COROLLARY 6. *If $f(z)$ is as in Corollary 2, then $[y'_m\{f^{(n)}(z\zeta_n)\}z^n]$ is complete in $P(E;U)$ if (ζ_n) is a sequence in U such that $\zeta_n\to\zeta_0\in U$ and*

$$\sum_{n=1}^{\infty}|\zeta_{n+1}-\zeta_n|<\infty.$$

All these results on analytic functions hold for more complicated domain spaces that U but we have avoided the technical complications which would result from more general statements since they do not add to the interest of the situation.

I would like to thank my colleague G. Bennett of St. John's College, Cambridge, for many helpful discussions on this work.

References

[1] V. L. Gončarov, *Recherches sur les dérivées successives des fonctions analytiques*, Thesis 1930.

[2] A. Grothendieck, *Sur certains espaces de fonctions holomorphes I, II*, Jour. reine angew. Math. 192 (1953), pp. 35-64 and 77-95.

[3] — *Produits tensoriel topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[4] A. I. Markuševič, *On bases in the space of analytic functions*. Amer. Math. Soc. Translations (2) 22, p. 1-42.

Reçu par la Rédaction le 22. 5. 1968

STUDIA MATHEMATICA, T. XXXIII. (1969)

Transforming bilinear vector integrals

by

JAMES K. BROOKS (Gainesville)

Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be Banach spaces with a continuous bilinear multiplication defined on $\mathfrak{X}\times\mathfrak{Y}$ into $\mathfrak{Z}$. Bartle [1] has developed a Lebesgue-type integration theory for $\mathfrak{X}$-valued integrands and $\mathfrak{Y}$-valued measures in which some of the classical integration theorems remain valid. This bilinear vector integral has many important applications, for example, numerous authors have used this integral (in restrictive forms) for establishing integral representations for linear operators defined on $L^p_{\mathfrak{X}}$ and $C_{\mathfrak{X}}(S)$ into $\mathfrak{Y}$ (e.g. see [3], [8], and [10]); consequently, it is of interest to consider the relationships between these integrals which arise from transforming one vector measure space into another. More precisely, let T be a transformation between the $\mathfrak{Y}$-vector measure spaces (S,Σ,μ) and (S',Σ',μ'). We are interested in finding conditions in order that $\mathfrak{X}$-valued measurable functions G' defined on S' satisfy the *transformation formula*

$$(*)\qquad \int_D G'\circ Tf\,d\mu=\int_{TD} G'W'(\cdot,D)\,d\mu',$$

where D belongs to a certain subfamily of Σ; W' and f are appropriate functions describing the change of measure induced by T.

In section 1 the standard hypotheses concerning the structure of the measure spaces in transformation theory are presented. By altering the usual definition used in transformation theory for a weight function, we are able to introduce in section 2 the notion of an absolutely continuous transformation relative to vector measures, along with the concept of a generalized Jacobian for T. $(*)$ is established in section 3 for absolutely continuous transformations. This development includes all the existing transformation formulas in the literature for the absolutely continuous case; in particular, it includes the theory developed by Reichelderfer [12] for positive measure spaces. The author wishes to express his gratitude to P. V. Reichelderfer for his helpful suggestions.

1. The setting. Throughout this paper, $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ will denote Banach spaces over the complex number field $\mathfrak{C}$. For terminology and concepts concerning vector measures and integration theory used in sections 1 and 2, see [2] and [9]. If (S, Σ, μ) is a $\mathfrak{Y}$-vector measure space, $\|\mu\|$ and $|\mu|$ denote respectively the semi-variation (with respect to $\mathfrak{C}$) and the total variation of μ. A set is *μ-null* if it is contained in a set $E \in \Sigma$, where $\|\mu\|(E) = 0$. There exists a finite positive measure α defined on Σ (called a *control measure* for μ) such that $\alpha(E) \to 0$ if and only if $\|\mu\|(E) \to 0$ ([9], p. 321).

We now list the hypotheses I-IX under which the theory is to be developed. For brevity, E, E', D, B', O' will be generic notations for sets belonging to $\Sigma, \Sigma', \mathfrak{D}, \mathfrak{B}', \mathfrak{O}'$ respectively.

I. (S, Σ, μ) is a $\mathfrak{Y}$-vector measure space with control measure α.

II. (S', Σ', μ') is a $\mathfrak{Y}$-vector measure space with control measure α'.

III. T is a function (transformation) from S onto S'.

IV. $\mathfrak{D}$ is a subfamily of Σ containing the empty set and S. $T\mathfrak{D} \subseteq \Sigma'$.

The intersection of two sets belonging to $\mathfrak{D}$ can be expressed as a countable union of disjoint sets from $\mathfrak{D}$. For every E and $\varepsilon > 0$ there exists a disjoint sequence of sets D_i such that $E \subseteq \bigcup D_i$ and $\alpha(\bigcup D_i - E) < \varepsilon$.

Definitions. E is a *$\mu\mu'$-null set* if $E = A_1 \cup A_2$, where A_1 is μ-null and TA_2 is μ'-null. An element D *is of type* γ if it belongs to a countable partition of S, where the partition consists of sets from $\mathfrak{D}$ and a $\mu\mu'$-null set.

V. Every member of $\mathfrak{D}$ can be expressed as the union of a monotone increasing sequence of sets of type γ.

VI. $\mathfrak{B}'$ is a sub-σ-algebra of Σ'. $T^{-1}\mathfrak{B}' \subseteq \Sigma$. For each E' there exist sets B_1', B_2' such that $B_1' \subseteq E' \subseteq B_2'$ and $\alpha'(B_2' - B_1') = 0$.

Definition. $\mathfrak{O}'$ denotes the family of subsets O' of S' such that $T^{-1}O'$ is a countable union of disjoint sets from $\mathfrak{D}$.

VII. For every E' and $\varepsilon > 0$ there exists an O' such that $E' \subseteq O'$ and $\alpha'(O' - E') < \varepsilon$.

Definitions. O' *is of type* γ' if it is a member of a countable partition of S', where the partition consists of sets from $\mathfrak{O}'$ and sets A', B', where A' is μ'-null and $T^{-1}B'$ is μ-null. E' *is of type* ι' if it is the union of a monotone increasing sequence of sets of type γ'.

VIII. For every B' there exists a monotone decreasing sequence of sets E_i', each of type ι', and μ'-null sets M', N' such that $(\bigcap E_i') \cup M' = B' \cup N'$.

The above hypotheses provide a setting in which absolute continuity can be defined. I-VIII imply the hypotheses H1-H8 in [12], when μ and μ' are positive. When S and S' are topological spaces and T is continuous, there is a theory [13] which includes a large class of topological spaces satisfying I-VIII.

IX. A *weight function* for T is a complex-valued function W' defined on $S' \times \mathfrak{D}$ satisfying the following conditions:

(i) $W'(\cdot, D) = 0$ on $S' - TD$.

(ii) If D is the union of a monotone increasing sequence of sets D_i, then $\lim W'(\cdot, D_i) = W'(\cdot, D)$ a.e. μ'.

(iii) If there exists a disjoint sequence of sets D_i contained in D such that $D - \bigcup D_i$ is $\mu\mu'$-null, then $\sum W'(\cdot, D_i) = W'(\cdot, D)$ a.e. μ'.

(iv) $W'(\cdot, D)$ is μ'-measurable for each D.

Remark. $W'(s', D)$ represents a weight assigned to the points in D which T sends into s'; this allows us to treat different possible definitions of absolute continuity simultaneously. A *non-negative weight function* W' is a non-negative function satisfying IX, except (iii) is replaced by the under additive requirement that $\sum W'(\cdot, D_i) \leqslant W'(\cdot, D)$ a.e. μ', whenever the D_i are disjoint subsets of D. If W' is a weight function whose range is the non-negative reals, then by partitioning D into sets of type γ and using (iii), one can show that W' is under additive a.e. μ'; consequently, in this case W' essentially satisfies H9 in [12]. Conditions are given in [4] in order that a real-valued weight function can be decomposed uniquely into non-negative weight functions.

We list some important examples of non-negative weight functions for continuous transformations T defined on R^n ($\mathfrak{D}$ in this case is the family of domains contained in the bounded domain S): (a) the essential multiplicity functions $K(s', T, D)$, $K^+(s', T, D)$, $K^-(s', T, D)$, which are generated by the topological index defined on indicator domains; (b) $N(s', T, D)$, the crude multiplicity function or the Banach indicatrix of T; (c) $k(s', T, D)$, which counts the number of essential maximal model continua for (s', T, D). These functions are studied in detail in [11]. Note that if T is essentially absolutely continuous and if $\mu_e(\cdot, D) = K^+(\cdot, T, D) - K^-(\cdot, T, D)$, then μ_e is a weight function. See [5] and [6] for additional properties concerning non-negative weight functions.

We shall always assume that I-VIII hold and that W' is a weight function for T.

2. Absolute continuity. The variation of W' is the non-negative extended real-valued function V' defined on $S' \times \mathfrak{D}$ as follows: $V'(s', D) = \sup \sum |W'(s', D_i)|$, where the supremum is taken over all finite partitions $\{D_i\}$ of D. T is of *bounded variation* with respect to W' (BVW') if $V'(\cdot, D)$ is μ'-integrable for each D. T is *absolutely continuous* with respect to W' (ACW') if T is BVW' and if there exists a complex valued μ-integrable function f defined on S such that for each D,

$$\int_D f\,d\mu = \int_{S'} W'(\cdot, D)\,d\mu'.$$

f is called a *gauge* or a *generalized Jacobian* for T (relative to W', μ, μ').

Remark. The above integrals are those defined by Bartle, Dunford, and Schwartz [2]. By IX, the integral on the right, which exists since $V'(\cdot, D)$ is integrable, need only be taken over TD. One can show that gauges are uniquely defined in the sense of equality a. e. μ. The concepts of essential absolute continuity, absolute continuity in the Banach sense, and strong absolute continuity in the Banach sense as described in [11], and absolute continuity as defined in [12] are special cases of the above definition.

In the sequel, we assume T is ACW' and f is a gauge for T.

LEMMA 1. *Let A' be a μ'-null set. Then $f = 0$ a. e. μ on $T^{-1}A'$.*

Proof. Let y^* belong to $\mathfrak{Y}^*$, the dual space of $\mathfrak{Y}$, and define

$$\lambda(E) = \int_E f d(y^*\mu), \quad E \in \Sigma,$$

where $y^*\mu$ is now a scalar measure with total variation $|y^*\mu|$. The tota variation of λ is given by

$$|\lambda|(E) = \int_E |f|\, d\,|y^*\mu|.$$

Fix D and $\varepsilon > 0$; choose $\delta > 0$ such that $|y^*\mu|(E) < \delta$ implies $|\lambda|(E) < \varepsilon$. Pick a set $E \subseteq D$ satisfying $|\lambda|(D) \leqslant 4|\lambda(E)| + \varepsilon$. We can find a disjoint sequence $\{D_i\}$ such that $E \subseteq \bigcup D_i$ and $|y^*\mu|(\bigcup D_i - E) < \delta$. By IV, for each i there exists a disjoint sequence $\{D_{ij}\}_{j=1}^{\infty}$ such that $D \cap D_i = \bigcup_j D_{ij}$. Thus $|\lambda(\bigcup_{ij} D_{ij}) - \lambda(E)| < \varepsilon$. Consequently,

$$\begin{aligned}|\lambda|(D) &\leqslant 4\sum_{ij} |\lambda(D_{ij})| + 5\varepsilon \\ &= 4\sum_{ij}\Big|\int_{S'} W'(\cdot, D_{ij})\, d(y^*\mu')\Big| + 5\varepsilon \\ &\leqslant 4\int_{S'} \sum_{ij} |W'(\cdot, D_{ij})|\, d|y^*\mu'| + 5\varepsilon \\ &\leqslant 4\int_{S'} V'(\cdot, D)\, d|y^*\mu'| + 5\varepsilon.\end{aligned}$$

Since ε was arbitrary, we conclude

$$(1) \qquad \int_D |f|\, d|y^*\mu| \leqslant 4\int_{S'} V'(\cdot, D)\, d|y^*\mu'| = 4\int_{TD} V'(\cdot, D)\, d|y^*\mu'|.$$

The integrability of $V'(\cdot, S)$ implies that for each positive integer n there exists a $\delta_n > 0$ such that $|\int_{E'} V'(\cdot, S)\, d\mu'| < 1/n$ whenever $\alpha'(E') < \delta_n$.

If y^* belongs to the unit sphere of $\mathfrak{Y}^*$ and $\alpha'(E') < \delta_n$, then

$$\int_{E'} V'(\cdot, S)\, d|y^*\mu'| \leqslant 4 \sup_{Q' \subseteq E'} \Big|y^* \int_{Q'} V'(\cdot, S)\, d\mu'\Big| \leqslant 4/n.$$

For each n choose an $O_n' \supseteq A'$ such that $\alpha'(O_n') < \delta_n$. Let $Z_n = T^{-1}O_n' = \bigcup_i D_i^n$, where the $\{D_i^n\}_{i=1}^{\infty}$ are disjoint. By the under additivity of V' and (1), we have

$$\int_{Z_n} |f|\, d|y^*\mu| \leqslant 4\sum_i \int_{TD_i^n} V'(\cdot, D_i^n)\, d|y^*\mu'| \leqslant 4\int_{O_n'} V'(\cdot, S)\, d|y^*\mu'| \leqslant 16/n.$$

Thus if $Z = \bigcap Z_n$, $\int_Z |f|\, d|y^*\mu| = 0$. Write $E^* = Z \cap \{s : f(s) \neq 0\}$; then $|y^*\mu|(E^*) = 0$. The control measure α can be chosen so that $\alpha(E) \leqslant \sup |y^*\mu|(E)$, $|y^*| \leqslant 1$, for each E ([9], Cor. 4.9.3). Hence $\alpha(E^*) = 0$. Since $T^{-1}A' \subseteq Z$, we conclude that $f = 0$ a.e. μ on $T^{-1}A'$.

THEOREM 1. *Assume that $H' : S' \to \mathfrak{C}$ is μ'-measurable. Then $H' \circ Tf$ is μ-measurable. Let $D \in \mathfrak{D}$. If $H'W'(\cdot, D)$ is μ'-integrable and $H' \circ Tf$ is μ-integrable on D, then*

$$(2) \qquad \int_D H' \circ Tf\, d\mu = \int_{S'} H'W'(\cdot, D)\, d\mu'.$$

Proof. To prove the first part of the conclusion, use VI and construct a B'-measurable function K' such that $K' = H'$ a. e. μ'. $H' \circ Tf$ is μ-measurable since $K' \circ Tf$ is μ-measurable and $K' \circ Tf = H' \circ Tf$ a. e. μ by lemma 1.

To establish (2), note that for $y^* \in \mathfrak{Y}^*$, $(S, \Sigma, |y^*\mu|)$, $(S', \Sigma', |y^*\mu'|)$, $T, \mathfrak{D}, \mathfrak{B}', \mathfrak{D}'$ satisfy H1-H8 in [12]. By modifying the techniques used in section 4 of [12], it can be shown that

$$(3) \qquad \int_D H' \circ Tf\, d(y^*\mu) = \int_{S'} H'W'(\cdot, D)\, d(y^*\mu').$$

In the course of deriving (3), lemma 1, property (iii) of W', and the Lebesgue dominated convergence theorem involving W' and V' are used; we omit the lengthy details. (2) then follows from (3) and the Hahn-Banach theorem.

3. The bilinear vector case. Assume that there exists a continuous bilinear mapping from $\mathfrak{X} \times \mathfrak{Y}$ into $\mathfrak{Z}$, denoted by juxtaposition, such that

(#) if $xy = 0$ for all $x \in \mathfrak{X}$, then $y = 0$.

We are now able to consider integrals of the form $\int H'\, d\mu'$, where H' is $\mathfrak{X}$-valued. For definitions and theorems concerning bilinear integration theory, see [1]. As in [1], we make the following assumption: there exist non-negative finite measures β and β' defined on Σ and Σ' respectively such that $\beta(E) \to 0$ $(\beta'(E') \to 0)$ if and only if $\tilde{\mu}(E) \to 0$ $(\tilde{\mu}'(E') \to 0)$, where $\tilde{\mu}$ and $\tilde{\mu}'$ denote the semi-variations, with respect to $\mathfrak{X}$, of μ and μ'. A set is μ-null in this integration theory if it is contained in a set E, where $\tilde{\mu}(E) = 0$. The following theorem gives sufficient conditions in order that the transformation formula holds for this general integral.

THEOREM 2. *Fix D in $\mathfrak{D}$. Assume that $H': S' \to \mathfrak{X}$ is a μ'-measurable function such that $H'W'(\cdot, D)$ is μ'-integrable and $H' \circ Tf$ is μ-integrable on D. If there exists a sequence of μ'-simple functions $H'_n: S' \to \mathfrak{X}$ such that*

(i) $\lim \int_D H'_n \circ Tf\, d\mu = \int_D H' \circ Tf\, d\mu$,

(ii) $\lim \int_{S'} H'_n W'(\cdot, D)\, d\mu' = \int_{S'} H'W'(\cdot, D)\, d\mu'$,

then $\int_D H' \circ Tf\, d\mu = \int_{S'} H'W'(\cdot, D)\, d\mu'$.

Proof. Applying (#), one can show that $a(E) = 0$ $(a'(E') = 0)$ if and only if $\beta(E) = 0$ $(\beta'(E') = 0)$. Hence, the null sets in the bilinear vector integration theory and the null sets in the integration theory used in section 2 coincide; thus lemma 1 and theorem 1 can be used to ensure the measurability of the functions involved. If G' denotes the characteristic function of a set E', then by theorem 1 and the continuity of the bilinear multiplication, it follows that for $x \in \mathfrak{X}$,

$$\int_D xG' \circ Tf\, d\mu = \int_{S'} xG'W'(\cdot, D)\, d\mu'.$$

Thus the transformation formula holds for μ'-simple $\mathfrak{X}$-valued functions. Consequently, conditions (i) and (ii) imply the conslusion of the theorem.

Now we restrict our attention to the important case when μ and μ' have finite total variation. This case is the one usually considered in representation theory for operators. For this case set $a = \beta = |\mu|$ and $a' = \beta' = |\mu'|$. A function $g: S \to \mathfrak{X}$ is *μ-strongly integrable* if g is Bochner integrable with respect to $|\mu|$. Note that a strongly integrable function is integrable in Bartle's sense.

THEOREM 3. *Fix D in $\mathfrak{D}$. Assume that $H': S' \to \mathfrak{X}$ is a μ'-measurable function. If $H'W'(\cdot, D)$ is μ'-strongly integrable and $H' \circ Tf$ is μ-strongly integrable on D, then*

$$\int_D H' \circ Tf\, d\mu = \int_{S'} H'W'(\cdot, D)\, d\mu'.$$

Proof. Since H' is measurable, we can construct a sequence of μ'-simple functions $H'_n: S' \to \mathfrak{X}$ such $\lim H'_n = H'$ a. e. μ' and $|H'_n(s')| \leqslant 2|H'(s')|$, $s' \in S'$. Hence, $\lim H'_n W'(\cdot, D) = H'W'(\cdot, D)$ a. e. μ' and, by lemma 1, $\lim H'_n \circ Tf = H' \circ Tf$ a. e. μ. Moreover, $|H'_n \circ Tf| \leqslant 2|H' \circ Tf|$ and $|H'_n W'(\cdot, D)| \leqslant 2|H'W'(\cdot, D)|$. It can be shown that a Lebesgue dominated convergence theorem holds for strongly integrable functions ([7], p. 136). By applying this result to the sequences $\{H'_n \circ Tf\}$ and $\{H'_n W'(\cdot, D)\}$, we see that conditions (i) and (ii) of theorem 2 are satisfied. This in turn implies that H' satisfies the transformation formula.

Remarks. 1. Assume μ and μ' are complex-valued and that $H': S' \to \mathfrak{X}$ is μ'-measurable. By using inequality (1) in the proof of lemma 1, it can be shown that if $H'V'(\cdot, D)$ is integrable, then $H' \circ Tf$ is μ-integrable on D. Reichelderfer and the author have recently shown that the integrability of $H' \circ Tf$ on D implies the integrability of $H'W'(\cdot, D)$. Examples show that the integrability of $H'W'(\cdot, D)$ does not imply the integrability of $H' \circ Tf$ on D.

2. For applications to representation theory for operators defined on $C_{\mathfrak{X}}(S)$, where S is a locally compact Hausdorff space, the transformation formula can be derived in a topological setting described in [13].

References

[1] R. G. Bartle, *A general bilinear vector integral*, Studia Math. 15 (1956), p. 337-352.

[2] — N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7 (1955), p. 289-305.

[3] W. M. Bogdanowicz, *Integral representations of linear continuous operators from the space of Lebesgue-Bochner summable functions into any Banach space*, Proc. Nat. Acad. Sci. 54 (1965), p. 351-354.

[4] J. K. Brooks and P. V. Reichelderfer, *A Jordan decomposition for weight functions* (to appear).

[5] R. W. Chaney, *Measurability theorems in the transformation theory for measure space*, Rend. Circ. Math. di Palermo 14 (1965), p. 309-323.

[6] — *On the transformation of integrals in measure space*, Pacific J. Math. (2) 19 (1966), p. 229-242.

[7] N. Dinculeanu, *Vector measures*, Berlin 1967.

[8] — and C. Foiaş, *Sur la representation intégrale de certaines opérations linéaires IV*, Cand. J. Math. (13) (1961), p. 529-556.

[9] N. Dunford and J. Schwartz, *Linear operators*, Part I: General theory, New York 1958.

[10] A. Ionescu Tulcea and C. Ionescu Tulcea, *On the decomposition and integral representation of continuous linear operators*, Ann. Mat. Pura ed Appl. 53 (1961), p. 63-87.

[11] T. Rado and P. V. Reichelderfer, *Continuous transformations in analysis*, Berlin — Göttingen — Heidelberg 1955.

[12] P. V. Reichelderfer, *A transformation theory for measure space*, Rend. Circ. Math. di Palermo 10 (1961), p. 283-313.

[13] — *A transformation theory for topological space*, Math. Japonicae 7 (1962), p. 181-194.

UNIVERSITY OF FLORIDA

Reçu par la Rédaction le 3. 6. 1968