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On Nikaidô's proof of the invariant mean-value theorem

by

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Dedicated to Professor S. Mazur and Professor W. Orlicz

The proof [4] mentioned in the title is actually a proof of what might be called "almost a fixed-point theorem", in this case one that asserts that for a semigroup Φ of continuous affine maps in a compact convex set K there will be, under certain circumstances, some $\bar{\varphi}$ in Φ and some $\bar{x}$ in K such that $\bar{\varphi}(\varphi(\bar{x})) = \bar{\varphi}(\bar{x})$ for every φ in Φ. Such theorems were established earlier by Peck [5] and Klee [3]; after Nikaidô, Cohen and Collins [2] were the first to observe that his proof established a theorem of the above sort. Here, using the same proof, we present a slightly more general version, Theorem 0 below [6], from which follow the above theorems as well as some others, including Kakutani's on equicontinuous groups.

Throughout, except in Theorem 0, K is a compact convex set in a real linear topological space E that is separated by its dual E^*, K^K is the set of all functions on K to K and has the product topology, Φ is a subsemigroup of K^K and has all its elements affine, and Ψ is the closure of Φ in K^K. We recall that under these circumstances Ψ is compact, has its elements affine, and as a set of maps in K has the same fixed points as Φ. If Φ is equicontinuous, then so is Ψ, and hence the elements of Ψ are continuous and affine, Ψ is a subsemigroup of K^K, and the maps $(\psi, x) \to \psi(x)$ and $(\psi, \psi') \to \psi(\psi')$ are continuous on $\Psi \times K$ to K and on $\Psi \times \Psi$ to Ψ; in particular, Ψ is a compact topological subsemigroup of K^K.

THEOREM 1. *If Φ is equicontinuous there exist $\bar{\psi}$ in Ψ and $\bar{x}$ in K such that $\bar{\psi}(\bar{x}) = \bar{\psi}(\psi(\bar{x}))$ for every ψ in Ψ.*

From this there follow these fixed point results.

COROLLARY 1. *If Φ is equicontinuous and if given x in K, φ and φ' in Φ, and U a nucleus in E there is some φ'' in Φ such that $\varphi''(\varphi'(\varphi(x))) - \varphi(x) \in U$ and $\varphi''(\varphi'(x)) - x \in U$, then Ψ has a fixed point.*

Kakutani's theorem that Φ has a fixed point if it is an equicontinuous group follows immediately.

COROLLARY 2. *If Φ is equicontinuous and if given x in K, φ and φ' in Φ, and U any nucleus in E there is some φ'' in Φ such that $\varphi(\varphi''(x))-\varphi'(\varphi(x))\in U$, then Ψ has a fixed point.*

Corollary 2 is a variant of the Markoff-Kakutani theorem; commutativity has been replaced by equicontinuity plus a weak form of commutativity. It implies I.2.13 of [1].

To obtain these from Theorem 1 we use the following, the proof of which is postponed to the end of this section:

LEMMA. *If Φ satisfies the hypotheses of Corollary 1 or of Corollary 2, then so does Ψ.*

Thus under the conditions of Corollary 1 we know from Theorem 1 that there are $\bar{x}$ and $\bar{\psi}$ such that $\bar{\psi}(\bar{x})=\bar{\psi}(\psi(\bar{x}))$ for every ψ in Ψ, and from the lemma that given any ψ in Ψ and any nucleus U in E there is some ψ'' in Ψ such that $\psi''(\bar{\psi}(\psi(\bar{x})))-\psi(\bar{x})\in U$ and $\psi''(\bar{\psi}(\bar{x}))-\bar{x}\in U$. Thus $\psi(\bar{x})-\bar{x}\in -U+U$, and so $\psi(\bar{x})=\bar{x}$ for every ψ in Ψ. For Corollary 2 an even simpler proof establishes that $\bar{\psi}(\bar{x})$ is a fixed point for Ψ.

Theorem 1, in view of the remarks preceding it, will clearly result from the following

THEOREM 0. *Let K be a compact convex set in a real linear topological space E having in its dual a subset K^* that separates points of K. Let Ψ be a semigroup of continuous affine maps of K into K and suppose Ψ has a compact topology such that $f(\psi(x))$ is, for each f in K^*, continuous on $\Psi\times K$. Then there exist $\bar{\psi}$ in Ψ and $\bar{x}$ in K such that $\bar{\psi}(\bar{x})=\bar{\psi}(\psi(\bar{x}))$ for every ψ in Ψ.*

For each finite set γ in K^* and each finite set δ in Ψ let

$$A(\gamma,\delta)=\{(\bar{\psi},\bar{x}):\ f(\bar{\psi}(\bar{x}))=f(\bar{\psi}(\psi(\bar{x})))\text{ for every } f \text{ in } \gamma \text{ and every } \psi \text{ in } \delta\}.$$

Since $\Psi\times K$ is compact and K^* separates K, it is enough to show that each $A(\gamma,\delta)$ is closed and non-void. But $A(\gamma,\delta)$ is closed since each ψ is continuous and since $f(\bar{\psi}(\bar{x}))$ is continuous in $(\bar{\psi},\bar{x})$ for each f. To show that it is non-void suppose $\gamma=\{f_1,\ldots,f_m\}$ and $\delta=\{\psi_1,\ldots,\psi_n\}$. Let

$$\sigma=\frac{1}{n}\sum_{j=1}^{n}\psi_j;$$

then σ is continuous and affine in K and hence has a fixed point $\bar{x}$. Define T on Ψ to R^m by $T(\psi)=(f_1(\psi(\bar{x})),\ldots,f_m(\psi(\bar{x})))$; clearly, T is continuous.

Since

$$f_i(\psi(\bar{x}))=f_i(\psi(\sigma(\bar{x})))=f_i\Big(\psi\Big(\frac{1}{n}\sum_{j=1}^{n}\psi_j(\bar{x})\Big)\Big)=\frac{1}{n}\sum_{j=1}^{n}f_i(\psi\psi_j(\bar{x})),$$

we have

$$T(\psi)=\frac{1}{n}\sum_{j=1}^{n}T(\psi\psi_j).$$

The function $\|T(\psi)\|$, being continuous on compact Ψ, attains its maximum at some $\bar{\psi}$; then

$$\|T(\bar{\psi})\|=\Big\|\frac{1}{n}\sum_{j=1}^{n}T(\bar{\psi}\psi_j)\Big\|\leqslant\frac{1}{n}\sum_{j=1}^{n}\|T(\bar{\psi}\psi_j)\|\leqslant\frac{1}{n}\sum_{j=1}^{n}\|T(\bar{\psi})\|$$

and hence $T(\bar{\psi})=T(\bar{\psi}\psi_j)$ for $j=1,\ldots,n$. Thus $f_i(\bar{\psi}(\bar{x}))=f_i(\bar{\psi}\psi_j(\bar{x}))$ for all i,j and so $A(\gamma,\delta)$ is not void.

Reverting to the proof of the lemma, suppose Φ satisfies the hypotheses of Corollary 1. Then Ψ is equicontinuous; and given x in K, ψ and ψ' in Ψ, and any nucleus U in E, some ψ'' in Ψ must be found such that $\psi''(\psi'(\psi(x)))-\psi(x)\in U$ and $\psi''(\psi'(x))-x\in U$. We may suppose U to be closed. Choose nets $\{\varphi_a\}$ and $\{\varphi'_a\}$ in Φ converging to ψ and ψ' in K^K. From the assumptions on Φ there is for each a some φ''_a in Φ such that $\varphi''_a(\varphi'_a(\varphi_a(x)))-\varphi_a(x)\in U$ and $\varphi''_a(\varphi'_a(x))-x\in U$. Since Ψ is compact, the net $\{\varphi''_a\}$ clusters at some ψ'' in Ψ; we may presume that $\{\varphi''_a\}$ converges to ψ''. From the remarks preceding Theorem 1, Ψ is a topological semigroup; hence $\varphi''_a(\varphi'_a(\varphi_a))$ converges to $\psi''(\psi'(\psi))$ and $\varphi''_a(\varphi'_a)$ to $\psi''(\psi')$ in K^K; thus $\varphi''_a(\varphi'_a(\varphi_a(x)))$ converges to $\psi''(\psi'(\psi(x)))$ and $\varphi''_a(\varphi'_a(x))$ to $\psi''(\psi'(x))$ in K, so that $\varphi''_a(\varphi'_a(\varphi_a(x)))-\varphi_a(x)$ converges to $\psi''(\psi'(\psi(x)))-\psi(x)$ and $\varphi''_a(\varphi'_a(x))-x$ to $\psi''(\psi'(x))-x$. Since U is closed, we have $\psi''(\psi'(\psi(x)))-\psi(x)$ in U and $\psi''(\psi'(x))-x$ in U, as desired.

A similar proof covers the case of Corollary 2.

From Theorem 0 there also follow immediately earlier theorems due to Peck [5] and Klee (4.3, 4.4, and 3.1 (d) of [3]) and the Cohen and Collins theorem (Theorem 2 of [2], and II.3.14 of [1]).

For other applications let S be a set and Φ a semigroup of transformations in S. Let E be a linear topological space of functions on S and for each x in E and each φ in Φ let $T_\varphi(x)=x(\varphi)$. Suppose K is a compact convex set in E such that $T_\varphi(K)\subset K$ for every φ in Φ. If there is a compact topology for $\mathcal{T}=\{T_\varphi:\ \varphi \text{ in } \Phi\}$ such that $(T,x)\to T(x)$ is continuous on

$T \times K$ to K, then there are $\bar{x}$ in K and $\bar{\varphi}$ in Φ such that $\bar{x}(\bar{\varphi}(s)) = \bar{x}\left(\varphi(\bar{\varphi}(s))\right)$ for every φ in Φ and every s in S. If, moreover, given any s, s' in S there are φ, φ' in Φ such that $\varphi(s) = \varphi'(s')$, then $\bar{x}(\bar{\varphi})$ is a constant element of K.

More particularly, if S is compact regular and Φ is a semigroup of transformations in S such that Φ has a compact regular topology such that $(\varphi, s) \to \varphi(s)$ is continuous, and if F is a complete locally convex linear topological space and E is the space of continuous functions on S to F topologized by uniform convergence, let $O_x = \{x(\varphi)\colon \varphi \text{ in } \Phi\}$ for each x in E and $K_x =$ the closed convex cover of O_x. Then for each x in E there exists y_x in K_x and φ_x in Φ such that $y_x(\varphi_x(s)) = y_x\left(\varphi(\varphi_x(s))\right)$ for all φ and s; if, moreover, given s, s' there are φ, φ' such that $\varphi(s) = \varphi'(s')$, then for each x the function y_x is constant. For, the map $(\varphi, s) \to \varphi(s)$ is uniformly continuous and hence the map $(x, \varphi) \to x(\varphi)$ is continuous on $E \times \Phi$ to E. Clearly O_x is then compact for each x and so therefore is K_x since E is complete. Applying the preceding paragraph to K_x for each x yields the above conclusion [4].

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On equations with reflection

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If an equation contains together with the unknown function $x(t)$ the value $x(-t)$, then it will be called an *equation with reflection*. For example, the differential equation

$$a_0 x(t) + b_0 x(-t) + a_1 x'(t) + b_1 x'(-t) = y(t) \tag{1}$$

is an equation with reflection.

Let us denote the reflection by S. Since $S^2 = I$, where I is identity operator, S is an involution. The differentiation operator D is anticommuting with S. Indeed,

$$(SDx)(t) = x'(-t), \quad (DSx)(t) = x(-t)' = -x'(-t) = (-SDx)(t).$$

Hence $SD + DS = 0$.

In this paper we shall consider a linear equation

$$(a_0 I + b_0 S)x + (a_1 I + b_1 S)Dx = y, \tag{2}$$

where S is an involution on a linear space X, D is a linear operator acting in X and anticommuting with S, and a_0, b_0, a_1, b_1 are scalars.

As examples we shall consider equation (1) and an integral equation of form (2).

1. Let X be a linear space (over complex scalars). Let S be an involution: $S^2 = I$ on X. Let

$$P^+ = \frac{1}{2}(I + S), \quad P^- = \frac{1}{2}(I - S).$$

The following properties of an involution, shown in [1] (see also [2]) will be used further:

1° The operators P^+ and P^- are disjoint projectors giving a partition of unity:

$$P^+P^- = P^-P^+ = 0, \quad (P^+)^2 = P^+, \quad (P^-)^2 = P^-, \quad P^+ + P^- = I \tag{1.1}$$

Moreover, $P^+ - P^- = S$, $SP^+ = P^+$, $SP^- = -P^-$.