STUDIA MATHEMATICA, T. XXXIII. (1969)

Unbounded integrally positive definite functions

by

R. E. EDWARDS (Canberra)

1. Throughout this note G will denote a Hausdorff locally compact group, e its neutral element, λ a chosen left Haar measure on G, $L^p = L^p(G, \lambda)$, and $C_c = C_c(G)$ is the set of continuous complex-valued functions on G having compact supports. For brevity we shall write dx in place of $d\lambda(x)$ and $d(x, y)$ in place of $d(\lambda\times\lambda)(x, y)$.

Given a set F of complex-valued measurable functions on G, a locally integrable complex-valued function φ on G will be said to be F-PD ($=F$-positive definite) if and only if

(PD) $$\int_{G\times G} |\varphi(y^{-1}x)\overline{f(y)}f(x)|\,d(x, y) < \infty$$

and

(PD′) $$\int_{G\times G} \varphi(y^{-1}x)\overline{f(y)}f(x)\,d(x, y) \geqslant 0$$

for every $f\in F$.

A continuous complex-valued function φ on G is said to be (B)-PD ($=$ positive definite in Bochner's sense) if and only if

(B) $$\sum_{i,j} \varphi(x_i^{-1}x_j)\,\overline{\alpha_i}\alpha_j \geqslant 0$$

for every finite complex-valued sequence (α_i) and every finite G-valued sequence (x_i). Such a function is necessarily bounded: $|\varphi(x)| \leqslant \varphi(e)$ for all $x\in G$.

If G happens to be discrete, and if $F\supset C_c$, any F-PD function is obviously (B)-PD and therefore belongs to L^∞. Even if G is non-discrete, it is known that any C_c-PD function which is essentially bounded on some neighbourhood of e is equal l. a. e. to a (B)-PD function (see, for example, [1], p. 715-720); and that an L^1-PD function belongs to L^∞ (see [1], p. 490 and the proof of Theorem 10.3.3).

Hewitt and Ross [2] have raised the question of the existence of F-PD functions not in L^∞, and they have indicated how to construct Borel functions φ on any non-discrete group G which are L^2-PD but not in L^∞. Here we shall give a somewhat more general discussion referring to L^r-PD functions and a construction leading to a stronger result.

We shall begin with a few simple remarks indicating some special cases which can be discarded from the outset.

From this point on we assume that G is *non-discrete.*

2. Some special cases.

2.1. If $r = \infty$, (PD) asserted for every $f \in L^\infty$ amounts simply to the demand that the function $(x, y) \to \varphi(y^{-1}x)$ be integrable for $\lambda \times \lambda$. Unless G is compact, this is so only if $\varphi = 0$ a.e.

On the other hand, if G is compact, it is evident that an integrable φ is L^∞-PD provided only that (PD′) holds for each $f \in L^\infty$. It is furthermore easy to see that this is the case if and only if the (generally operator-valued) Fourier transform $\hat{\varphi}$ is non-negative. A simple category argument suffices to show that there exist always functions φ of this type which are not in L^∞.

In the sequel we may therefore suppose that $1 \leqslant r < \infty$.

2.2. Supposing now that $1 \leqslant r < \infty$, general functional analytic principles ([1], p. 490) show that a locally integrable φ is L^r-PD if and only if

(A) $$f * \varphi \in L^{r'} \quad \text{and} \quad \|f*\varphi\|_{r'} \leqslant \text{const}\,\|f\|_r$$

and

(A′) $$(f*\varphi \mid f) \geqslant 0$$

for every $f \in C_c$, where $r' = r/(r-1)$ and

$$(u \mid v) = \int_G u\bar{v}\,d\lambda.$$

In this connection we should perhaps remark that, if $f \in L^r$ for some $r < \infty$, then f vanishes off a subset of G which is sigma-finite for λ; hence the function

$$(x, y) \to \varphi(y^{-1}x)\overline{f(y)}f(x)$$

vanishes off a subset of $G \times G$ which is sigma-finite for $\lambda \times \lambda$ (this is a consequence of [1], 4.17.2); consequently the theorems of Fubini and Tonelli ([1], Theorems 4.17.4 and 4.17.8) are applicable and show that the appropriate integrals $\int_{G\times G} \ldots d(x, y)$ can be replaced by either of the associated repeated integrals $\int_G \{\int_G \ldots dx\}\,dy$ and $\int_G \{\int_G \ldots dy\}\,dx$.

If $r = 1$, then (as was noted in section 1) (A) alone suffices to show that $\varphi \in L^\infty$. Consequently, the interest lies in the case where $1 < r < \infty$.

2.3. If G is compact and $r \geqslant 2$, it is evident that any $\varphi \in L^1$ such that $\hat{\varphi} \geqslant 0$ is L^r-PD, and that there exist many such functions outside L^∞ (see 2.1).

On the other hand, if $r > 2$ and G is non-compact, (A) alone forces φ to be zero l. a. e.: this is proved by Hörmander in case $G = R^n$ ([4], Theorem 1.1) and his argument extends at once to any non-compact G.

Accordingly we shall in the sequel be concerned primarily with the remaining case, where $1 < r \leqslant 2$. For this purpose se shall use the following lemma:

2.4. Lemma. *Suppose that* $1 < r \leqslant 2$ *and put* $a = \frac{1}{2}r'$. *If* φ *is measurable and* $\|\varphi\|_a = \|\varphi^*\|_a$, *where* $\varphi^*(x) = \varphi(x^{-1})$ *for all* $x \in G$, *then* (A) *holds with* const $\leqslant \|\varphi\|_a$.

Proof. This is a special case of Young's inequality; see [3], p. 296, Theorem (20.18), or, more generally but less directly, [1], p. 655, Theorem 9.5.1.

3. The main result.

3.1. Supposing G to be non-discrete, let $(U_n)_{n=1}^\infty$ be any sequence of closed neighbourhoods of e in G and write $P = \bigcap_{n=1}^\infty U_n$. Put

$$F_0 = \bigcup \{L^r : 1 < r \leqslant 2\}.$$

We describe the construction of functions φ on G with the following properties:

(1) $\varphi \geqslant 0$, $\varphi = \varphi^*$, φ is lower semicontinuous, φ has a compact support and vanishes on $G \cap U_1'$;

(2) φ is continuous at all points of $G \cap P'$;

(3) $\lim_{x\to e} \varphi(x) = \infty$, so that $\varphi \notin L^\infty$;

(4) $\varphi \in \bigcap \{L^p : 1 \leqslant p < \infty\}$;

(5) φ is F_0-PD.

3.2. The construction is based on the fact that, if u is a bounded measurable complex-valued function on G which has its support lying within a relatively compact symmetric open neighbourhood V of e in G, then

$$\varphi_u = u * u^\sim,$$

where $u^\sim = \bar{u}^*$, belongs to C_c and satisfies

(6) $$\operatorname{supp}\varphi_u \subset V^2, \quad \varphi_u^* = \bar{\varphi}_u, \quad \|\varphi_u\|_p \leqslant \lambda(V)\lambda(V^2)^{1/p}$$

for $0 < p < \infty$. Plainly, φ_u is real and non-negative whenever u has these properties. By (6), Lemma 2.4, and the fact that

$$\int_{G\times G} \varphi_u(y^{-1}x)\overline{f(y)}f(x)\,d(x, y) = \int_G |\bar{f} * u|^2\,d\lambda,$$

it is visible that φ_u is F_0-PD. Functions φ of the desired type will now be obtained as suitable infinite sums of such functions φ_u.

To do this, choose relatively compact symmetric open neighbourhoods V_n $(n = 1, 2, \ldots)$ of e such that

(7) $$V_n^2 \subset U_1 \cap U_2 \cap \ldots \cap U_n, \quad 0 < \lambda(V_n^2) \leqslant 2^{-n};$$

this choice is possible since G is non-discrete. Next choose functions $u_n \in C_c$ such that

(8) $$\operatorname{supp} u_n \subset V_n, \quad 0 \leqslant u_n \leqslant 1,$$

(9) $$u_n * \tilde{u_n}(e) = \int_G u_n^2 d\lambda \geqslant \tfrac{1}{2}\lambda(V_n).$$

Put $\varphi_n = \varphi_{u_n}$ and consider the non-negative lower semicontinuous function φ defined by

(10) $$\varphi = \sum_{n=1}^{\infty} \lambda(V_n)^{-1}\varphi_n.$$

Relations (6), together with the choice of the u_n, show at once that (1) is true. Statement (3) follows easily from (7) and (10). If K is any compact subset of G not meeting P, (7) and (8) show that K meets $\operatorname{supp}\varphi_n$ for at most a finite set of positive integers n, and (2) follows at once from this. Fatou's lemma combines with (6) (with $u = u_n$) and (7) to show that for $1 \leqslant p < \infty$ we have

(11) $$\|\varphi\|_p \leqslant \sum_{n=1}^{\infty} \lambda(V_n)^{-1}\|\varphi_n\|_p \leqslant \sum_{n=1}^{\infty} 2^{-n/p} < \infty,$$

which establishes (4). Finally, (5) follows from (4), Lemma 2.4, and the identity

$$\int_{G\times G} \varphi(y^{-1}x)\overline{f(y)}f(x)\,d(x,y) = \sum_{n=1}^{\infty} \lambda(V_n)^{-1} \int_G |\bar{f} * u_n|^2 d\lambda.$$

3.3. Supplements.

(a) If G is first countable, we may arrange that $P = \{e\}$.

(b) If G is a Lie group, we can arrange that $P = \{e\}$ and that $u_n \in C_c^{\infty}(G)$ for every n; accordingly, (2) may be replaced by

(2′) $$\varphi \in C^{\infty}(G \cap \{e\}').$$

(c) Condition (4) can be strengthened by making $\lambda(V_n^2)$ tend to zero sufficiently rapidly. For, by (11),

(12) $$\|\varphi\|_p \leqslant \sum_{n=1}^{\infty} \lambda(V_n^2)^{1/p}.$$

So, for example, if we take V_n so small that $\lambda(V_n^2) \leqslant \exp(-e^{n^2})$ and assume that U_1 is integrable, calculations show that

(13) $$\int_{U_1} \exp(e^{\varphi})\,d\lambda < \infty.$$

(Recall that φ vanishes on $G \cap U_1'$.) In the same way one can arrange even that

(14) $$\int_{U_1} E(\varphi)\,d\lambda < \infty,$$

where E is any preassigned iterated exponential function.

(d) *The Abelian case.* Suppose henceforth that G is also Abelian (and additively written), and denote by X the non-compact character group of G and by $\hat{\varphi}$ the Fourier transform of φ. The Lebesgue spaces $L^q(X)$ are to be formed with that Haar measure μ on X which is dual to λ, so that the Parseval formula assumes the form $\int_G |g|^2 d\lambda = \int_X |\hat{g}|^2 d\mu$.

From (4) it follows that

$$\hat{\varphi} \in C_0(X) \cap L^2(X);$$

(10) shows that $\hat{\varphi} \geqslant 0$; and from (3) it follows that

(15) $$\hat{\varphi} \notin L^1(X).$$

On the other hand, (10) shows that

(16) $$\|\hat{\varphi}\|_q \leqslant \sum_{n=1}^{\infty} \lambda(V_n)^{-1}\|\hat{u}_n\|_{2q}^2$$

for any $q \in [1, \infty)$. Now, if $1 \leqslant p \leqslant 2$, the Hausdorff-Young inequality gives

$$\|\hat{u}_n\|_{p'} \leqslant \|u_n\|_p \leqslant \lambda(V_n)^{1/p},$$

the last step by (8). Accordingly, taking $p = 2q/(2q-1)$, (16) leads to

$$\|\hat{\varphi}\|_q \leqslant \sum_{n=1}^{\infty} \lambda(V_n)^{-1}\lambda(V_n)^{(2q-1)/q}$$

for $q \in [1, \infty)$. The second clause of (7) shows that the last-written sum is finite provided $q > 1$. Thus, in spite of (15), we do have the relation

(17) $$\hat{\varphi} \in C_0(X) \cap \bigcap \{L^q(X) : 1 < q < \infty\}.$$

It is moreover possible to show that φ may be chosen so that also

(18) $$\int_X D\hat{\varphi}\,d\mu < \infty,$$

where D denotes any preassigned bounded (and measurable) non-negative function on X which satisfies

$$\lim_{\chi\to\infty} D(\chi) = 0. \tag{19}$$

This may be done by modifying slightly the construction appearing in 3.2, considering functions φ of the type

$$\varphi = \sum_{n=1}^{\infty} c_n \varphi_n, \tag{20}$$

where the non-negative real numbers c_n are to be chosen so that

$$\sum_{n=1}^{\infty} \lambda(V_n) c_n = \infty, \tag{21}$$

$$\sum_{n=1}^{\infty} \lambda(V_n)\lambda(V_n^2)^{1/p} c_n < \infty \quad (1 \leqslant p < \infty), \tag{22}$$

$$\sum_{n=1}^{\infty} c_n \int_X D\hat{\varphi}_n d\mu < \infty. \tag{23}$$

For, if such a choice is possible, (21) will combine with (9) to show that $\varphi(0) = \infty$; (22) will combine with (6) to yield (4); (23) will ensure that (18) holds; and it is clear that the remainder of properties (1)-(5) will remain intact.

Concerning the existence of a sequence (c_n) satisfying (21), (22) and (23), it will follow once it is known that

$$\lim_{n\to\infty} \lambda(V_n)^{-1} \int_X D\hat{\varphi}_n d\mu = 0. \tag{24}$$

Indeed, given (24) and writing $d_n = \lambda(V_n)c_n$, our demands take the following form:

$$\sum_{n=1}^{\infty} d_n = \infty,$$

$$\sum_{n=1}^{\infty} \lambda(V_n^2)^{1/p} d_n < \infty \quad (1 \leqslant p < \infty),$$

$$\sum_{n=1}^{\infty} \lambda(V_n)^{-1} d_n \int_X D\hat{\varphi}_n d\mu < \infty,$$

to satisfy which it suffices to first choose positive integers $n_1 < n_2 < n_3 < \ldots$ such that

$$\lambda(V_{n_k})^{-1} \int_X D\hat{\varphi}_{n_k} d\mu \leqslant k^{-2} \quad (k = 1, 2, \ldots),$$

a possibility which is vouchsafed by (24), and then define d_n to be 1 if $n = n_k$ for some k and to be 0 for all other positive integers n. (Recall equation (7).)

It thus remains merely to verify (24), to do which we first show that

$$\lim_{n\to\infty} \lambda(V_n)^{-1} \int_K \hat{\varphi}_n d\mu = 0 \tag{25}$$

for any compact set $K \subset X$. In fact, choosing $f \epsilon C_c$ so that $\hat{f} \geqslant 0$ and $\hat{f}(\chi) \geqslant 1$ for $\chi \epsilon K$, we have

$$\lambda(V_n)^{-1} \int_K \hat{\varphi}_n d\mu \leqslant \lambda(V_n)^{-1} \int_X \hat{f}\hat{\varphi}_n d\mu = \lambda(V_n)^{-1} f * \varphi_n(0)$$
$$\leqslant \lambda(V_n)^{-1} \|f\|_2 \|\varphi_n\|_2 \leqslant \|f\|_2 \lambda(V_n^2)^{\frac{1}{2}},$$

the last step by (6), so that (7) leads to (25). So, given any $\varepsilon > 0$, first choose a compact $K \subset X$ such that $D(\chi) \leqslant \varepsilon$ for $\chi \epsilon X \cap K'$; this is possible by (19). Then, since (8) gives

$$\int_X \hat{\varphi} d\mu = \varphi_n(0) = \int_G u_n^2 d\lambda \leqslant \lambda(V_n),$$

we have

$$\lambda(V_n)^{-1} \int_X D\hat{\varphi}_n d\mu \leqslant \lambda(V_n)^{-1} \int_K + \lambda(V_n)^{-1} \int_{X \cap K'}$$
$$\leqslant \lambda(V_n)^{-1} \int_K + \varepsilon\lambda(V_n)^{-1} \int_X \hat{\varphi}_n d\mu$$
$$\leqslant \lambda(V_n)^{-1} \int_K D\varphi_n d\mu + \varepsilon.$$

Accordingly, (25) shows that

$$\limsup_{n\to\infty} \lambda(V_n)^{-1} \int_X D\hat{\varphi}_n d\mu \leqslant \varepsilon,$$

and the arbitrary choice of ε indicates that (24) holds.

References

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[2] E. Hewitt and K. A. Ross, *Integrally positive-definite functions on groups* (to appear in Studia Math).

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Reçu par la Rédaction le 1. 7. 1968