On Nikaidō's proof of the invariant mean-value theorem

by

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Dedicated to Professor S. Mazur
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The proof [4] mentioned in the title is actually a proof of what might be called "almost a fixed-point theorem", in this case one that asserts that for a semigroup \( \Phi \) of continuous affine maps in a compact convex set \( K \) there will be, under certain circumstances, some \( \bar{\varphi} \) in \( \Phi \) and some \( \bar{z} \) in \( K \) such that \( \bar{\varphi}(\varphi(z)) = \bar{\varphi}(\bar{z}) \) for every \( \varphi \) in \( \Phi \). Such theorems were established earlier by Peck [5] and Klee [3]; after Nikaidō, Cohen and Collins [2] were the first to observe that his proof established a theorem of the above sort. Here, using the same proof, we present a slightly more general version, Theorem 0 below [6], from which follow the above theorems as well as some others, including Kakutani's on equicontinuous groups.

Throughout, except in Theorem 0, \( K \) is a compact convex set in a real linear topological space \( E \) that is separated by its dual \( E' \), \( K^E \) is the set of all functions on \( K \) to \( K \) and has the product topology, \( \Phi \) is a subsemigroup of \( K^E \) and has all its elements affine, and \( \mathcal{V} \) is the closure of \( \Phi \) in \( K^E \). We recall that under these circumstances \( \mathcal{V} \) is compact, has its elements affine, and as a set of maps in \( K \) has the same fixed points as \( \Phi \). If \( \Phi \) is equicontinuous, then so is \( \mathcal{V} \), and hence the elements of \( \mathcal{V} \) are continuous and affine, \( \mathcal{V} \) is a subsemigroup of \( K^E \), and the maps \( (\varphi, x) \to \varphi(x) \) and \( (\varphi, \varphi') \to \varphi(\varphi') \) are continuous on \( \mathcal{V} \times K \) to \( K \) and on \( \mathcal{V} \times \mathcal{V} \) to \( \mathcal{V} \); in particular, \( \mathcal{V} \) is a compact topological subsemigroup of \( K^E \).

**Theorem 1.** If \( \Phi \) is equicontinuous there exist \( \bar{\varphi} \) in \( \mathcal{V} \) and \( \bar{z} \) in \( K \) such that \( \bar{\varphi}(\varphi(z)) = \bar{\varphi}(\bar{z}) \) for every \( \varphi \) in \( \mathcal{V} \).

From this there follow these fixed point results.

**Corollary 1.** If \( \Phi \) is equicontinuous and if given \( x \) in \( K \), \( \varphi \) and \( \varphi' \) in \( \Phi \), and \( U \) a nucleus in \( E \) there is some \( \varphi'' \) in \( \Phi \) such that \( \varphi''(\varphi(\varphi(x))) \to \varphi(x) \ast U \) and \( \varphi''(\varphi'(\varphi(x))) \to x \ast U \), then \( \Phi \) has a fixed point.
Kakutani's theorem that $\Phi$ has a fixed point if it is an equicontinuous group follows immediately.

**Corollary 2.** If $\Phi$ is equicontinuous and if given $x$ in $K$, $\varphi$ and $\varphi'$ in $\Phi$, and $U$ any nucleus in $E$ there is some $\varphi''$ in $\Phi$ such that $\varphi'\varphi''(x)$ $=$ $-\varphi'(\varphi''(x))$ $= U$, then $\Phi$ has a fixed point.

Corollary 2 is a variant of the Markoff-Kakutani theorem; commutativity has been replaced by equicontinuity plus a weak form of commutativity. It implies I.2.13 of [1].

To obtain these from Theorem 1 we use the following, the proof of which is postponed to the end of this section:

**Lemma.** If $\Phi$ satisfies the hypotheses of Corollary 1 or of Corollary 2, then so does $\Psi$.

Thus under the conditions of Corollary 1 we know from Theorem 1 that there are $x$ and $\varphi$ such that $\varphi(x) = \varphi'(\varphi''(x))$ for every $\varphi$ in $\Psi$, and from the lemma that given any $y$ in $\Psi$ and any nucleus $U$ in $E$ there is some $\varphi'$ in $\Psi$ such that $\varphi'(\varphi''(x)) = y U$. Thus $\varphi(x) = \varphi'(\varphi''(x))$ for every $\varphi$ in $\Psi$.

**Theorem 3.** In view of the remarks preceding it, will clearly result from the following:

**Theorem 0.** Let $K$ be a compact convex set in a real linear topological space $E$ having in its dual subset $K^*$ that separates points of $K$. Let $\Psi$ be a semigroup of continuous affine maps of $K$ into $K$ and suppose $\Psi$ has a compact topology such that $f(\varphi)$ is, for each $f$ in $K^*$, continuous on $K \times K$. Then there exist $\Psi$ in $\Psi$ and $x$ in $K$ such that $\varphi(x) = \varphi'(\varphi''(x))$ for every $\varphi$ in $\Psi$.

For each finite set $y$ in $K^*$ and each finite set $\delta$ in $\Psi$ let

$$\mathcal{A}(\gamma, \delta) = \left\{ \varphi(x) : f(\varphi(x)) = f(\varphi'(\varphi''(x))) \right\}$$

for every $\varphi$ in $\Psi$. Define $\mathcal{A}(\gamma, \delta)$ as the set of all $\varphi(x)$ for which there exists a $\varphi'$ in $\Psi$ such that $f(\varphi'(\varphi''(x))) = f(\varphi(x))$ for every $\varphi$ in $\Psi$.

Since $\Psi \times K$ is compact and $K^*$ separates $K$, it is enough to show that each $\mathcal{A}(\gamma, \delta)$ is closed and non-void. But $\mathcal{A}(\gamma, \delta)$ is closed since $f(\varphi'(\varphi''(x)))$ is continuous in $\Psi$ for each $\varphi$ and $f(\varphi''(x))$ is continuous in $\Psi$ for each $\varphi$.

To show that it is non-void suppose $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$. Let $\sigma = \frac{1}{n} \sum \gamma_i \delta_j$. Then $\sigma$ is continuous and affine in $K$ and hence has a fixed point $x$. Define $T$ on $\Psi$ to $K^*$ by $T(\varphi) = \left\{ f(\varphi(x)) : f(\varphi'(\varphi''(x))) \right\}$; clearly, $T$ is continuous.

Since

$$f_i(\varphi(x)) = f_i(\varphi'(\varphi''(x))) = f_i\left( \left\{ \varphi(x) \right\} \right) = \frac{1}{n} \sum_{j=1}^{n} f_i(\varphi_j(x)),$$

we have

$$T(x) = \frac{1}{n} \sum_{j=1}^{n} \left\{ \mathcal{A}(\gamma, \delta) \right\}.$$

The function $\|T(x)\|$ being continuous on compact $\Psi$, attains its maximum at some $\varphi$; then

$$\|T(x)\| = \frac{1}{n} \sum_{j=1}^{n} \|T(\varphi_j(x))\| \leq \frac{1}{n} \sum_{j=1}^{n} \|\mathcal{A}(\gamma, \delta)\|$$

and hence $T(x) = T(\varphi_j(x))$ for all $\varphi_j$. Thus $f_i(\varphi_j(x)) = f_i(\varphi(x))$ for all $\varphi_j$ and so $\mathcal{A}(\gamma, \delta)$ is not void.

Reverting to the proof of the lemma, suppose $\Phi$ satisfies the hypotheses of Corollary 1. Then $\Psi$ is equicontinuous; and given $x$ in $K$, $\varphi$ and $\varphi'$ in $\Psi$, and any nucleus $U$ in $E$, some $\varphi''$ in $\Phi$ must be such that $\varphi''(x) = \varphi'(x) U$. We may suppose $U$ to be closed. Choose nets $(\varphi_n)$ and $(\varphi'_n)$ in $\Psi$ converging to $\varphi$ and $\varphi'$ in $K^*$.

From the assumptions on $\Phi$ there is for each $\varphi''$ some $\varphi''$ in $\Phi$ such that $\varphi''(\varphi''(x)) = (\varphi''(x)) U$. Thus $\varphi''(\varphi''(x)) = (\varphi''(x)) U$. Since $\Psi$ is compact, the net $(\varphi'_n)$ converges to some $\varphi''$. From the remarks preceding Corollary 1, $\Psi$ is a topological semigroup; hence $\varphi''$ converges to $\varphi''$. For $\varphi''$ in $K^*$; thus $\varphi''(\varphi''(x))$ converges to $\varphi''(\varphi''(x))$ and $\varphi''(\varphi''(x))$ to $\varphi''(\varphi''(x))$ in $E$, so that $\varphi''(\varphi''(x)) = \varphi''(x) U$. Since $U$ is closed, we have $\varphi''(\varphi''(x)) = \varphi''(x) U$. It follows from this that $\varphi''(\varphi''(x)) = \varphi''(x) U$.}

A similar proof covers the case of Corollary 2.

From Theorem 0 there also follow earlier, simpler theorems due to Peck ([5], 2.13), and due to Peck ([5], 2.13) and Collins theorem (Theorem 2 of [2], and II.3.14 of [1]).

For other applications let $S$ be a set and $\Phi$ a semigroup of transformations in $S$. Let $E$ be a linear topological space of functions on $S$ and for each $x$ in $S$ and each $\varphi$ in $\Phi$ let $T(x) = x(\varphi)$. Suppose $K$ is a compact convex set in $E$ such that $T(x) = x(x)$ for every $x$ in $K$. If there is a compact topology for $\mathcal{F} = \{ T : \varphi \in \Phi \}$ such that $(T, \varphi) \rightarrow (x, \varphi)$ is continuous on}

$$\|T(x)\| = \frac{1}{n} \sum_{j=1}^{n} \|T(\varphi_j(x))\| \leq \frac{1}{n} \sum_{j=1}^{n} \|\mathcal{A}(\gamma, \delta)\|$$

and hence $T(x) = T(\varphi_j(x))$ for all $\varphi_j$. Thus $f_i(\varphi_j(x)) = f_i(\varphi(x))$ for all $\varphi_j$ and so $\mathcal{A}(\gamma, \delta)$ is not void.

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T \times K to K, then there are \exists in K and \varphi \in \Phi such that \exists \varphi(s) = \exists \varphi'(s') for every \varphi in \Phi and every s in S. If, moreover, given any s, s' in S there are \varphi, \varphi' in \Phi such that \varphi(s) = \varphi'(s'), then \exists \varphi is a constant element of K.

More particularly, if S is compact regular and \Phi is a semigroup of transformations in S such that \Phi has a compact regular topology such that \varphi \times s \to \varphi(s) is continuous, and if F is a complete locally convex topological space and E is the space of continuous functions on S to F topologized by uniform convergence, let \Omega_s = \{ \varphi(s) : \varphi \in \Phi \} for each s \in S and \Omega = \{ \Omega_s : s \in S \} = \Omega. Then for each \varphi in \Phi and \varphi in \Phi such that \varphi \varphi(s) \to \varphi(s) for all \varphi and s; if, moreover, given \varphi, \varphi' in \Phi such that \varphi(s) = \varphi'(s'), then for each \varphi the function \varphi_{s} is constant. For, the map \varphi \to \varphi_{s} is uniformly continuous and hence the map \varphi \to \varphi_{s} is continuous on \Phi \times \Phi to E. Clearly \Omega is then compact for each s and so therefore is \Omega since E is complete. Applying the preceding paragraph to \Omega_{s} for each \varphi yields the above conclusion [4].

References

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On equations with reflection

by

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If an equation contains together with the unknown function \varphi(t) the value \varphi(-t), then it will be called an equation with reflection. For example, the differential equation

\begin{equation}
\frac{d\varphi(t)}{dt} + b_1 \varphi(-t) + a_1 \varphi'(t) + b_2 \varphi''(-t) = y(t)
\end{equation}

is an equation with reflection.

Let us denote the reflection by S. Since S^2 = I, where I is identity operator, S is an involution. The differentiation operator D is anticommuting with S. Indeed,

\begin{equation}
(SDx)(t) = \varphi'(-t), \quad (DS\varphi)(t) = \varphi(-t) = -\varphi'(-t) = -(SD\varphi)(t).
\end{equation}

Hence SD + DS = 0.

In this paper we shall consider a linear equation

\begin{equation}
(a_1 I + b_2 S)x + (a_1 I + b_2 S)Dx = y,
\end{equation}

where S is an involution on a linear space X, D is a linear operator acting in X and anticommuting with S, and \alpha_1, b_2, a_1, b_2 are scalars.

As examples we shall consider equation (3) and an integral equation of form (2).

1. Let X be a linear space (over complex scalars). Let S be an involution: S^2 = I on X. Let

\begin{equation}
P^+ = \frac{1}{2}(I + S), \quad P^- = \frac{1}{2}(I - S).
\end{equation}

The following properties of an involution, shown in [1] (see also [2]) will be used further:

\begin{itemize}
\item 1st The operators P^+ and P^- are disjoint projectors giving a partition of unity:
\end{itemize}

\begin{equation}
(P^+)^2 = P^+, \quad (P^-)^2 = P^-, \quad P^+ + P^- = I
\end{equation}

Moreover, P^+ = P^+, S P^+ = P^+, S P^- = -P^-.