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Generalized operational functions

by

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Introduction. Mikusiński [3] has defined a new type of functions, namely operational functions as mappings from the field R of real numbers to the field Q of operators of Mikusiński. His definition of continuity and differentiability in this class of operational functions has a defect, for there are continuous operational functions that are not differentiable. So it is desirable to construct a new class of functions which contains all continuous operational functions and in which the derivative is defined in such a way that every element of this new class has derivatives of all orders. Following Mikusiński and Sikorski [4], we construct and study one such class, called *generalized operational functions*.

In section 1 we deal with the definition of a generalized operational function and discuss a few elementary properties of those functions. In section 2 we define the value of generalized operational functions and prove an existence theorem.

1. We begin with some definitions.

Definition 1.1 [3]. An *operational function* is a function f which assigns an operator $f(\theta)$ to each non-negative real number θ.

Definition 1.2 [3]. An operational function f is said to be a *parametric operational function* if each value $f(\theta)$ is itself an operator of a special kind, namely a function of the real variable, say t.

Definition 1.3 [3]. An operational function f is called *continuous in* $0 \leqslant \theta < \infty$, if it can be represented in $[0, \infty)$ as a ratio $f_1(\theta) \circledast a$ (1), of a parametric operational function $f_1(\theta) = \{f_1(\theta, t)\}$ and an operator a equal to a continuous function $\{a(t)\}$, $0 \leqslant t < \infty$, where the function $a(t)$ is not identically equal to zero, such that the function $f_1(\theta, t)$ is continuous in the domain $D(0 \leqslant \theta < \infty, 0 \leqslant t < \infty)$.

Definition 1.4. Two continuous operational functions f and g are said to be *related* — in symbols $f \sim g$ — where $f(\theta) = \{f(\theta, t)\} \circledast \{a(t)\}$ (2) and $g(\theta) = \{g(\theta, t)\} \circledast \{b(t)\}$ if and only if $f(\theta, t) * b(t) = g(\theta, t) * a(t)$.

(1) $\circledast$ stands for convolution quotient.

(2) For the sake of typographical convenience we omit the braces hereafter.

The relation $\sim$ can be seen to be an equivalence relation which divides the class of all continuous operational functions into mutually disjoint classes.

Hereafter, whenever we say *a continuous operational function* f, we mean an equivalence class of elements of the form $f(\theta) \overset{*}{*} a$ representing the function.

Remark 1.5. Obviously, all continuous operational functions form a real vector space with the usual addition and multiplication by real scalars.

Definition 1.6. A sequence f_n of continuous operational functions is said *to converge to a continuous operational function* f if there exist a sequence of parametric operational functions $f_n(\theta, t)$, a parametric operational function $f(\theta, t)$ and a continuous function $a(t)$ such that $f_n(\theta) = f_n(\theta, t) \overset{*}{*} a(t)$, $f(\theta) = f(\theta, t) \overset{*}{*} a(t)$ and $f_n(\theta, t)$ converges almost uniformly to $f(\theta, t)$, i.e. converges uniformly over every bounded rectangle in the domain D.

As in the case of continuous functions, we say that a continuous operational function f has the *continuous operational function g as a derivative* if $(\tau_h f - f)/h$ tends to g as h tends to zero, where $\tau_h f(\theta) = f(\theta + h)$. It is easy to see that there exist continuous operational functions which are not differentiable. To meet this situation, we construct generalized operational functions.

Definition 1.7. A sequence f_n of continuous operational functions is said to be *fundamental* if and only there exist a non-negative integer k and a sequence F_n of continuous operational functions such that

(i) $f_n(\theta) = F_n^{(k)}(\theta)$;

(ii) $F_n(\theta)$ converges in the sense of definition 1.6; or, equivalently,

(i) $f_n(\theta) = \dfrac{\partial^k}{\partial \theta^k} F_n(\theta, t) \overset{*}{*} a(t)$;

(ii) $F_n(\theta, t)$ converges almost uniformly.

The following are immediate consequences of the above definition.

1. For $k = 0$ all convergent sequences of continuous operational functions are fundamental.

2. For $k = 0$ and $a = 1$, all almost uniformly convergent sequences of continuous functions in the variables θ and t are fundamental.

3. If f_n has a continuous m-th derivative and if f_n is fundamental, then the m-th derivative of f_n is also fundamental.

4. The order k can be replaced by any greater order m.

Indeed,

$$G_n(\theta) = \underbrace{\int_0^\theta \ldots\ldots \int_0^\theta}_{(m-k)\text{ times}} F_n(\theta)\, d\theta$$

also satisfies (i) and (ii) of definition 1.7 if F_n does.

Lemma 1.8. *For any two fundamental sequences f_n and g_n, the following statements are equivalent:*

(i) $f_1, g_1, f_2, g_2, \ldots$ *is fundamental;*

(ii) $f_n(\theta) = \dfrac{\partial^k}{\partial \theta^k} F_n(\theta, t) \overset{*}{*} a(t)$, $\quad g_n(\theta) = \dfrac{\partial^k}{\partial \theta^k} G_n(\theta, t) \overset{*}{*} a(t)$ *and* $F_n(\theta, t) \rightrightarrows\leftleftarrows G_n(\theta, t)$ (*where* $\rightrightarrows\leftleftarrows$ *means converging almost uniformly to the same limit*);

(iii) $f_n(\theta) = \dfrac{\partial^k}{\partial \theta^k} F_n(\theta, t) \overset{*}{*} a(t)$, $\quad g_n(\theta) = \dfrac{\partial^k}{\partial \theta^k} G_n(\theta, t) \overset{*}{*} b(t)$ *and* $F_n(\theta, t) * b(t) \rightrightarrows\leftleftarrows G_n(\theta, t) * a(t)$.

Proof. (i) $\Rightarrow$ (ii). Indeed, by hypothesis $f_1, g_1, f_2, g_2, \ldots$ is fundamental. This asserts that we can find a sequence $F_1(\theta, t), G_1(\theta, t), F_2(\theta, t), G_2(\theta, t), \ldots$ of continuous operational functions, a continuous function $a(t)$ and a non-negative integer k such that

$$f_1(\theta) = \frac{\partial^k}{\partial \theta^k} F_1(\theta, t) \overset{*}{*} a(t), \quad g_1(\theta) = \frac{\partial^k}{\partial \theta^k} G_1(\theta, t) \overset{*}{*} a(t), \ldots$$

and $F_1(\theta, t), G_1(\theta, t), F_2(\theta, t), G_2(\theta, t), \ldots$ converges almost uniformly, which is precisely (ii).

(ii) $\Rightarrow$ (iii) by putting $a(t) = b(t)$ in the representation of $g_n(\theta)$.

(iii) $\Rightarrow$ (i). Assuming (iii) we have

$$f_n(\theta) = \frac{\partial^k}{\partial \theta^k} F_n(\theta, t) \overset{*}{*} a(t)$$

$$= \left(b(t) * \frac{\partial^k}{\partial \theta^k} F_n(\theta, t)\right) \overset{*}{*} \left(a(t) * b(t)\right)$$

$$= \left(\frac{\partial^k}{\partial \theta^k} \left(F_n(\theta, t) * b(t)\right)\right) \overset{*}{*} \left(a(t) * b(t)\right)$$

$$= \left(\frac{\partial^k}{\partial \theta^k} \overline{F}_n(\theta, t)\right) \overset{*}{*} c(t),$$

where $\bar{F}_n = b * F_n$, $c = a * b$, and

$$g_n(\theta) = \frac{\partial^k}{\partial\theta^k} G_n(\theta, t) \circledast b(t)$$

$$= \left[a(t) * \frac{\partial^k}{\partial\theta^k} G_n(\theta, t)\right] \circledast [a(t) * b(t)]$$

$$= \frac{\partial^k}{\partial\theta^k} \big(G_n(\theta, t) * a(t)\big) \circledast a(t) * b(t)$$

$$= \left(\frac{\partial^k}{\partial\theta^k} \bar{G}_n(\theta, t)\right) \circledast c(t),$$

where $\bar{G}_n = G_n * a$ and $\bar{G}_n \rightrightarrows\!\leftleftarrows \bar{F}_n$. This is precisely the definition of $f_1, g_1, f_2, g_2, \ldots$ being fundamental.

Definition 1.9. We say that two fundamental sequences f_n and g_n are *related* — in symbols $f_n \sim g_n$ — if one of the three conditions of the above lemma holds.

It is easy to see that the relation $\sim$ is an equivalence relation dividing the class of all fundamental sequences of continuous operational functions into mutually disjoint classes.

Definition 1.10. A *generalized operational function* is a class of equivalent fundamental sequences of continuous operational functions and in symbol we write $f = [f_n]$.

Remark 1.11. Every continuous operational function f can be viewed as a generalized operational function since $f = [f_n]$ where $f_n \equiv f$.

LEMMA 1.12. *If two fundamental sequences f_n and g_n have continuous m-th derivatives and if $f_n \sim g_n$, then $f_n^{(m)} \sim g_n^{(m)}$.*

Proof. If f_n and g_n satisfy one of the equivalent conditions of Lemma 1.8, say condition (ii), then $f_n^{(m)}$ and $g_n^{(m)}$ satisfy that condition by replacing k by $k+m$.

Algebraic operations on generalized operational functions.

Definition 1.13. *Multiplication by an operator a:*

(i) *If f_n is fundamental, so is af_n.*

By this property we can extend this operation onto arbitrary generalized operational functions $f = [f_n]$ by assuming $af = [af_n]$.

(ii) *This representation is independent of the choice of f_n.*

Indeed, if $f_n \sim g_n$, then $af_n \sim ag_n$, since, taking into account that $f_1, g_1, f_2, g_2, \ldots$ is fundamental, we find that $af_1, ag_1, af_2, ag_2, \ldots$ is also fundamental by (i).

Definition 1.14. *Addition of two generalized functions:*

(i) *If f_n and g_n are fundamental, so is $f_n + g_n$.*

Indeed, $f_n(\theta) = \big(\partial^k F_n(\theta, t)/\partial\theta^k\big) \circledast a(t)$ and $F_n(\theta, t)$ converges almost uniformly; $g_n(\theta) = \big(\partial^m G_n(\theta, t)/\partial\theta^m\big) \circledast b(t)$ and $G_n(\theta, t)$ converges almost uniformly.

We can assume $k = m$ since each of the orders k and m can be arbitrarily enlarged. Then

$$f_n(\theta) + g_n(\theta) = \left(\frac{\partial^k}{\partial\theta^k} \{F_n(\theta, t) * b(t) + G_n(\theta, t) * a(t)\}\right) \circledast \big(a(t) * b(t)\big)$$

and $F_n * b + G_n * a$ converges almost uniformly.

This operation can be extended onto the class of generalized operational functions in virtue of this property by assuming $f + g = [f_n + g_n]$, where $f = [f_n]$, $g = [g_n]$.

(ii) *This representation is independent of the choice of f_n and g_n by* (i).

Definition 1.15. *Multiplication of two generalized operational functions:*

(i) *If f_n and g_n are two fundamental sequences, so is $f_n g_n$.*

Indeed,

$$f_n(\theta) g_n(\theta) = \left(\frac{\partial^k}{\partial\theta^k} F_n(\theta, t) * \frac{\partial^m}{\partial\theta^m} G_n(\theta, t)\right) \circledast \big(a(t) * b(t)\big)$$

$$= \left(\frac{\partial^{k+m}}{\partial\theta^{k+m}} \big(F_n(\theta, t) * G_n(\theta, t)\big)\right) \circledast \big(a(t) * b(t)\big)$$

and $F_n * G_n$ converges almost uniformly.

This operator can be extended onto arbitrary generalized operational functions by assuming $fg = [f_n g_n]$, where $f = [f_n]$, $g = [g_n]$.

That this representation is independent of f_n and g_n can be proved by (i).

Definition 1.16. By the *translation* of a generalized operational function $f = [f_n]$ through a distance h we mean the generalized operational function $\tau_h f = [\tau_h f_n]$.

LEMMA 1.17. *A sequence of polynomials $\sum_{j=0}^{k-1} a_{nj}(t)\theta^j$ of degree less than a positive integer k converges almost uniformly to a polynomial $\sum_{j=0}^{k-1} a_j(t)\theta^j$ of degree less than k if and only if the sequence of continuous functions $\{a_{nj}(t)\}$ converges almost uniformly.*

The proof follows from the well-known Weierstrass approximation theorem, where the constants are replaced by functions and the convergence of the coefficients is taken to be the almost uniform convergence.

Definition 1.17. A continuous operational function p is called a *polynomial operational function of degree less than k*, where

$$p(\theta) = p(\theta, t) \circledast a(t), \quad \text{if } p(\theta, t) = \sum_{j=0}^{k-1} a_j(t)\theta^j,$$

where the coefficients $a_j(t)$ are continuous functions of the variable t, is a polynomial of degree less than k.

THEOREM 1.18. *A sequence p_n of polynomial operational functions is fundamental if and only if it converges in the sense of definition* 1.6.

Proof. By definition 1.7 (1), the sufficiency follows. To prove the necessity, since p_n is fundamental, we have

$$p_n(\theta) = \frac{\partial^k}{\partial\theta^k} P_n(\theta, t) \circledast a(t)$$

and $p_n(\theta, t)$ converges almost uniformly.

By Lemma 1.17, the coefficients of $P_n(\theta, t)$ converge almost uniformly. Therefore, the coefficients of $\partial^k P_n(\theta, t)/\partial\theta^k$ converge almost uniformly, i.e. p_n converges in the sense of definition 1.6.

THEOREM 1.19. *Every fundamental sequence of continuous operational functions has an equivalent fundamental sequence of smooth (differentiable any number of times) operational functions.*

Proof. Since f_n is fundamental, we have

$$f_n(\theta) = \frac{\partial^k}{\partial\theta^k} F_n(\theta, t) \circledast a(t)$$

and $F_n(\theta, t)$ converges almost uniformly to $F(\theta, t)$. Let $p_n(\theta, t)$ be a sequence of polynomials in θ, t, converging almost uniformly to $F(\theta, t)$. Then

$$p_n(\theta) = \frac{\partial^k}{\partial\theta^k} p_n(\theta, t) \circledast a(t)$$

is the required sequence of smooth operational functions.

By Theorem 1.19, we find that every generalized operational function f can be represented by $[p_n]$, where p_n are equivalent fundamental sequences of smooth operational functions.

As in the case of functions, the derivative f' of a generalized operational function f could be defined as

$$\operatorname*{Lt}_{h\to 0} \frac{\tau_h f - f}{h},$$

and we notice that

$$\operatorname*{Lt}_{h\to 0} \frac{\tau_h f - f}{h} = \operatorname*{Lt}_{h\to 0} \frac{[\tau_h p_n - p_n]}{h}$$

$$= \operatorname*{Lt}_{h\to 0} \left(\frac{\tau_h p_n(\theta, t) - p_n(\theta, t)}{h}\right) \circledast a(t)$$

$$= \frac{\partial}{\partial\theta} p_n(\theta, t) \circledast a(t).$$

Therefore $f' = \operatorname*{Lt}_{h\to 0} \frac{\tau_h f - f}{h} = [p_n']$. Hence we have

THEOREM 1.20. *Every generalized operational function f has a derivative f' represented by $[p_n']$.*

LEMMA 1.21. *If a sequence of continuous operational functions F_n converges to F in the sense of Definition* 1.6, *then $F = [F_n]$.*

Indeed, since F is a continuous operational function and F_n and F satisfy Lemma 1.8 (ii) with $k = 0$, the lemma follows.

THEOREM 1.22. *Every generalized operational function is the derivative of some order of a continuous operational function.*

Proof. We have $f = [f_n]$, where

$$f_n(\theta) = \frac{\partial^k}{\partial\theta^k} F_n(\theta, t) \circledast a(t)$$

and $F_n(\theta, t)$ converges almost uniformly to $F(\theta, t)$. In other words, $f = F_n^{(k)}$ and F_n converges to F in the sense of definition 1.6.

By Lemma 1.21, $F = [F_n]$. Therefore, $f = [F_n^{(k)}] = [F_n]^{(k)} = F^{(k)}$.

Definition 1.23. A sequence f of generalized operational functions is said *to converge to a generalized operational function f* if and only if there exist a non-negative integer k and a sequence F_n of continuous operational functions and a continuous operational function F such that $f = F_n^{(k)}$; $f = F^{(k)}$ and F_n converges to F in the sense of definition 1.6.

The convergence defined in the class of generalized operational functions is a Hausdorff convergence. In other words, if f_n converges to f and also to g, then $f = g$.

Indeed,

$$f_n(\theta) = \frac{\partial^k}{\partial\theta^k} F_n(\theta, t) \circledast a(t), \qquad f(\theta) = \frac{\partial^k}{\partial\theta^k} F(\theta, t) \circledast a(t)$$

and $F_n(\theta, t)$ converges almost uniformly to $F(\theta, t)$. Also

$$f_n(\theta) = \frac{\partial^m}{\partial\theta^m} G_n(\theta, t) \circledast b(t), \qquad g(\theta) = \frac{\partial^m}{\partial\theta^m} G(\theta, t) \circledast b(t)$$

and $G_n(\theta, t)$ converges almost uniformly to $G(\theta, t)$.

Let $m > k$. On account of definition 1.7 (4), there exist $\overline{F}_n(\theta)$ and $\overline{F}(\theta)$ such that

$$f_n(\theta) = \frac{\partial^m}{\partial\theta^m}\overline{F}_n(\theta, t) \circledast a(t), \qquad f(\theta) = \frac{\partial^m}{\partial\theta^m}\overline{F}(\theta, t) \circledast a(t)$$

and $\overline{F}_n(\theta, t)$ converges almost uniformly to $\overline{F}(\theta, t)$. Since

$$\left[\frac{\partial^m}{\partial\theta^m}\overline{F}_n(\theta, t) \circledast a(t)\right] - \left[\frac{\partial^m}{\partial\theta^m}G_n(\theta, t) \circledast b(t)\right] = 0,$$

we have $\overline{F}_n * b - G_n * a$, which is a polynomial $\sum_{j=0}^{m-1} a_{nj}(t)\theta^j$ of degree less than m. By Lemma 1.17, its limit is also a polynomial of degree less than m. Therefore

$$\overline{F} * b - G * a = \sum_{j=0}^{k-1} a_j(t)\theta^j.$$

So $(\overline{F} * b)^{(m)} - (G * a)^m = 0$ and hence $f = g$.

The following are immediate consequences of definition 1.23.

1. If a sequence of continuous operational functions converges in the sense of definition 1.6, it also converges in the generalized sense.

2. For $k = 0$ and $a = 1$, all almost uniformly convergent sequences of continuous functions in the variable θ, t are convergent in the generalized sense.

3. If f_n converges to f and g_n converges to g, then
 (i) $f_n + g_n$ converges to $f + g$,
 (ii) $f_n \cdot g_n$ converges to $f \cdot g$.

2. Now we introduce the notion of the value of a generalized operational function.

Definition 2.1. A generalized operational function f is said *to have a value at* θ_0 if $\operatorname{Lt}_{\alpha\to 0} f(\alpha\theta + \theta_0)$ exists. If the limit exists, then it is an operator.

LEMMA 2.2. *Let f be a continuous operational function defined in the neighbourhood of zero, and suppose that*

$$\operatorname{Lt}_{\alpha\to 0} \frac{f(\alpha\theta) - a_0(\alpha) - a_1(\alpha)(\theta\alpha) - \ldots - a_{n-1}(\alpha)(\theta\alpha)^{n-1}}{\alpha^n} = 0$$

(recall that the convergence is as given in definition 1.6) and the coefficients $a_i(\alpha)$ are continuous operational functions defined for $\alpha > 0$. Then the $a_i(\alpha)$ will converge for $\alpha \to 0$ and we have $a_i(\alpha) = a_i + o(\alpha^{n-1})$ and in consequence $f(\theta) = a_0 + a_1\theta + \ldots + a_{n-1}\theta^{n-1} + o(\theta^n)$ for θ tending to zero.

Proof. Let

$$\text{(I)} \qquad \frac{f(\alpha\theta) - a_0(\alpha) - \ldots - a_{n-1}(\alpha)(\theta\alpha)^{n-1}}{\alpha^n}$$

tend to zero as α tends to zero.

Now $f(\alpha\theta), a_0(\alpha), \ldots, a_{n-1}(\alpha)$ can all be written as

$$f(\alpha\theta, t) \circledast b_0(t),\ a_0(\alpha, t) \circledast b_1(t),\ \ldots,\ a_{n-1}(\alpha, t) \circledast b_n(t).$$

Without loss of generality we can assume $b_i(t) \equiv b(t)$.

(I) means that

$$\text{(II)} \qquad \frac{1}{a_n}\left(f(\alpha\theta, t) - a_0(\alpha, t) - \ldots - a_{n-1}(\alpha, t)(\theta\alpha)^{n-1}\right)$$

converges to zero almost uniformly, i.e.

$$|f(\alpha\theta, t) - a_0(\alpha, t) - \ldots - a_{n-1}(\alpha, t)(\theta\alpha)^{n-1}| < \varepsilon(\alpha)\alpha^n,$$

where $\varepsilon(\alpha)$ is an increasing function of α such that $\varepsilon(\alpha)$ converges to zero as α tends to zero.

Let δ, β be such that $0 < \delta < \beta$; choise v such that $0 < v\beta \leqslant \delta < \beta$, where $0 < v < 1$. Let (a, b) be an interval in which (II) converges uniformly. Fix $a < \theta_0 < \theta_1 < \ldots < \theta_n < b$. Write in (II)

$$\theta = \theta_i, \qquad \alpha = \delta, \qquad \theta = \frac{\delta\theta_i}{\beta}, \qquad \alpha = \beta,$$

$$|f(\delta\theta_i, t) - a_0(\delta, t) - \ldots - a_{n-1}(\delta, t)(\delta\theta_i)^{n-1}| < \varepsilon(\delta)\delta^n,$$

$$|f(\delta\theta_i, t) - a_0(\beta, t) - \ldots - a_{n-1}(\beta, t)(\delta\theta_i)^{n-1}| < \varepsilon(\beta)\beta^n.$$

Subtracting

$$\left|\left(a_0(\beta, t) - a_0(\delta, t)\right) - \ldots - \left(\left(a_{n-1}(\beta, t) - a_{n-1}(\delta, t)\right)(\delta\theta_i)^{n-1}\right)\right| < 2\varepsilon(\beta)\beta^n$$

and denoting by $p(\theta_i, t)$ the polynomial within the modulus sign, we can write,

$$\delta^i|a_i(\beta, t) - a_i(\delta, t)| = \frac{1}{A}\left(A_{1i}p(\theta_1, t) - \ldots - A_{ni}p(\theta_n, t)\right),$$

where A_{ji} are the minors of

$$A = \begin{vmatrix} 1 & \theta_1 & \ldots & \theta_1^{n-1} \\ 1 & \theta_2 & \ldots & \theta_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \theta_n & \ldots & \theta_n^{n-1} \end{vmatrix}.$$

There exists a constant k such that

$$\delta^i |a_i(\beta, t) - a_i(\delta, t)| < k\varepsilon(\beta)\beta^n.$$

Since $0 < v\beta \leqslant \delta < \beta$,

(III) $$|a_i(\beta, t) - a_i(\delta, t)| < kv^{-i}\varepsilon(\beta)\beta^{n-i}$$

Let $\delta = v^{r+1}$, $\beta = v^r$; then

$$|a_i(v^{r+1}, t) - a_i(v^r, t)| < kv^{-i}\varepsilon(v^i)v^{(n-i)r}$$

and for integers p and q such that $p \leqslant q$

(IV) $$|a_i(v^p, t) - a_i(v^q, t)| < kv^{-i}\varepsilon(v^p)v^{(n-i)p/(1-v^{n-i})}.$$

Since δ and β can be arbitrarily small, there exist integers p and q, $p \leqslant q$, such that $v^q \leqslant \delta \leqslant v^{q-1}$, $v^p \leqslant \beta \leqslant v^{p-1}$; we have in virtue of (III) and (IV) a constant k_0 such that

$$|a_i(\beta, t) - a_i(\delta, t)| < k_0 v^{-i}\varepsilon(\beta)\beta^{n-i}.$$

The limits $a_i = \operatorname{Lt}_{\delta\to 0} a_i(\delta)$ exist. Hence the lemma.

THEOREM 2.3 [1]. *For a generalized operational function f to have a value c (operator) at a point θ_0 it is necessary and sufficient that $f = F^{(k)}$, where F is a continuous operational function and*

$$\operatorname{Lt}_{\theta\to\theta_0} \frac{F(\theta)}{(\theta-\theta_0)^k} = \frac{c}{k!}.$$

Proof. Without loss of generality we can take $\theta_0 = 0$. Let $\alpha^{-k}F(\alpha\theta)$ tend to $c\theta^k/k!$ as α tends to zero. Since $f(\alpha\theta) = [\alpha^{-k}F(\alpha\theta)]^{(k)}$, we see that $f(\alpha\theta)$ tends to c, i.e. $f(\theta)$ has a value c at the point zero.

To prove the converse, let $f(\alpha\theta)$ tend to c as α tends to zero. Consider $f(\alpha\theta) = \varphi_\alpha^{(k)}(\theta)$, where $\operatorname{Lt}_{\alpha\to 0}\varphi_\alpha(\theta) = c\theta^k/k!$

Now the generalized operational function $f(\theta)$ is the derivative of order k of the continuous operational function $\alpha^k\varphi_\alpha(\theta/\alpha)$, i.e. there exists a continuous operational function $F_0(\theta)$ such that $f(\theta) = F_0^{(k)}(\theta)$. The difference $F_0(\theta) - \alpha^k\varphi_\alpha(\theta/\alpha)$ and in consequence the difference $F_0(\alpha\theta) - \alpha^k\varphi_\alpha(\theta) = \omega_\alpha(\theta)$, where

$$\omega_\alpha(\theta) = a_0(\alpha) + a_1(\alpha)\cdot\theta + \ldots + a_{k-1}(\alpha)\cdot\theta^{k-1}$$

and $a_i(\alpha)$ are continuous operational functions. As α tends to zero, $\big(F_0(\alpha\theta) - \omega_\alpha(\theta)\big)/\alpha^k$ tends to $c\theta^k/k!$.

Assume $f(\theta) = F_0(\theta) - c\theta^k/k!$. $\big(f(\alpha\theta) - \omega_\alpha(\theta)\big)/\alpha^k$ tends to zero. Therefore, by Lemma 2.2,

$$f(\theta) = a_0 + \ldots + a_{k-1}\theta^{k-1} + o(\theta^k),$$

and assuming

$$F(\theta) = F_0(\theta) - a_0 - \ldots - a_{n-1}\theta^{k-1} = c\theta^k/k! + o(\theta)^k,$$

we find that this $F(\theta)$ satisfies the conditions of the theorem.

THEOREM 2.4 [1, 2]. *If the generalized operational function f' has a value at θ_0, then the generalized operational function f has a value at θ_0.*

Proof. There exist a non-negative integer k and a continuous operational function F such that $f' = F^{(k)}$ and $\operatorname{Lt}_{\theta\to\theta_0} F(\theta)/(\theta-\theta_0)^k$ exists. If $k = 0$, then f' is a continuous operational function as so is f. Therefore the theorem is true. If $k > 0$, then $\operatorname{Lt}_{\theta\to\theta_0} F(\theta)/(\theta-\theta_0)^{k-1} = 0$. It follows from the previous theorem that the generalized operational function $F^{(k-1)}$ has a value at θ_0. The generalized operational function f differs from $F^{(k-1)}$ by a constant (operator). Therefore f has a value at θ_0. This completes the proof of the theorem.

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