Conversely, if $1 < \liminf p_i$ and $\limsup p_i < \infty$, then lemma 2 shows us that $l(p_i) \cong l(p_i)^{**}$ and, since this isomorphism is the natural imbedding, $l(p_i)$ is reflexive.

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Boundedness in certain topological linear spaces

by

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1. Introduction. Throughout this paper we assume that $\{p_k\}$ is a sequence of real numbers such that $0 < p_k \leqslant 1$ for all $k \geqslant 1$. We also write this sequence as $\{p(k)\}$ when this is convenient. Several authors have considered the topological linear space $l(p_k)$ of complex sequences $\{b_k\}$ with the property that

$$\varrho(\{b_k\}) = \sum_{k=1}^{\infty} |b_k|^{p(k)} < +\infty,$$

where the function ϱ defines an invariant metric on $l(p_k)$ by $d(\{b_k\}, \{a_k\}) = \varrho(\{b_k - a_k\})$ (see [4] and the references of [4]). $l(p_k)$ is a complete metric linear space with this metric by [4], Lemma 1, p. 423. Most of the interest in the spaces $l(p_k)$ has been confined to the cases where $\inf p_k > 0$. Then $l(p_k)$ is a locally bounded topological linear space in its metric topology by [4], Theorem 6, p. 430. Also in this case a set is bounded if and only if it is bounded in metric by the same theorem. The space $l(p_k)$ has quite different topological properties when $\inf p_k = 0$. In this paper we investigate the bounded sets of $l(p_k)$ in the case $\lim p_k = 0$ and the weakly bounded sets in $l(p_k)$ with a slightly stronger assumption on $\{p_k\}$. Our results contrast sharply with those concerning boundedness and weak boundedness in the case $\inf p_k > 0$. We prove in Section 2 that if $\lim p_k = 0$, then a bounded set in $l(p_k)$ is always totally bounded. In Section 3, with a slightly stronger hypothesis on $\{p_k\}$, we prove that a weakly bounded set in $l(p_k)$ is always totally weakly bounded. The last section is devoted to the consideration of questions concerning boundedness with respect to k-pseudometrics.

After this paper was sent for publication, we learnt that S. Rolewicz had considered some of the matter presented here in an earlier paper [2].

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2. Bounded sets. In this section we assume that $\lim p_k = 0$. As we note following Theorem 2.2, this assumption is necessary for the conclusions of this section to hold. Given any non-empty set B in $l(p_k)$, define

$$M_k(B) = \sup_{\{b_j\}\in B} |b_k|^{p(k)}.$$

We shall always assume that the sets B we discuss in this paper are non-empty so that $M_k(B)$ is always defined. Now we characterize the bounded sets in $l(p_k)$.

THEOREM 2.1. *Assume that* $\lim p_k = 0$. *Then a set* B *is bounded in* $l(p_k)$ *if and only if*

(1) $M_k(B) < +\infty$ *for all* $k \geqslant 1$.

(2) *Given any* $\varepsilon > 0$, *there exists an integer* N *such that* $\sum_{k=N}^{+\infty} |b_k|^{p(k)} < \varepsilon$ *for all* $\{b_k\}\in B$.

Proof. First assume that B is bounded. Then B is bounded in the metric of $l(p_k)$. (1) follows immediately from this remark. Suppose now that (2) does not hold. Then there exists a number $\varepsilon > 0$, a sequence of positive integers $\{m(j)\}$, and a sequence of elements $\{b_k^{(j)}\} \subset B$ such that $m(j) \to +\infty$ as $j \to +\infty$ and

$$\sum_{k=m(j)}^{\infty} |b_k^{(j)}|^{p(k)} \geqslant \varepsilon \quad \text{for all } j \geqslant 1.$$

Let U be the open set $\{b\in l(p_k) \mid \varrho(b) < \varepsilon/2\}$. Since B is topologically bounded, there exists a non-zero number λ such that $\lambda B \subset U$. Since $\lim p_k = 0$, we can choose K so large that $k \geqslant K$ implies $|\lambda|^{p(k)} \geqslant 1/2$. Choose j such that $m(j) \geqslant K$. Then

$$\varrho(\{\lambda b_k^{(j)}\}) \geqslant \sum_{k=m(j)}^{\infty} |\lambda|^{p(k)} |b_k^{(j)}|^{p(k)} \geqslant \varepsilon/2.$$

This contradiction proves that (2) must hold.

Conversely assume that (1) and (2) hold for the set B. Given any $\delta > 0$, let $U = \{b\in l(p_k) \mid \varrho(b) < \delta\}$. We shall choose $a > 0$ such that $aB \subset U$. First by (2) we can choose N so large that $\sum_{k>N} |b_k|^{p(k)} < \delta/2$ for all $\{b_k\}\in B$. Next by (1) we can choose a number $M \geqslant M_k(B)$ for $1 \leqslant k \leqslant N$. Let $p = \min(p_1, p_2, \ldots, p_N)$. Finally choose a such that $0 < a < 1$ and $a^p < \delta/2NM$. Then for any $\{b_k\}\in B$,

$$\varrho(\{ab_k\}) = \sum_{k=1}^{N} |ab_k|^{p(k)} + \sum_{k>N} |ab_k|^{p(k)} < \sum_{k=1}^{N} a^p M + \sum_{k>N} |b_k|^{p(k)} < \delta/2 + \delta/2 = \delta.$$

When $\inf p_k > 0$, Simons has shown ([3], Theorem 6, p. 430) that a set in $l(p_k)$ is bounded if and only if it is bounded in metric. In particular, the open balls

$$\Big\{\{b_k\}\in l(p_k) \Big| \sum_{k=1}^{\infty} |b_k|^{p(k)} = \varrho(\{b_k\}) < \varepsilon\Big\}$$

are always bounded in this case. In contrast when $\lim p_k = 0$, every bounded set in $l(p_k)$ is nowhere dense. For assume that B is a closed and bounded set and B has a non-empty interior. Then there is an $\varepsilon > 0$ and $\{a_k\}\in B$ such that the open ball

$$\big\{\{b_k\}\in l(p_k) \big| \varrho(\{a_k - b_k\}) < \varepsilon\big\} \subset B.$$

Define for each $j \geqslant 1$, $b_k^{(j)} = a_k$, $k \neq j$, $b_j^{(j)} = a_j - (\varepsilon/2)^{1/p(j)}$. Then $\{b_k^{(j)}\}\in B$, $j = 1, 2, \ldots$ Now given any positive integer N, then whenever $j \geqslant N$,

$$\sum_{k=N}^{\infty} |b_k^{(j)}|^{p(k)} \geqslant \varepsilon/2 - \sum_{k=N}^{\infty} |a_k|^{p(k)}.$$

Since $\sum_{k=N}^{\infty} |a_k|^{p(k)} \to 0$ as $N \to +\infty$, this contradicts Theorem 2.1 (2).

Now we prove that bounded sets are always totally bounded when $\lim p_k = 0$.

THEOREM 2.2. *Assume that* $\lim p_k = 0$. *Then if* $B \subset l(p_k)$ *is bounded,* B *is totally bounded.*

Proof. Assume that B is a bounded set in $l(p_k)$, and $\varepsilon > 0$ is given. By Theorem 2.1 we can choose N so large that $\sum_{k>N} |b_k|^{p(k)} < \varepsilon/4$ for all $\{b_k\}\in B$. Let π be the projection of $l(p_k)$ onto N-dimensional complex Euclidean space C^N given by $\pi(\{b_k\}) = \{b_1, b_2, \ldots, b_N\}$. Then the metric determined by

$$\varrho_0(\{b_1, \ldots, b_N\}) = \sum_{k=1}^{N} |b_k|^{p(k)}$$

is equivalent to the usual Euclidean metric on C^N. Therefore $\pi(B)$ is a totally bounded set in C^N and we can choose $a_1, a_2, \ldots, a_n \in \pi(B)$ such that whenever $b\in\pi(B)$, then $\varrho_0(b - a_j) < \varepsilon/2$ for some j, $1 \leqslant j \leqslant n$. Choose a sequence $\{b_k^{(j)}\}\in \pi^{-1}(a_j) \cap B$ for each j, $1 \leqslant j \leqslant n$.

Now assume that $\{b_k\}\in B$. Choose j such that $\varrho_0(\pi(\{b_k\}) - a_j) < \varepsilon/2$. Then

$$\varrho(\{b_k - b_k^{(j)}\}) = \sum_{k=1}^{N} |b_k - b_k^{(j)}|^{p(k)} + \sum_{k>N} |b_k - b_k^{(j)}|^{p(k)} < \varepsilon/2 + (\varepsilon/4 + \varepsilon/4) = \varepsilon.$$

This completes the proof.

COROLLARY 2.3. *Assume* $\lim p_k = 0$. *Then a set* $B \subset l(p_k)$ *is compact if and only if* B *is bounded and closed.*

Proof. $l(p_k)$ is a complete metric space by [4], Lemma 1, p. 423. Then if B is closed and bounded, B is totally bounded and complete. Hausdorff's Theorem on Total Boundedness ([1], Theorem 7.6, p. 61) implies that B is compact. The converse is immediate.

We remark that Theorem 2.2 and Corollary 2.3 do not hold when $\limsup p_k > 0$. For assume $\limsup p_k > 0$. Then there exists a sequence of distinct positive integers $\{m(k)\}$ and a number $\varepsilon > 0$ such that $p_{m(k)} \geqslant \varepsilon$ for all $k \geqslant 1$. Let $b^{(k)}$ be the sequence which is 1 in the $m(k)$-th place and 0 everywhere else. Let $B = \{b^{(k)} \mid k \geqslant 1\}$. Given any number λ, $|\lambda| < 1$, then $\varrho(\lambda b^{(k)}) = |\lambda|^{p(m(k))} \leqslant |\lambda|^{\varepsilon}$ for all $k \geqslant 1$. Therefore B is bounded in $l(p_k)$. Also B is closed in $l(p_k)$. But $\varrho(b^{(n)} - b^{(m)}) = 2$ whenever $n \neq m$, so that B is not compact. Thus B is also not totally bounded.

3. Weakly bounded sets. The space of continuous linear functionals on $l(p_k)$ can be identified in the natural way with the set of all complex sequences $\{\Phi_k\}$ such that $\sup_k |\Phi_k|^{p(k)} < +\infty$; see [4], Section 3. This space is denoted by $m(p_k)$. Now we characterize weak boundedness in $l(p_k)$. To prove the first part of the characterization only the condition $\lim p_k = 0$ is required. The converse is proved assuming a stronger condition which is discussed following the proof of the next theorem.

THEOREM 3.1. *Assume that* $\lim p_k = 0$. *If* B *is a weakly bounded subset of* $l(p_k)$, *then* $M_k(B)$ *is finite for all* $k \geqslant 1$ *and* $\lim M_k(B) = 0$.

Proof. Assume that B is a weakly bounded subset of $l(p_k)$. Then it is obvious that $M_k(B)$ is finite for all $k \geqslant 1$. Suppose there is a number ε with $0 < \varepsilon < 1$ such that $M_k(B) > \varepsilon$ for an infinite number of k. In this case we shall prove by induction that there exists an infinite sequence $\{b_k^{(j)}\} \subset B$ and a strictly increasing sequence of positive integers $\{q(j)\}$ such that for every $m \geqslant 1$:

$$|b_{q(m)}^{(m)}|^{p(q(m))} > \varepsilon, \tag{1}$$

$$\sum_{j=1}^{m-1} M_{q(j)}(B)^{1/p(q(j))}(1/\varepsilon)^{2/p(q(j))} < \varepsilon^{-1/p(q(m))}/4, \tag{2}$$

and

$$|b_{q(k)}^{(j)}|(1/\varepsilon)^{2/p(q(k))} < (1/2)^k \varepsilon^{-1/p(q(j))} \tag{3}$$

for all j, k such that $j < k \leqslant m$.

First since $M_k(B) > \varepsilon$ for some k, we can choose $\{b_k^{(1)}\} \in B$ and $q(1)$ a positive integer such that $|b_{q(1)}^{(1)}|^{p(q(1))} > \varepsilon$. Now assume that $\{b_k^{(1)}\}, \ldots, \{b_k^{(m)}\}$ and $q(1), \ldots, q(m)$ have been chosen satisfying (1)-(3). For any $\{b_k\} \in l(p_k)$, $\lim |b_k|(1/\varepsilon)^{2/p(k)} = 0$. Therefore there exists $M > 0$ such that whenever $k \geqslant M$, then $|b_k^{(j)}|(1/\varepsilon)^{2/p(k)} < (1/2)^{m+1}\varepsilon^{-1/p(q(j))}$ for $1 \leqslant j \leqslant m$. Since $p_k \to 0$ and $\varepsilon < 1$, there exists $N > 0$ such that whenever $n \geqslant N$, then

$$\sum_{j=1}^{m} M_{q(j)}(B)^{1/p(q(j))}(1/\varepsilon)^{2/p(q(j))} < \varepsilon^{-1/p(n)}/4.$$

Let $K = \max(N, M, q(m)+1)$. By hypothesis we can choose $\{b_k\} \in B$ and $n \geqslant K$ such that $|b_n|^{p(n)} > \varepsilon$. Let $\{b_k^{(m+1)}\} = \{b_k\}$ and $q(m+1) = n$. By this choice obviously $|b_{q(m+1)}^{(m+1)}|^{p(q(m+1))} > \varepsilon$. Also since $q(m+1) \geqslant K \geqslant N$,

$$\sum_{j=1}^{m} M_{q(j)}(B)^{1/p(q(j))}(1/\varepsilon)^{2/p(q(j))} < \varepsilon^{-1/p(q(m+1))}/4.$$

This verifies (1) and (2) for $m+1$. Now (3) holds by the induction hypothesis whenever $j < k \leqslant m$. Therefore it remains to prove (3) when $j < k = m+1$. In this case $q(m+1) \geqslant K \geqslant M$, so that

$$|b_{q(m+1)}^{(j)}|(1/\varepsilon)^{2/p(q(m+1))} < (1/2)^{m+1}\varepsilon^{-1/p(q(j))}$$

whenever $1 \leqslant j \leqslant m$. This completes the induction.

We define an element $\{\Phi_k\} \in m(p_k)$ by $\Phi_{q(j)} = (1/\varepsilon)^{2/p(q(j))}$ for $j \geqslant 1$ and $\Phi_k = 0$ for all other values of k. Then for any $m > 1$,

$$\begin{aligned}
\Big|\sum_{n=1}^{\infty} b_n^{(m)}\Phi_n\Big| &= \Big|\sum_{j=1}^{\infty} b_{q(j)}^{(m)}\Phi_{q(j)}\Big| \\
&\geqslant |b_{q(m)}^{(m)}\Phi_{q(m)}| - \sum_{j\neq m} |b_{q(j)}^{(m)}\Phi_{q(j)}| \\
&\geqslant \varepsilon^{-1/p(q(m))} - \sum_{j\neq m} |b_{q(j)}^{(m)}\Phi_{q(j)}|, \quad \text{by (1)} \\
&\geqslant \varepsilon^{-1/p(q(m))} - \frac{1}{4}\varepsilon^{-1/p(q(m))} - \sum_{j=m+1} |b_{q(j)}^{(m)}\Phi_{q(j)}|, \quad \text{by (2)} \\
&\geqslant \frac{3}{4}\varepsilon^{-1/p(q(m))} - \sum_{j=m+1} \left(\frac{1}{2}\right)^j \varepsilon^{-1/p(q(m))}, \quad \text{by (3)} \\
&\geqslant \frac{1}{4}\varepsilon^{-1/p(q(m))}.
\end{aligned}$$

Therefore

$$\Big|\sum_{n=1}^{\infty} b_n^{(m)}\Phi_n\Big| \to +\infty \quad \text{as } m \to +\infty.$$

This contradicts the assumption that B is weakly bounded. Therefore the theorem follows.

Since $0 < p_k \leqslant 1$ for all $k \geqslant 1$, then $q_k = 1/(1+p_k)$ has the property $0 < q_k < 1$ for all $k \geqslant 1$. Therefore it makes sense to consider $l(q_k)$. In order to prove a converse of Theorem 3.1, a stronger condition than $\lim p_k = 0$ is necessary. This condition is the requirement that $l(q_k) = l_1$, where l_1 is the space of absolutely convergent sequences. By [4], Theorem 3, p. 426, a necessary and sufficient condition that $l(q_k) = l_1$ is that there exists an integer $N > 1$, such that $\sum_{k=1}^{\infty} N^{\pi(k)} < +\infty$ where $1/\pi_k + 1/q_k = 1$. $1/\pi_k + 1/q_k = 1$ if and only if $1/\pi_k + (1+p_k) = 1$ or $\pi_k = -1/p_k$. Therefore $l(1/(1+p_k)) = l_1$ if and only if there exists an integer $N > 1$ such that $\sum_{k=1}^{\infty} N^{-1/p(k)} < +\infty$. In particular, note that if $l(q_k) = l_1$, then $\lim p_k = 0$. Sufficient conditions on $\{q_k\}$ such that $l(q_k) = l_1$ are given in [4], Corollaries 1 and 2, p. 427. Now we prove a converse of Theorem 3.1.

THEOREM 3.2. *Assume that* $l(1/(1+p_k)) = l_1$. *If* B *is a subset of* $l(p_k)$ *such that* $M_k(B)$ *is finite for all* $k \geqslant 1$ *and* $\lim M_k(B) = 0$, *then* B *is weakly bounded.*

Proof. Let $\{\Phi_k\} \in m(p_k)$ and $\varepsilon > 0$ be given. We shall prove that there exists a non-zero number λ such that

$$\Big|\sum_{k=1}^{\infty} \lambda b_k \Phi_k\Big| < \varepsilon \quad \text{whenever } \{b_k\} \in B.$$

Since $\{\Phi_k\} \in m(p_k)$, there exists $M > 0$ such that $|\Phi_k|^{p(k)} \leqslant M$ for all $k \geqslant 1$. Also by the remarks preceding this theorem, there exists a positive integer N such that $\sum_{k=1}^{\infty} N^{-1/p(k)} < +\infty$. Choose δ such that $0 < \delta < 1$ and $\delta M < 1/N$. Since $M_k(B) \to 0$, we can choose K so large that $k \geqslant K$ implies that $M_k(B) < \delta$. Then whenever $\{b_k\} \in B$ and λ is any number,

$$\Big|\sum_{k=1}^{\infty} \lambda b_k \Phi_k\Big| \leqslant |\lambda| \sum_{k=1}^{K} |b_k \Phi_k| + |\lambda| \sum_{k>K} (\delta M)^{1/p(k)}$$
$$\leqslant |\lambda| \Big(\sum_{k=1}^{K} (M_k(B) M)^{1/p(k)} + \sum_{k=1}^{\infty} N^{-1/p(k)} \Big)$$

clearly we can choose λ small enough so the right hand side of this inequality is less than ε whenever $\{b_k\} \in B$.

We remark that the condition $l(1/(1+p_k)) = l_1$ is necessary for Theorem 3.2 to hold. For suppose that $l(1/(1+p_k)) \neq l_1$. Then by the remarks preceding Theorem 3.2, the series $\sum_{k=1}^{\infty} (1/n)^{1/p(k)}$ diverges for every positive integer n. Then it is obvious that we can choose a strictly increasing sequence of positive integers $\{m(n)\}$ such that $\sum_{k>m(n-1)}^{m(n)} (1/n)^{1/p(k)}$ $\geqslant n$ for all $n > 1$. Given any $n > 1$, define $b_k^{(n)} = 0$, when $k \leqslant m(n-1)$, $b_k^{(n)} = (1/n)^{1/p(k)}$, when $m(n-1) < k \leqslant m(n)$, and $b_k^{(n)} = 0$, when $k > m(n)$. For every $n > 1$, $\{b_k^{(n)}\} \in l(p_k)$. Let $B = \{\{b_k^{(n)}\} \mid n > 1\}$. Then clearly $M_k(B) = 1/n$ whenever $m(n-1) < k \leqslant m(n)$. Therefore $M_k(B) \to 0$ as $k \to \infty$. Now let $\{\Phi_k\} \in m(p_k)$ be defined by $\Phi_k = 1$ for all $k \geqslant 1$. Then

$$\sum_{k=1}^{\infty} b_k^{(n)} \Phi_k = \sum_{k>m(n-1)}^{m(n)} (1/n)^{1/p(k)} \geqslant n \quad \text{for all } n > 1.$$

Thus the set B is not weakly bounded.

Also we note here that these results concerning weak boundedness are in sharp contrast to the results which hold in the case where $\inf p_k > 0$. For if $\inf p_k > 0$, then the open balls $\{\{b_k\} \in l(p_k) \mid \varrho(\{b_k\}) < \varepsilon\}$ are always weakly bounded. This follows easily from the fact that the topological dual of $l(p_k)$ in this case is the space of all bounded sequences; see [4], Theorem 9, p. 434. But an argument similar to the one following Theorem 2.1, proves that when $\lim p_k = 0$, then every weakly bounded set is nowhere dense. In particular, the open balls given above are not weakly bounded.

Now we prove that every weakly bounded set is totally weakly bounded when $l(1/(1+p_k)) = l_1$.

THEOREM 3.3. *Assume* $l(1/(1+p_k)) = l_1$. *Then if a set* $B \subset l(p_k)$ *is weakly bounded,* B *is totally weakly bounded.*

Proof. Assume that B is a weakly bounded set in $l(p_k)$. Suppose $\varepsilon > 0$, and $\{\Phi_k^{(j)}\} \in m(p_k)$, $1 \leqslant j \leqslant n$, are given. First there exists $M > 0$ such that $|\Phi_k^{(j)}|^{p(k)} \leqslant M$ whenever $k \geqslant 1$ and $1 \leqslant j \leqslant n$. Next since $l(1/(1+p_k)) = l_1$, we can choose a positive integer N such that $\sum_{k=1}^{\infty} N^{-1/p(k)} < +\infty$. Since B is weakly bounded, there exists $K > 0$ such that $k \geqslant K$ implies $M_k(B) < 1/NM$. Then choose and fix a positive integer L such that $L \geqslant K$ and $\sum_{k>L} N^{-1/p(k)} < \varepsilon/4$. By an argument similar to part of the proof of Theorem 2.2, we can choose $\{b_k^{(1)}\}, \ldots, \{b_k^{(m)}\} \in B$ with the property that given any $\{b_k\} \in B$, there exists i, $1 \leqslant i \leqslant m$, such that

$$\Big|\sum_{k=1}^{L} (b_k - b_k^{(i)}) \Phi_k^{(j)}\Big| < \varepsilon/2 \quad \text{for } 1 \leqslant j \leqslant n.$$

Then given any $\{b_k\} \in B$, we choose $\{b_k^{(i)}\}$ with the property above, and then for $1 \leqslant j \leqslant n$,

$$\Big|\sum_{k=1}^{\infty} (b_k - b_k^{(i)}) \Phi_k^{(j)}\Big| < \Big|\sum_{k=1}^{L} (b_k - b_k^{(i)}) \Phi_k^{(j)}\Big| + \sum_{k>L} 2 M_k(B)^{1/p(k)} M^{1/p(k)}$$
$$< \varepsilon/2 + 2 \sum_{k>L} N^{-1/p(k)}$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves that B is totally weakly bounded.

COROLLARY 3.4. *Assume $l(1/(1+p_k)) = l_1$. Then a set $B \subset l(p_k)$ is weakly compact if and only if B is weakly bounded and weakly complete.*

This Corollary follows directly from Theorem 3.3 and [1], Theorem 7.6, p. 61.

4. Sets bounded with respect to k-pseudometrics. We take the following definitions from Simons' paper [3]: Let X be a real or complex linear space. Then a function $d: X \to R^+$ (the non-negative real numbers) is an *invariant pseudometric* if d is not identically zero on X and $d(0) = 0$, $d(-x) = d(x)$ for all $x \in X$, and $d(x+y) \leqslant d(x)+d(y)$ for all $x, y \in X$. A function d as above is called a *k-pseudometric*, where $0 < k \leqslant 1$, if d is an invariant pseudometric and $d(\lambda x) = |\lambda|^k d(x)$ for all $x \in X$ and all scalars λ. Finally, a topological linear space is an *upper bound space* if there is a family $\{d_a\}$ of continuous k-pseudometrics on X which define (in the usual manner) the topology of X. When $\inf p_k > 0$, the space $l(p_k)$ is an upper bound space by [4], Theorem 6, p. 430, and [3], Theorem 1, p. 170. In any upper bound space a set is bounded if and only if it is bounded with respect to every k-pseudometric; see [3], Theorem 6, p. 179. When $\inf p_k = 0$, it can be shown that $l(p_k)$ is not an upper bound space. When $l(1/(1+p_k)) = l_1$, not only is $l(p_k)$ not an upper bound space, but we have the following result (compare with the remarks above).

THEOREM 4.1. *Assume $l(1/(1+p_k)) = l_1$. A set $B \subset l(p_k)$ is weakly bounded if and only if B is bounded with respect to every continuous k-pseudometric $(0 < k \leqslant 1)$ on $l(p_k)$.*

Proof. Given $\{\Phi_k\} \in m(p_k)$, then

$$d(\{b_k\}) = \Big|\sum_{k=1}^{\infty} b_k \Phi_k\Big|$$

is a continuous 1-pseudometric on $l(p_k)$. Thus if B is bounded with respect to every continuous k-pseudometric, then B is weakly bounded. Now we prove the converse. Let d be a continuous k-pseudometric on $l(p_k)$. Let e_n be the sequence with 1 at the n-th place and 0 everywhere else. Then

(*) *There exists $M > 0$ such that $d(e_n) < M^{1/p(n)}$ for all $n \geqslant 1$.*

For suppose (*) does not hold. Then there exists a strictly increasing sequence of positive integers $\{m(n)\}$ and a sequence of positive numbers M_n where $M_n \to +\infty$, such that $d(e_{m(n)}) \geqslant M_n^{k/p(m(n))}$ for all $n \geqslant 1$. Then $d((1/M_n)^{1/p(m(n))} e_{m(n)}) \geqslant 1$ for all $n \geqslant 1$. But $\varrho((1/M_n)^{1/p(m(n))} e_{m(n)}) = 1/M_n \to 0$. This contradicts the continuity of d, and hence (*) must hold.

Now assume that B is weakly bounded in $l(p_n)$. First choose an integer $N > 1$ such that $\sum_{n=1}^{\infty} N^{-1/p(n)} < +\infty$. Next, since B is a weakly bounded set in $l(p_n)$, by Theorem 3.1 we can choose an integer J such that $n \geqslant J$ implies $M_n(B)^k M < 1/N$. Then whenever $\{b_n\} \in B$,

$$\begin{aligned} d(\{b_n\}) &\leqslant \sum_{n=1}^{\infty} |b_n|^k d(e_n) \\ &\leqslant \sum_{n=1}^{\infty} M_n(B)^{k/p(n)} M^{1/p(n)} \\ &\leqslant \sum_{n=1}^{J} M_n(B)^{k/p(n)} M^{1/p(n)} + \sum_{n>J} N^{-1/p(n)} \\ &< +\infty. \end{aligned}$$

This proves that B is bounded with respect to d.

An upper bound space has the property that a set is bounded in the space if and only if it is bounded with respect to every continuous k-pseudometric by [3], Theorem 6, p. 179. To our knowledge it is an open question whether this property characterizes upper bound spaces; see [3], p. 180. Next we consider this general question in the special context of the spaces $l(p_k)$. If $l(p_k)$ is not an upper bound space, we construct an example of a set B in $l(p_k)$ which is bounded with respect to every continuous k-pseudometric on $l(p_k)$, but which is not bounded.

Assume $l(p_k)$ is not an upper bound space. Then $\inf p_k = 0$. We can choose a strictly increasing sequence of positive integers $\{m(k)\}$ such that $\sum_{k=1}^{\infty} (1/2)^{1/p(m(k))} < +\infty$. Set $q_k = p_{m(k)}$ for all $k \geqslant 1$. Then by the remarks preceding Theorem 3.2, $l(1/(1+q_k)) = l_1$. $l(q_k)$ is imbedded isometrically in a natural way in $l(p_k)$. It is enough for us to give an example of a set B in $l(q_k)$ which is bounded with respect to every continuous k-pseudometric on $l(q_k)$, but which is not bounded in $l(q_k)$. To this end we define $b_k^{(n)} = 0$ for $k \leqslant n$, $b_k^{(n)} = (1/n)^{1/q(n)}$ for $n < k \leqslant 2n$, and $b_k^{(n)} = 0$ for $k > 2n$. The sequence $\{b_k^{(n)}\} \in l(q_k)$ for all $n \geqslant 2$. Let now $B = \{\{b_k^{(n)}\} \mid n \geqslant 2\}$. Let N be a positive integer. Take $n \geqslant N$. Then

$$\sum_{k=N}^{\infty} |b_k^{(n)}|^{q(k)} = \sum_{k=n+1}^{2n} 1/n = 1.$$

Thus by Theorem 2.1, B is not bounded. Furthermore, $M_k(B) = \sup_n |b_k^{(n)}|^{q(k)} = 1/k$ for $k \geqslant 2$. Therefore, by Theorem 3.2, B is weakly bounded. Finally, by Theorem 4.1, B is bounded with respect to every continuous k-pseudometric on $l(q_k)$.

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Bases and complete systems for analytic functions

by

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If E is a topological linear space, a sequence (x_k) in E is *complete* if its linear span is dense in E. It is a *Schauder basis* if for every x in E there is a unique sequence of scalars (ξ_k) such that $x = \sum_{k=1}^{\infty} \xi_k x_k$, and the maps $x \to \xi_k$ are continuous for each k. Perhaps the most natural example of a space with a basis is the space of functions which are analytic on the unit disc $U = \{z: |z| < 1\}$ in the complex plane. We give this space the topology of uniform convergence on compact subsets of U and denote it by $P(U)$. Then (z^n) is a basis for $P(U)$.

If E is any locally convex space we will denote by $P(E; U)$ the space of analytic functions from U into E again with the topology of uniform convergence on compact subsets of U. The reader is referred to Grothendieck [2] for definitions and basic properties of vector-valued analytic functions. Then if E is complete, $P(E; U) = P(U)\hat{\otimes}E$ the projective topological tensor product of $P(U)$ and E. The purpose of this paper is to derive some theorems in bases and complete systems in tensor products of locally convex spaces and to use them to extend known results for $P(U)$ to $P(E; U)$. We will find it convenient to assume E complete and barrelled although more general hypotheses could be used for some results.

Firstly we give a criterion for a sequence in a locally convex space to be a basis. This is a generalisation of Grundblum's well-known result for Banach spaces.

PROPOSITION 1. *Let (x_n) be a complete sequence in a barrelled locally convex space E. Then the following are equivalent:*

(1) *(x_n) is a Schauder basis for E;*

(2) *(x_n) is a Schauder basis for $(E, \sigma(E, E'))$:*

(3) *for every continuous seminorm q on E there is a continuous seminorm p on E such that for all positive integers r, s and all scalars $(t_1, \ldots, t_{r+s})$,*

$$q\Big\{\sum_{i=1}^{r} t_i x_i\Big\} \leqslant p\Big\{\sum_{i=1}^{r+s} t_i x_i\Big\}. \tag{A}$$