Unbounded integrally positive definite functions

by

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1. Throughout this note \( \mathcal{G} \) will denote a Hausdorff locally compact group, \( e \) its neutral element, \( \lambda \) a chosen left Haar measure on \( \mathcal{G} \), \( \mathcal{D} = \mathcal{L}^\prime(\mathcal{G}, \lambda) \), and \( \mathcal{C} = \mathcal{C}(\mathcal{G}) \) is the set of continuous complex-valued functions on \( \mathcal{G} \) having compact supports. For brevity we shall write \( dx \) in place of \( d\mu(x) \) and \( d(x, y) \) in place of \( d(\lambda \times \lambda)(x, y) \).

Given a set \( F \) of complex-valued measurable functions on \( \mathcal{G} \), a locally integrable complex-valued function \( \varphi \) on \( \mathcal{G} \) will be said to be \( F \)-PD (\( = \) \( F \)-positive definite) if and only if

\[
\int_{\mathcal{G} \times \mathcal{G}} \varphi(y^{-1} x) f(y) f(x) \, d(x, y) < \infty
\]

and

\[
\int_{\mathcal{G} \times \mathcal{G}} \varphi(y^{-1} x) f(y) f(x) \, d(x, y) \geq 0
\]

for every \( f \in F \).

A continuous complex-valued function \( \varphi \) on \( \mathcal{G} \) is said to be (B)-PD (\( = \) positive definite in Bochner’s sense) if and only if

\[
\sum_{\mathcal{G}} \varphi(x^{-1} x) \leq 0
\]

for every finite complex-valued sequence \( (a_i) \) and every finite \( \mathcal{G} \)-valued sequence \( (a_i) \). Such a function is necessarily bounded: \( \varphi(x) \leq \varphi(e) \) for all \( x \in \mathcal{G} \).

If \( \mathcal{G} \) happens to be discrete, and if \( F \supset C_\mathcal{G} \), any \( F \)-PD function is obviously (B)-PD and therefore belongs to \( \mathcal{L}^\infty \). Even if \( \mathcal{G} \) is non-discrete, it is known that any \( C_\mathcal{G} \)-PD function which is essentially bounded on some neighbourhood of \( e \) is equal l. a. e. to a (B)-PD function (see, for example, [1], p. 715-720); and that an \( \mathcal{L}^1 \)-PD function belongs to \( \mathcal{L}^\infty \) (see [1], p. 490 and the proof of Theorem 10.3.3).

Hewitt and Ross [2] have raised the question of the existence of \( F \)-PD functions not in \( \mathcal{L}^\infty \), and they have indicated how to construct Borel functions \( \varphi \) on any non-discrete group \( \mathcal{G} \) which are \( \mathcal{L}^\infty \)-PD but not
in $L^\infty$. Here we shall give a somewhat more general discussion referring to $L^r$-PD functions and a construction leading to a stronger result.

We shall begin with a few simple remarks indicating some special cases which can be discarded from the outset.

From this point on we assume that $G$ is non-discrete.

2. Some special cases.

2.1. If $r = \infty$, (PD) asserts that for every $f \in L^\infty$ amounts simply to the demand that the function $(x, y) \rightarrow f(x, y)$ be integrable for $\lambda \times \lambda$. Unless $G$ is compact, this is so only if $f = 0$ a.e.

On the other hand, if $G$ is compact, it is evident that an integrable $f$ is $L^\infty$-PD provided only that (PD) holds for each $f \in L^\infty$. It is furthermore easy to see that this is the case if and only if the (generally operator-valued) Fourier transform $\hat{\varphi}$ is non-negative. A simple category argument suffices to show that there exist always functions $\varphi$ of this type which are not in $L^\infty$.

In the sequel we may therefore suppose that $1 \leq r < \infty$.

2.2. Supposing now that $1 \leq r < \infty$, general analytic approximations (\cite{1}, p. 490) show that a locally integrable $\varphi$ is $L^r$-PD if and only if

(A) $f * \varphi \in L^r$ and $\|f * \varphi\|_r \leq \text{const} \|f\|_r$

and

(A') $(f * \varphi)(f) \geq 0$

for every $f \in C^\infty$, where $r' = r/(r-1)$ and

$$ (u|v) = \int u \overline{v} d\lambda. $$

In this connection we should perhaps remark that, if $f \in L^r$ for some $r < \infty$, then $f$ vanishes a subset of $G$ which is sigma-finite for $\lambda$; hence the function

$$ (x, y) \rightarrow \varphi(y^{-1} x) \overline{f(y)} f(x) $$

vanishes off a subset of $\lambda \times \lambda$ which is sigma-finite for $\lambda \times \lambda$ (this is a consequence of \cite{1}, 4.17.3); consequently the theorems of Fubini and Tonelli (\cite{1}, Theorems 4.17.4 and 4.17.5) are applicable and show that the appropriate integrals

$$ \int \int f(x, y) d\lambda(x) dy $$

and

$$ \int \int f(x, y) d\lambda(y) dx $$

can be replaced by either of the associated repeated integrals

$$ \int \int \int f(x, y) d\lambda(x) dy dx $$

and

$$ \int \int \int f(x, y) d\lambda(y) dx dy $$

If $r = 1$, then (as was noted in section 1) (A) alone suffices to show that $\varphi \in L^1$. Consequently, the interest lies in the case where $1 < r < \infty$.

2.3. If $G$ is compact and $r \geq 2$, it is evident that any $\varphi \in L^r$ such that $\varphi \geq 0$ is $L^r$-PD, and that there exist many such functions outside $L^\infty$ (see 2.1).

On the other hand, if $r > 2$ and $G$ is non-compact, $\varphi$ alone forces $\varphi$ to be zero. a. e.; this is proved by Hörmander in case $G = \mathbb{R}^n$ (\cite{1}, Theorem 1.1) and his argument extends at once to any non-compact $G$.

Accordingly we shall in the sequel be concerned primarily with the remaining case, where $1 < r \leq 2$. For this purpose we shall use the following lemma:

2.4. Lemma. Suppose that $1 < r \leq 2$ and put $a = \varphi^r$. If $\varphi$ is measurable and $\|\varphi\|_r = \|\varphi^r\|_a$, where $\varphi^r(x) = \varphi(x^{-1})$ for all $x \in G$, then $\varphi$ holds with const $\leq \|\varphi\|_a$.

Proof. This is a special case of Young's inequality; see \cite{3}, p. 396, Theorem (20.16), or, more generally but less directly, [1, p. 655, Theorem 9.5.1].

3. The main result.

3.1. Supposing $G$ to be non-discrete, let $(U_n)_{n=1}^\infty$ be any sequence of closed neighbourhoods of $e$ in $G$ and write $P = \bigcap_{n=1}^\infty U_n$. Put

$$ F_0 = \bigcup \{L^r : 1 < r < 2\}. $$

We describe the construction of functions $\varphi$ on $G$ with the following properties:

(1) $\varphi \geq 0$, $\varphi = \varphi^r$, $\varphi$ is lower semicontinuous, $\varphi$ has a compact support and vanishes on $G \cap U_1$;

(2) $\varphi$ is continuous at all points of $G \cap F'$;

(3) $\lim_{r \to a} \varphi(r) = \infty$, so that $\varphi \in L^\infty$;

(4) $\varphi \in \{L^p : 1 \leq p < \infty\}$;

(5) $\varphi$ is $F_0$-PD.

3.2. The construction is based on the fact that, if $u$ is a bounded measurable complex-valued function on $G$ which has its support lying within a relatively compact symmetric open neighbourhood $V$ of $e$ in $G$, then

$$ \varphi_u = u * u^* = u \cdot u^* $$

where $u^* = u^r$, belongs to $C_0$ and satisfies

$$ \sup_p \varphi_u \in V, \quad \varphi_u^r = \varphi_u, \quad \|\varphi\|_p \leq \lambda(F) \lambda(V)^{1/p} $$

for $0 < p < \infty$. Plainly, $\varphi_u$ is real and non-negative whenever $u$ has these properties. By (6), Lemma 2.4, and the fact that

$$ \int \varphi_u(y^{-1} x) f(y) \overline{f(x)} d(x, y) = \int u \overline{u} d\lambda, $$

for $u \in V$, we conclude that $\varphi_u$ is a $\mathcal{B}$.
it is visible that \( \varphi_n \) is \( F_r\)-PD. Functions \( \varphi \) of the desired type will now be obtained as suitable infinite sums of such functions \( \varphi_n \).

To do this, choose relatively compact symmetric open neighbourhoods \( V_n \) \((n = 1, 2, \ldots)\) of \( e \) such that

\[
V_n^2 \subset U_1 \cap U_2 \cap \ldots \cap U_n, \quad 0 < \lambda(V_n^2) \ll 2^{-n};
\]

this choice is possible since \( G \) is non-discrete. Next choose functions \( u_n \in C_c \) such that

\[
\supp u_n \subset V_n, \quad 0 \leq u_n \leq 1,
\]

\[
u_n \in C_c^0(G \smallsetminus U^2_1) = \int \nu_n^\ast \, d\lambda \gg \frac{1}{2} \lambda(V_n^2).
\]

Put \( \varphi_n = \varphi u_n \) and consider the non-negative lower semicontinuous function \( \varphi \) defined by

\[
\varphi = \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \varphi_n.
\]

Relation (6), together with the choice of the \( u_n \), show at once that (1) is true. Statement (3) follows easily from (7) and (10). If \( K \) is any compact subset of \( G \) not meeting \( E \), (7) and (8) show that \( K \) meets \( \supp \varphi_n \) for at most a finite set of positive integers \( n \), and (2) follows at once from this. Fatou’s lemma combines with (6) (with \( u = u_n \)) and (7) to show that for \( 1 \leq p < \infty \) we have

\[
\|\varphi\|_p \leq \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \|\varphi_n\|_p \leq \sum_{n=1}^\infty 2^{-np} < \infty,
\]

which establishes (4). Finally, (5) follows from (4), Lemma 2.4, and the identity

\[
\int_{\partial \chi^2} \varphi(y^{-1}x)f(y)f(x)\, d(x, y) = \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \int_{\partial \chi^2} \varphi \ast u_n\, d\lambda.
\]

3.3. Supplements.

(a) If \( G \) is first countable, we may arrange that \( F = \{e\} \).

(b) If \( G \) is a Lie group, we can arrange that \( F = \{e\} \) and that \( u_n \in C_c^0(G \smallsetminus \{e\}) \) for every \( n \); accordingly, (2) may be replaced by

\[
\varphi \in C_c^0(G \smallsetminus \{e\}).
\]

(c) Condition (4) can be strengthened by making \( \lambda(V_n^2) \) tend to zero sufficiently rapidly. For, by (11),

\[
\|\varphi\|_p \leq \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \|\varphi_n\|_p.
\]

So, for example, if we take \( V_n \) so small that \( \lambda(V_n^2) \leq \exp(-n^2) \) and assume that \( U_1 \) is integrable, calculations show that

\[
\int_{U_1} \exp(\varphi) \, d\lambda < \infty.
\]

(Recall that \( \varphi \) vanishes on \( G \smallsetminus \{e\} \).) In the same way one can arrange even that

\[
\int_{U_1} \exp(\varphi) \, d\lambda < \infty,
\]

where \( E \) is any preassigned iterated exponential function.

(d) The Abelian case. Suppose henceforth that \( G \) is also Abelian (and additively written), and denote by \( X \) the non-compact character group of \( G \) and by \( \hat{\varphi} \) the Fourier transform of \( \varphi \). The Lebesgue spaces \( L^p(X) \) are to be formed with that Haar measure \( \mu \) on \( X \) which is dual to \( \lambda \), so that the Parseval formula assumes the form \( \int |\hat{\varphi}|^2 \, d\mu = \int |\hat{\varphi}|^2 \, d\mu \).

From (4) it follows that

\[
\hat{\varphi} \in C_c(X) \cap L^1(X);
\]

(10) shows that \( \hat{\varphi} \geq 0 \); and from (3) it follows that

\[
\hat{\varphi} \in L^1(X).
\]

On the other hand, (10) shows that

\[
\|\hat{\varphi}\|_p \leq \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \|\varphi_n\|_p
\]

for any \( q \in [1, \infty) \). Now, if \( 1 \leq p \leq 2 \), the Hausdorff-Young inequality gives

\[
\|\hat{\varphi}\|_q \leq \|\varphi\|_p \leq \lambda(V_n^2)^{1/p},
\]

the last step by (8). Accordingly, taking \( p = 2q/(2q-1) \), (16) leads to

\[
\|\hat{\varphi}\|_q \leq \sum_{n=1}^\infty \lambda(V_n^2)^{-1} \lambda(V_n^2)^{2q/(2q-1)};
\]

for \( q \in (1, \infty) \). The second clause of (7) shows that the last-written sum is finite provided \( q > 1 \). Thus, in spite of (15), we do have the relation

\[
\hat{\varphi} \in C_c(X) \cap \bigcap \{L^q(X) : 1 < q < \infty\};
\]

(17)

It is moreover possible to show that \( \varphi \) may be chosen so that also

\[
\int_X D\hat{\varphi} \, d\mu < \infty,
\]

(18)
where $D$ denotes any preassigned bounded (and measurable) non-negative function on $X$ which satisfies

$$
\lim_{n \to \infty} D(\xi) = 0.
$$

This may be done by modifying slightly the construction appearing in 3.2, considering functions $\psi \in \mathcal{F}$ of the type

$$
\varphi = \sum_{n=1}^{\infty} c_n \phi_n,
$$

where the non-negative real numbers $c_n$ are to be chosen so that

$$
\sum_{n=1}^{\infty} \lambda(V_n) c_n = \infty,
$$

$$
\sum_{n=1}^{\infty} \lambda(V_n)^{1/p} c_n < \infty \quad (1 \leq p < \infty),
$$

$$
\sum_{n=1}^{\infty} \int_X D\hat{\psi}_n d\mu < \infty.
$$

For, if such a choice is possible, (21) will combine with (9) to show that $\varphi(0) = \infty$; (22) will combine with (6) to yield (4); (23) will ensure that (18) holds; and it is clear that the remainder of properties (1)-(5) will remain intact.

Concerning the existence of a sequence $(c_n)$ satisfying (21), (22) and (23), it will follow once it is known that

$$
\lim_{n \to \infty} \lambda(V_n)^{-1} \int_X D\hat{\psi}_n d\mu = 0.
$$

Indeed, given (24) and writing $d_n = \lambda(V_n) c_n$, our demands take the following form:

$$
\sum_{n=1}^{\infty} d_n = \infty,
$$

$$
\sum_{n=1}^{\infty} \lambda(V_n)^{1/p} d_n < \infty \quad (1 \leq p < \infty),
$$

$$
\sum_{n=1}^{\infty} \lambda(V_n)^{-1} d_n \int_X D\hat{\psi}_n d\mu < \infty,
$$

to satisfy which it suffices to first choose positive integers $n_1 < n_2 < n_3 < \ldots$

such that

$$
\lambda(V_{n_k})^{-1} \int_X D\hat{\psi}_{n_k} d\mu \leq k^{-2} \quad (k = 1, 2, \ldots),
$$

a possibility which is vouchsafed by (24), and then define $d_n$ to be 1 if $n = n_k$ for some $k$ and to be 0 for all other positive integers $n$. (Recall equation (7).)

It thus remains merely to verify (24), to do which we first show that

$$
\lim_{n \to \infty} \lambda(V_n)^{-1} \int_X D\hat{\psi}_n d\mu = 0
$$

for any compact set $K \subset X$. In fact, choosing $f \in C_c$ so that $f \geq 0$ and $f(\xi) \geq 1$ for $\xi \in K$, we have

$$
\lambda(V_n)^{-1} \int_X \hat{\psi}_n d\mu \leq \lambda(V_n)^{-1} \int_X f\hat{\psi}_n d\mu = \lambda(V_n)^{-1} \int_X f \lambda(V_n)\psi_n(0) d\mu \leq \lambda(V_n)^{-1} \|f\|_{L^1} \|\psi_n\|_{L^1} \leq \|f\|_{L^1} \lambda(V_n)^{-1} \|\hat{\psi}_n\|_{L^1},
$$

the last step by (6), so that (7) leads to (25). So, given any $\epsilon > 0$, first choose a compact $K \subset X$ such that $D(\xi) \leq \epsilon$ for $\xi \in X \cap K$; this is possible by (19). Then, since (8) gives

$$
\int_X \hat{\varphi} d\mu = \varphi(0) = \int_X \hat{\varphi} d\lambda \leq \lambda(V_n),
$$

we have

$$
\lambda(V_n)^{-1} \int_X D\hat{\psi}_n d\mu \leq \lambda(V_n)^{-1} \int_X + \lambda(V_n)^{-1} \int_X \hat{\varphi} d\mu \leq \lambda(V_n)^{-1} \int_X \hat{\varphi} d\mu \leq \lambda(V_n)^{-1} \int_X D\hat{\psi}_n d\mu + \epsilon.
$$

Accordingly, (25) shows that

$$
\limsup_{n \to \infty} \lambda(V_n)^{-1} \int_X D\hat{\psi}_n d\mu \leq \epsilon,
$$

and the arbitrary choice of $\epsilon$ indicates that (24) holds.

References


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