THEOREM 2.7. *Let $A = [(a_0 I + b_0 S) + (a_1 I + b_1 S)D]$, where $S^2 = I$ and $SD + DS = 0$ on X, $a_0^2 - b_0^2 \neq 0$, $a_1^2 - b_1^2 \neq 0$. Let $\tilde{x}$ be a solution of the equation $(D^2 - \lambda I)\tilde{x} = y$. Then any solution of the equation $Ax = y$ is of the form*

$$x = R_A\tilde{x} + \mu[(a_0 - a_1\sqrt{\lambda})I - (b_0 + b_1\sqrt{\lambda})S]z_1,$$

where z_1 is a solution of the equation $(D - \sqrt{\lambda}I)z_1 = 0$, μ is an arbitrary scalar, $\lambda = (a_0^2 - b_0^2)(a_1^2 - b_1^2)^{-1}$, $R_A = -(a_1^2 - b_1^2)^{-1}[(a_0 I - b_0 S) - (a_1 I + b_1 S)D]$.

Any solution of the equation $R_A u = y$ is

$$u = A\tilde{x} + \mu[(a_0 + a_1\sqrt{\lambda})I - (b_0 + b_1\sqrt{\lambda})S]z_1.$$

This follows immediately from Theorem 2.5, Proposition 2.6 and from the linearity of A.

3. So far we have considered the case where $a_0^2 - b_0^2 \neq 0$, $a_1^2 - b_1^2 \neq 0$. In order to study other cases we remark that according to (1.1)

$$A = (a_0 I + b_0 S) + (a_1 I + b_1 S)D = (a_0 I + b_0 S)(P^+ + P^-) + (a_1 I + b_1 S)(P^+ + P^-)D = (a_0 + b_0)P^+ + (a_0 - b_0)P^- + (a_1 + b_1)P^+ D + (a_1 - b_1)P^- D. \tag{3.1}$$

We have the following few particular cases, which will be solved in different manners.

I. If $a_0 + b_0 = a_0 - b_0 = a_1 + b_1 = a_1 - b_1 = 0$, then $A = 0$.

II. If $a_1 + b_1 = a_1 - b_1 = 0$, we have a case completely solved in [1]. Namely,

(a) If $a_0^2 - b_0^2 \neq 0$, then the equation $Ax = y$ has a unique solution $x = (a_0^2 - b_0^2)^{-1}(a_0 I - b_0 S)y$.

(b) If $a_0 + b_0 = 0$ but $a_0 - b_0 \neq 0$, a necessary and sufficient condition of solvability of the equation $Ax = y$ is $(I + S)y = 0$, and under this condition the solution is $x = \frac{1}{2}a_0^{-1}y + x_1^+$, where x_1^+ is an arbitrary element of X^+.

(c) If $a_0 - b_0 = 0$, but $a_0 + b_0 \neq 0$, we obtain similarly the solution $x = \frac{1}{2}a_0^{-1}y + x_1^-$, $x_1^- \in X^-$ under the condition $(I - S)y = 0$.

III. If $a_0 + b_0 = a_0 - b_0 = 0$ but $a_1^2 - b_1^2 \neq 0$, then $A = (a_1 I + b_1 S)D$, and we can solve the equation $Ax = y$ with respect to the unknown Dx in the same manner as in II. Then we obtain an equation without involution, $Dx = y_1$, where $y_1 = (a_1^2 - b_1^2)^{-1}(a_1 I - b_1 S)y$.

IV. Let $a_1 + b_1 = 0$ but $a_1 - b_1 \neq 0$. Then

$$A = (a_0 + b_0)P^+ + (a_0 - b_0)P^- + (a_1 - b_1)P^- D.$$

Since the space X is a direct sum of spaces X^+ and X^-, the equation $Ax = y$ is equivalent to two independent equations:

$$P^+ Ax = P^+ y, \quad P^- Ay = P^- y.$$

But

$$P^+ A = (a_0 + b_0)P^+,$$

$$P^- A = (a_0 - b_0)P^- + (a_1 - b_1)P^- D = (a_0 - b_0)P^- + (a_1 - b_1)DP^+.$$

Hence we have two equations of the form

$$(a_0 + b_0)x^+ = y^+, \tag{3.2}$$

$$(a_0 - b_0)x^- + (a_1 - b_1)Dx^+ = y^-. \tag{3.3}$$

(a) Let $a_0^2 - b_0^2 \neq 0$. Then $x^+ = (a_0 + b_0)^{-1}y^+$, and $(a_0 - b_0)x^- = y^- - (a_1 - b_1)D(a_0 + b_0)^{-1}y^+$. Hence

$$x^- = (a_0 - b_0)^{-1}y^- - (a_1 - b_1)(a_0^2 - b_0^2)^{-1}Dy^+$$

and

$$\begin{aligned} x = x^+ + x^- &= (a_0 + b_0)^{-1}y^+ + (a_0 - b_0)^{-1}y^- - (a_1 - b_1)(a_0^2 - b_0^2)^{-1}Dy^+ \\ &= (a_0^2 - b_0^2)^{-1}[(a_0 - b_0)P^+ + (a_0 + b_0)P^- - (a_1 - b_1)DP^+]y \\ &= (a_0^2 - b_0^2)^{-1}[(a_0 I - b_0 S) - (a_1 - b_1)P^- D]y. \end{aligned}$$

(b) Let $a_0 + b_0 = 0$ but $a_0 - b_0 \neq 0$. Then a solution exists if and only if $y^+ = 0$ (compare with II.b), i.e. if $(I + S)y = 0$. In this case the solution is of the form

$$x = x_1^+ + x^- = x_1^+ + (a_0 - b_0)^{-1}y - (a_1 - b_1)Dx_1^+,$$

where $x_1^+ \in X^+$ is arbitrary.

(c) Let $a_0-b_0=0$ but $a_0+b_0\neq 0$. Then $x^+=(a_0+b_0)^{-1}y^+$ and equation (3.3) is of the form

$$(a_0-b_0)x^-=y^--(a_0+b_0)^{-1}(a_1-b_1)Dy^+. \tag{3.4}$$

A solution of (3.4) exists (compare with II.b) if and only if

$$y^--(a_0+b_0)^{-1}(a_1-b_1)Dy^+=0.$$

This condition can be written as follows:

$$(I-S)[I-(a_0+b_0)^{-1}(a_1-b_1)D]y=0.$$

Under this condition the solution is

$$x=x^++x^-=(a_0+b_0)^{-1}y^++x_1^-,$$

where $x_1^-\in X^-$ is arbitrary.

(d) $a_0+b_0=a_0-b_0=0$.

By similar considerations to those of (b) and (c) we find that a solution of system (3.2), (3.3) exists if and only if $y\in X^-$ and

$$y-(a_0+b_0)^{-1}(a_1-b_1)Dy\in X^+. \tag{3.5}$$

If $y\in X^-$, then $Dy\in X^+$ and from (3.5) it follows that $y\in X^+$, which implies $y=0$. Therefore a necessary and sufficient condition of solvability of system (3.2) (3.3) is that $y=0$. In this case from (3.3) we obtain $(a_1-b_1)Dx^+$. Since $a_1-b_1\neq 0$, we have $Dx^+=0$, which means that $P^-Dx=0$ and $Dx\in X^+$, whence x is a solution of the equation $Dx=z^+$, where z^+ is an arbitrary element of X^+.

V. Let $a_1-b_1=0$ but $a_1+b_1\neq 0$. Changing the roles of a_1-b_1 and a_1+b_1 and respectively of the spaces X^- and X^+, we obtain an analogous result to that of IV.

4. Examples.

I. Let us now consider the differential equation

$$a_0x(t)+b_0x(-t)+a_1x'(t)+b_1x'(-t)=y(t), \tag{4.1}$$

where the given continuous function $y(t)$ is defined on the real line (may be, on a subset of the line real symmetric with respect to zero), and a_0, b_0, a_1, b_1 are reals. Let $(Sx)(t)=x(-t)$. Then $S^2=I$. It is obvious that

$$x^+(t)=\frac{1}{2}[x(t)+x(-t)]$$

is an even function and

$$x^-(t)=\frac{1}{2}[x(t)-x(-t)]$$

is an odd function.

Hence the decomposition by the involution S of the space of continuous functions on the set in question is corresponding to the decomposition of every function into the sum of an even and an odd function. As has been shown, the differentiation $Dx=x'$ is anticommutative with S. Then we can apply all the preceding considerations to solve (4.1).

For example, let us assume that $a_1^2-b_1^2\neq 0$ and that $\lambda=(a_0^2-b_0^2)\times(a_1^2-b_1^2)^{-1}>0$. The general form of the solution z_1 of the equation $(D-\sqrt{\lambda})z_1=0$ is $z_1(t)=\mu e^{\sqrt{\lambda}t}$. If we find a solution $\tilde{x}$ of the equation $(D^2-\lambda I)\tilde{x}=y$, we will obtain according to Theorem 2.9 a general form of the solution of (4.1). The function $\tilde{x}(t)$ can be determined by the method of variation of constants. Since $e^{\sqrt{\lambda}t}$ and $e^{-\sqrt{\lambda}t}$ are two linearly independent solutions of the equation $(D^2-\lambda I)z=0$ and the Wronskian $W(t)$ of these functions is $W(t)=-2\sqrt{\lambda}\neq 0$, we obtain

$$\tilde{x}(t)=-\frac{1}{2\sqrt{\lambda}}\Big[e^{\sqrt{\lambda}t}\int -y(t)e^{-\sqrt{\lambda}t}\,dt+e^{-\sqrt{\lambda}t}\int y(t)e^{\sqrt{\lambda}t}\,dt\Big]$$
$$=\frac{1}{2\lambda}\Big[e^{\sqrt{\lambda}t}\int y(t)e^{-\sqrt{\lambda}t}\,dt-e^{-\sqrt{\lambda}t}\int y(t)e^{\sqrt{\lambda}t}\,dt\Big].$$

By simple calculations we now find

$$\tilde{x}'(t)=\frac{1}{2\sqrt{\lambda}}\Big[e^{\sqrt{\lambda}t}\int y(t)e^{-\sqrt{\lambda}t}\,dt+e^{-\sqrt{\lambda}t}\int y(t)e^{\sqrt{\lambda}t}\,dt\Big]$$

and

$$(R_A\tilde{x})(t)=-(a_1^2-b_1^2)^{-1}[(a_0\tilde{x}(t)-b_0\tilde{x}(-t)-a_1\tilde{x}'(t)-b_1\tilde{x}'(-t)]$$
$$=-(a_1^2-b_1^2)^{-1}\frac{1}{2\sqrt{\lambda}}\Big[(a_0-a_1\sqrt{\lambda})e^{\sqrt{\lambda}t}\int y(t)e^{-\sqrt{\lambda}t}dt-$$
$$-(a_0+a_1\sqrt{\lambda})e^{-\sqrt{\lambda}t}\int y(t)e^{\sqrt{\lambda}t}dt-(b_0-b_1\sqrt{\lambda})e^{-\sqrt{\lambda}t}\Big(\int y(s)e^{\sqrt{\lambda}s}\,ds\Big)_{s=-t}+$$
$$+(b_0+b_1\sqrt{\lambda})e^{\sqrt{\lambda}t}\Big(\int y(s)e^{-\sqrt{\lambda}s}\,ds\Big)_{s=-t}\Big]$$

and finally

$$x(t)=(R_A\tilde{x})(t)+\mu[(a_0-a_1\sqrt{\lambda})e^{\sqrt{\lambda}t}-(b_0+b_1\sqrt{\lambda})e^{-\sqrt{\lambda}t}],$$

where μ is an arbitrary real constant.

II. Let

$$(Kx)(t)=\int_{-a}^{a}K(t,s)x(s)\,ds,$$

where $K(t,s)$ is an odd continuous function on the square $-a\leqslant t, s\leqslant a$. And let $(Sx)(t)=x(-t)$. Hence S is an involution on the space

$X = C[-a, a]$. The operator K is anticommutative with S. Indeed,

$$(SKx)(t) = \int_{-a}^{a} K(-t, s)x(s)\,ds,$$

$$(KSx)(t) = \int_{-a}^{a} K(t, s)x(-s)\,ds = -\int_{a}^{-a} K(t, -u)x(u)\,du$$
$$= \int_{-a}^{a} K(t, -u)x(u)\,du = -\int_{-a}^{a} K(-t, u)x(u)\,du = -(SKx)(t),$$

because $K(-t, s) = -K(t, -s)$.

Let us consider the equation $(a_0^2 - b_0^2 \neq 0 \neq a_1^2 - b_1^2)$

$$(4.2) \qquad a_0 x(t) + b_0 x(-t) + a_1 \int_{-a}^{a} K(t, s)x(s)\,ds + b_1 \int_{-a}^{a} K(-t, s)x(s)\,ds = y(t),$$
$$y \in C[-a, a].$$

According to Theorem 2.9, for solving (4.2) it is sufficient to know a solution of the equation $(K^2 - \lambda I)x = y$ and all the solutions of the equation $(K - \sqrt{\lambda} I)z_1 = 0$. But

$$(K^2 x)(t) = \int_{-a}^{a} K_1(t, s)x(s)\,ds, \quad \text{where } K_1(t, s) = \int_{-a}^{a} K(t, u)K(u, s)\,du.$$

This means that for solving (4.2) it is enough to solve the following equations:

$$-\lambda \tilde{x}(t) + \int_{-a}^{a} K_1(t, s)\tilde{x}(s)\,ds = y(t),$$

$$-\sqrt{\lambda} z_1(t) + \int_{-a}^{a} K(t, s)z_1(s)\,ds = 0.$$

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A uniform algebra with non-global peak points

by

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1. Introduction. By a *uniform algebra* A on a topological space X we mean an algebra of continuous complex-valued functions on X which contains the constants and is closed under uniform convergence on compact subsets of X. A point $p \in X$ is said to be a *local peak point* of A in X if there exists a neighborhood U of p in X and a function $a \in A$ such that $a(p) = 1$ and $|a(x)| < 1$ if $x \in U \setminus \{p\}$. If X is compact and the space $M(A)$ of non-zero continuous homomorphisms of A, with the w*-topology, is (homeomorphic to) X, it is known that every local peak point of A in X is a global peak point, i.e., U can be taken to be X [3]. It is the purpose of this paper to show that this result is not true for general uniform algebras. We exhibit a uniform algebra A on a σ-compact space M (which is set-wise just the complex numbers C) which has local peak points in $M = M(A)$ but has no global peak points. In fact, A contains no non-constant bounded functions.

2. The construction. We describe a sequence of subsets M_m of the plane which satisfy

(i) $M_m \subset M_{m+1}$;

(ii) $\bigcup_{m=1}^{\infty} M_m = C$;

(iii) each M_m is compact and non-separating;

(iv) for every z_0 on the positive y-axis, there is a sequence $\{z_m\}_{m=1}^{\infty}$, $z_m \in M_m - M_{m-1}$, such that $z_m \to z_0$;

(v) for every positive integer m and every z_0 on the positive y-axis with $|z_0| \leqslant m$ there is a sequence $\{z_n\}_{n=1}^{\infty} \subset M_m$ such that $z_n \to z_0$;

(vi) for every z not lying on the positive y-axis, z lies in the interior int M_m of M_m for some m.

Let B_m be the union of the lines $\{(1/n, y) : m/2n \leqslant y \leqslant m\}$ and the line segments joining the following pairs of points:

$$(0, -m) \text{ and } (m, -m); \quad (m, -m) \text{ and } (m, m);$$