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STUDIA MATHEMATICA, T. XXXIII. (1969)

Bases and complete systems for analytic functions

by

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If E is a topological linear space, a sequence (x_k) in E is *complete* if its linear span is dense in E. It is a *Schauder basis* if for every x in E there is a unique sequence of scalars (ξ_k) such that $x = \sum_{k=1}^{\infty} \xi_k x_k$, and the maps $x \to \xi_k$ are continuous for each k. Perhaps the most natural example of a space with a basis is the space of functions which are analytic on the unit disc $U = \{z: |z| < 1\}$ in the complex plane. We give this space the topology of uniform convergence on compact subsets of U and denote it by $P(U)$. Then (z^n) is a basis for $P(U)$.

If E is any locally convex space we will denote by $P(E; U)$ the space of analytic functions from U into E again with the topology of uniform convergence on compact subsets of U. The reader is referred to Grothendieck [2] for definitions and basic properties of vector-valued analytic functions. Then if E is complete, $P(E; U) = P(U)\hat{\otimes}E$ the projective topological tensor product of $P(U)$ and E. The purpose of this paper is to derive some theorems in bases and complete systems in tensor products of locally convex spaces and to use them to extend known results for $P(U)$ to $P(E; U)$. We will find it convenient to assume E complete and barrelled although more general hypotheses could be used for some results.

Firstly we give a criterion for a sequence in a locally convex space to be a basis. This is a generalisation of Grundblum's well-known result for Banach spaces.

PROPOSITION 1. *Let (x_n) be a complete sequence in a barrelled locally convex space E. Then the following are equivalent:*

(1) *(x_n) is a Schauder basis for E;*

(2) *(x_n) is a Schauder basis for $(E, \sigma(E, E'))$:*

(3) *for every continuous seminorm q on E there is a continuous seminorm p on E such that for all positive integers r, s and all scalars $(t_1, \ldots, t_{r+s})$,*

$$q\Big\{\sum_{i=1}^{r} t_i x_i\Big\} \leqslant p\Big\{\sum_{i=1}^{r+s} t_i x_i\Big\}. \tag{A}$$

Proof. (1) implies (2): clear.

(2) implies (3): if $x \in E$, then $x = \sum_{i=1}^{\infty} \xi_i x_i$ (convergence in $\sigma(E, E')$).

Then let

$$S_n(x) = \sum_{i=1}^{n} \xi_i x_i.$$

S_n is a continuous linear map from E into E. Also for each $x \in E$, the sequence $\{S_n(x)\}$ is weakly bounded and so bounded. Hence the the sequence (S_n) is equicontinuous by the Banach-Steinhaus theorem. Now let q be a continuous seminorm on E. Then $V = \{x: q(x) \leqslant 1\}$ is a neighbourhood in E and so there is a closed absolutely convex neighbourhood U such that

$$U \subseteq \bigcap_{n=1}^{\infty} S_n^{-1}(V).$$

Let p be the Minkowski functional of U. Then p satisfies (A).

(3) implies (1): By (A) the maps S_n where

$$S_n\Big(\sum_{i=1}^{r} \xi_i x_i\Big) = \begin{cases} \sum_{i=1}^{r} \xi_i x_i & (r < n), \\ \sum_{i=1}^{n} \xi_i x_i & (r \geqslant n) \end{cases}$$

are well defined and equicontinuous on $\operatorname{lin}(x_n)$. Hence they can be extended to an equicontinuous sequence of maps from $\hat{E}$ into $\hat{E}$ which we still denote by S_n. Clearly

$$x = S_1(x) + \sum_{n=2}^{\infty} S_n(x) - S_{n-1}(x)$$

is a basic expansion for x.

If E is a locally convex space a biorthogonal system $(x_n; f_n)$ on E is a sequence (x_n) of elements of E and (f_n) of elements of E' such that $\langle x_m, f_n\rangle = \delta_{mn}$ for all m, n. If $(x_n; f_n)$ is a biorthogonal system on E we can define the maps S_n by

$$S_n(x) = \sum_{i=1}^{n} \langle x, f_i\rangle x_i.$$

COROLLARY. *Suppose that $(x_n; f_n)$ is a biorthogonal system on a barrelled locally convex space E, and suppose that (x_n) is complete in E. Then (x_n) is a Schauder basis for E if and only if (S_n) is an equicontinuous sequence.*

If E and F are two locally convex spaces and if (x_i) (resp. (y_j)) is a sequence in E (resp. F), we define the sequence (z_k) in $E \otimes F$ as follows. We take the double sequence $x_i \otimes y_j$, enumerate it by squares i.e. $x_1 \otimes y_1$, $x_1 \otimes y_2$, $x_2 \otimes y_2$, $x_2 \otimes y_1$, $x_1 \otimes y_3$, $x_2 \otimes y_3$ etc., and denote the k-th element by z_k.

PROPOSITION 2. (1) *If (x_i) and (y_j) are complete in E and F resp., then (z_k) is complete in $E \hat{\otimes} F$.*

(2) *If $(x_i; f_i)$ (resp. $(y_j; g_j)$) is a biorthogonal system in E (resp. F), then $(z_k; h_k)$ is a biorthogonal system in $E \hat{\otimes} F$, where if $z_k = x_i \otimes y_j$, then $h_k = f_i \otimes g_j$.*

(3) *If (x_i) (resp. (y_j)) is a Schauder basis for E (resp. F) and if E and F are barrelled, then (z_k) is a Schauder basis for $E \hat{\otimes} F$.*

Proof. (1) and (2) are immediate.

(3): if $x \in E$, let

$$S_n(x) = \sum_{i=1}^{n} \langle x, f_i\rangle x_i,$$

where (f_i) is the biorthogonal sequence to (x_i) and if $y \in F$, let

$$T_n(y) = \sum_{j=1}^{n} \langle y, g_j\rangle y_j,$$

where (g_j) is the biorthogonal sequence to (y_j). Finally, if $z \in E \hat{\otimes} F$, let

$$U_n(z) = \sum_{k=1}^{n} \langle z, h_k\rangle z_k.$$

For any integer n, we have $z_n = x_i \otimes y_j$ for some i, j. There are three possibilities:

(1) $i = j$; then $U_n = S_{i-1} \otimes T_{i-1} + S_i \otimes (T_i - T_{i-1})$.

(2) $i > j$; then $U_n = S_i \otimes T_i - (S_i - S_{i-1}) \otimes T_{j-1}$.

(3) $i < j$; then $U_n = S_{j-1} \otimes T_{j-1} + S_i \otimes (T_j - T_{j-1})$.

Hence (U_n) is an equicontinuous since (S_n) and (T_n) are. Thus (z_k) is a Schauder basis for $E \hat{\otimes} F$.

Remark. This result remains true for any topological tensor product of E and F which satisfies the following properties:

(1) the natural bilinear mapping from $E \times F$ into $E \otimes F$ is separately continuous for the topology induced on $E \otimes F$.

(2) if $\mathscr{U}$ (resp. $\mathscr{V}$) is an equicontinuous sequence of linear maps from E into E (resp. F into F), then $\mathscr{U} \otimes \mathscr{V}$ is an equicontinuous sequence of linear maps from the topological tensor product into itself.

In particular, it is true for the tensor products $E \hat{\hat{\otimes}} F$ and $E \overline{\otimes} F$ introduced by Grothendieck [3].

PROPOSITION 3. *Let E be a complete barrelled locally convex space with a basis (x_i) and let the functions $\{f_j(z)\}$ form a basis for $P(U)$. Then the functions $\{f_j(z) x_i\}$, ordered by squares form a basis for $P(E; U)$.*

PROPOSITION 4. *Let the functions $\{\varphi_i(z)\}$ and $\{\psi_j(\zeta)\}$ be bases for $P(U)$. Then the sequence of functions $\{\varphi_i(z)\psi_j(\zeta)\}$ ordered by squares, form a basis for $P(U\times U)$.*

Proof. $P(U\times U) = P(U)\hat{\otimes}P(U)$.

Proposition 4 can be regarded as a generalisation of the extension of Taylor's theorem to functions of two variables. It can be clearly extended to any finite number of variables.

Markuševič [4] has introduced the following concept, which generalises the Borel transformation for analytic functions. Suppose that $F(z,\zeta)$ is an analytic function on $U\times U$. Then we can define subspaces O and Ω of $P(U)$ by

$$O = \{f(z)\in P(U)\colon f(z) = \Lambda_\zeta F(z,\zeta)\},$$

$$\Omega = \{\varphi(\zeta)\in P(U)\colon \varphi(\zeta) = L_z F(z,\zeta)\}.$$

Here Λ_ζ and L_z range over elements of the dual of $P(U)$. The subscript ζ is to emphasise that Λ_ζ acts on $F(z,\zeta)$ considered as a function of ζ.

A sequence of functions $\{f_n(z)\}$ in O is *relatively complete* in O if their closed linear span contains O. A sequence of linear functionals Λ_ζ^n in $P(U)'$ has the property of *relative uniqueness* on Ω if whenever $\varphi(\zeta)$ is an element of Ω such that $\Lambda_\zeta^n(\varphi) = 0$ for all n, $\varphi(\zeta) = 0$.

THEOREM (Markuševič). *The system of functions*

$$\{f_n(z)\} = \{\Lambda_\zeta^n(F(z,\zeta))\}$$

is relatively complete in O if and only if Λ_ζ^n has the property of relative uniqueness in Ω.

We can place these concepts in a more abstract setting and so give simpler proofs of Markuševič's result. First note that if E and F are complete locally convex spaces and if $y'\in F'$, then $(x,y)\to\langle y,y'\rangle x$ is a continuous bilinear transformation from $E\times F$ into E and so corresponds to a continuous linear map from $E\hat{\otimes}F$ into E. If $z\in E\hat{\otimes}F$ we will denote its image under this map by $y'(z)$. Now if z is a fixed element of $E\hat{\otimes}F$ we put

$$O = \{x\in E\colon x = y'(z) \text{ for some } y'\in F'\},$$

$$\Omega = \{y\in F\colon y = x'(z) \text{ for some } x'\in E'\}.$$

A sequence (x_n) in O is *relatively complete* (in O) if its closed linear span contains O. A sequence (y_n') in F' has the property of *relative uniqueness* in Ω if whenever $y\in\Omega$ is such that $\langle y, y_n'\rangle = 0$ for all n; $y = 0$.

PROPOSITION 5. *$(x_n) = \{y_n'(z)\}$ is relatively complete in O if and only if (y_n') has the property of relative uniqueness in Ω.*

Proof. First note that if $x'\in E'$, $y'\in F'$, $z\in E\hat{\otimes}F$, then

$$\langle y'(z), x'\rangle = \langle x'(z), y'\rangle.$$

For if $z = x\otimes y$, both sides are $\langle x,x'\rangle\langle y,y'\rangle$ and the result follows for general $z\in E\hat{\otimes}F$ by continuity.

Now suppose that (x_n) is relatively complete in O. Let $y = x'(z)\in\Omega$ be such that $\langle y, y_n'\rangle = 0$ for each n. Then if $y'\in F'$, $y'(z) = \lim_i P_i(z)$, where P_i is a linear combination of the y_n's. Thus

$$\langle y, y'\rangle = \langle x'(z), y'\rangle = \langle y'(z), x'\rangle = \lim_i \langle P_i(z), x'\rangle = 0$$

since $\langle y, y_n'\rangle = \langle x'(z), y_n'\rangle = \langle y_n'(z), x'\rangle = 0$ for each n and so $\langle P_i(z), x'\rangle = 0$.

Conversely, suppose that (y_n') has the property of relative uniqueness in Ω. If (x_n) is not complete in O, then there is an $x = y'(z)\in O$ such that $x\notin\overline{\lim}\{y_n'(z)\}$. Hence there is an $x'\in E'$ so that $\langle x, x'\rangle = 1$ but $\langle y_n'(z), x'\rangle = 0$ for each n. Now consider $x'(z)\in\Omega$. For each n,

$$\langle x'(z), y_n'\rangle = \langle y_n'(z), x'\rangle = 0.$$

Thus $x'(z) = 0$ and so

$$\langle x, x'\rangle = \langle y'(z), x'\rangle = \langle x'(z), y'\rangle = 0;$$

a contradiction.

EXAMPLE. Suppose that (x_n) and (y_n) are bases for the (F)-spaces E and F resp. We can assume that (x_n) and (y_n) are bounded sequences. For if for example (p_n) is an increasing sequence of seminorms defining the topology of E, we can replace x_n by $x_n/p_n(x_n)+1$. Then

$$z = \sum_{n=1}^{\infty}\frac{1}{n!}x_n\otimes y_n$$

is an element of $E\hat{\otimes}F$. We give a representation of O. If (x_n) and (y_n) are the Taylor bases for the entire functions then $z = \exp(z_1 z_2)$ and O is the set of functions which are Borel transforms of functions analytic at infinity and these are just the entire functions of exponential type. Using similar methods we can show that in our case O is the set of

$$x = \sum_{k=1}^{\infty}\xi_k x_k$$

as (ξ_k) runs through the set of sequence such that $\sum_{k=1}^{\infty}\xi_k x_k$ is convergent and $\sum_{k=1}^{\infty}k!\,a_k\xi_k$ is convergent for all sequences (a_k) such that $\sum_{k=1}^{\infty}a_k y_k\in F$.

We now give a version of Markuševič's theorem for vectorvalued functions. First we note that if E is a complete, barrelled locally convex space, then the dual of $P(E;\, U)$ is $R(E';\boldsymbol{b}U)$ the space of locally convex analytic functions on $\boldsymbol{b}U$ taking values in E' and vanishing at infinity. Here $\boldsymbol{b}U$ denotes the complements of U in the Riemann sphere. The duality is obtained as follows; if $f(z)\in P(E;\, U)$ and $g(z)\in R(E';\boldsymbol{b}U)$ then we put

$$\langle f, g\rangle = \int \langle f(z), g(z)\rangle dz$$

where the integral is taken round some circle $|z| = r$ $(r<1)$ on which $g(z)$ is defined. These results can be found in Grothendieck [2].

Proposition 6. *Suppose that $F(z,\zeta)$ is an analytic function from $U\times U$ into the space $E\hat{\otimes}F$ where E and F are complete, barrelled locally convex spaces. Let*

$$O = \Big\{f(z)\in P(E;\, U): f(z) = \int \varphi(\zeta)\{F(z,\zeta)\}d\zeta \text{ for some } \varphi(\zeta)\in R(F';\boldsymbol{b}U)\Big\},$$

$$\Omega = \Big\{g(\zeta)\in P(F;\, U): g(\zeta) = \int \psi(z)\{F(z,\zeta)\}dz \text{ for some } \psi(z)\in R(E';\boldsymbol{b}U)\Big\}.$$

Here, for fixed z and ζ, $\varphi(\zeta)\{F(z,\zeta)\}$ denotes the image of the element $F(z,\zeta)$ of $E\hat{\otimes}F$ in E when $\varphi(\zeta)$ is regarded as a map from $E\hat{\otimes}F$ into E. The integral is taken round some circle $|\zeta| = r$ $(r<1)$ on which $\varphi(\zeta)$ is defined.

Then a system of functions

$$f_n(z) = \int \varphi_n(\zeta)\{F(z,\zeta)\}d\zeta$$

is relatively complete in O if and only if the system of functions $\varphi_n(\zeta)$ has the property of relative uniqueness in Ω.

As a special case with $F = \boldsymbol{C}$ we have:

Proposition 7. *Suppose that $F(z,\zeta)$ is an analytic function from $U\times U$ into a complete, barrelled locally convex space E. Let*

$$O = \{f(z)\in P(E;\, U): f(z) = \int \varphi(\zeta)F(z,\zeta)d\zeta \text{ for some } \varphi(\zeta)\in R(\boldsymbol{b}U)\},$$

$$\Omega = \{g(\zeta)\in P(U): g(\zeta) = \int \langle F(z,\zeta), \psi(z)\rangle dz \text{ for some } \psi(z)\in R(E';\boldsymbol{b}U)\}.$$

Then a system of functions

$$f_n(z) = \int \varphi_n(\zeta)F(z,\zeta)d\zeta$$

is relatively complete in O if and only if the system of functions $\varphi_n(\zeta)$ has the property of relative uniqueness in Ω.

An element z of $E\hat{\otimes}F$ is said to be *complete* in E if O is dense in E. From the above result if E and F, have bases (x_n) and (y_n) resp., and if

$$z = \sum_{n=1}^{\infty} \xi_n x_n\otimes y_n$$

(this corresponds to a function of the form $f(z\zeta)$ for $E = F = P(U)$), then z is complete if and only if $\xi_k \neq 0$ for each k.

We now give a number of corollaries to Propositions 6 and 7 which place several of Markuševič's results on completeness and uniqueness in the context of vector-valued functions.

Corollary 1. *Suppose that $F(z,\zeta)$ is an analytic function from $U\times U$ into $E\hat{\otimes}F$ which is complete for z, i.e. such that $\bar{O} = P(E;\, U)$. Then the sequence of functions $y'_m\{F(z,\zeta_n)\}$ is complete in $P(E;\, U)$ for every sequence (y'_m) which is total on F and every sequence (ξ_n) in U which has a limit point in U.*

Corollary 2. *Let*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be a function in $P(E\hat{\otimes}F;\, U)$ such that c_n is complete on E for each n. Then if (ζ_n) is a sequence in U with a limit point in U, the system of functions $y'_m\{f(z\zeta_n)\}$ is complete in $P(E;\, U)$.

Corollary 3. *If $F(z,\zeta)$ is as in Corollary 1, then the system of functions*

$$\left[\frac{\partial^n}{\partial\zeta^n}\{y'_m(F(z,\zeta))\}\right]_{\zeta=\zeta_0}$$

is complete for any $\zeta_0\in U$.

Corollary 4. *If $f(z)$ is as in Corollary 2, then the system $\{y'_m(f^{(n)}(z))z^n\}$ is complete in $P(E;\, U)$.*

Proof. Apply Corollary 3 to $F(z,\zeta) = f(z\zeta)$.

Corollary 5. *If $F(z,\zeta)$ is as in Corollary 1, then the system of functions*

$$\left[\frac{\partial^n}{\partial\zeta^n}\{y'_m(F(z,\zeta))\}\right]_{\zeta=\zeta_n}$$

is complete in $P(E;\, U)$ if (ζ_n) is a sequence in U such that $\zeta_n\to\zeta_0\in U$ and

$$\sum_{n=1}^{\infty} |\zeta_{n+1}-\zeta_n| < \infty.$$

Proof. If $f(z)\in P(E;\, U)$ is such that $f^{(n)}(\zeta_n) = 0$ for all n, then $f = 0$. This has been proved for $E = \boldsymbol{C}$ by Gončarov [1]. For the general case, consider $\langle f(z), x'\rangle\in P(U)$ as x' runs through E'.

COROLLARY 6. *If $f(z)$ is as in Corollary 2, then $[y'_m\{f^{(n)}(z\zeta_n)\}z^n]$ is complete in $P(E;U)$ if (ζ_n) is a sequence in U such that $\zeta_n \to \zeta_0 \in U$ and*

$$\sum_{n=1}^{\infty} |\zeta_{n+1}-\zeta_n| < \infty.$$

All these results on analytic functions hold for more complicated domain spaces that U but we have avoided the technical complications which would result from more general statements since they do not add to the interest of the situation.

I would like to thank my colleague G. Bennett of St. John's College, Cambridge, for many helpful discussions on this work.

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Transforming bilinear vector integrals

by

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Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be Banach spaces with a continuous bilinear multiplication defined on $\mathfrak{X}\times\mathfrak{Y}$ into $\mathfrak{Z}$. Bartle [1] has developed a Lebesgue-type integration theory for $\mathfrak{X}$-valued integrands and $\mathfrak{Y}$-valued measures in which some of the classical integration theorems remain valid. This bilinear vector integral has many important applications, for example, numerous authors have used this integral (in restrictive forms) for establishing integral representations for linear operators defined on $L^p_{\mathfrak{X}}$ and $C_{\mathfrak{X}}(S)$ into $\mathfrak{Y}$ (e.g. see [3], [8], and [10]); consequently, it is of interest to consider the relationships between these integrals which arise from transforming one vector measure space into another. More precisely, let T be a transformation between the $\mathfrak{Y}$-vector measure spaces (S, Σ, μ) and (S', Σ', μ'). We are interested in finding conditions in order that $\mathfrak{X}$-valued measurable functions G' defined on S' satisfy the *transformation formula*

$$(*)\qquad \int_D G'\circ Tf\,d\mu = \int_{TD} G'W'(\cdot, D)\,d\mu',$$

where D belongs to a certain subfamily of Σ; W' and f are appropriate functions describing the change of measure induced by T.

In section 1 the standard hypotheses concerning the structure of the measure spaces in transformation theory are presented. By altering the usual definition used in transformation theory for a weight function, we are able to introduce in section 2 the notion of an absolutely continuous transformation relative to vector measures, along with the concept of a generalized Jacobian for T. $(*)$ is established in section 3 for absolutely continuous transformations. This development includes all the existing transformation formulas in the literature for the absolutely continuous case; in particular, it includes the theory developed by Reichelderfer [12] for positive measure spaces. The author wishes to express his gratitude to P. V. Reichelderfer for his helpful suggestions.

1. The setting. Throughout this paper, $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ will denote Banach spaces over the complex number field $\mathfrak{C}$. For terminology and concepts concerning vector measures and integration theory used in