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Extreme points in subalgebras of functions vanishing at infinity

by

ROBERT WHITLEY (Irvine)

Throughout, let S be a locally compact Hausdorff space which is not compact and let $C_0(S)$ be the space of all continuous scalar-valued functions which vanish at infinity, taken with the supremum norm. An important geometrical property of this Banach space is that its unit sphere contains no extreme points. We consider the circumstances under which subalgebras of $C_0(S)$ inherit this property and, to that extent at least, so resemble the algebra in which they are embedded. It is interesting that the conditions under which this occurs differ for the two scalar fields, i.e. for the space $C_0(S,R)$ of real-valued continuous functions vanishing at infinity and the space $C_0(S,C)$ of complex-valued continuous functions vanishing at infinity. For example, if S is connected, then $C_0(S,C)$ contains a closed subalgebra whose unit sphere has an extreme point, while $C_0(S,R)$ does not.

As in the case of real-valued functions, we call a subspace M of the complex space $C_0(S,C)$ a *linear sublattice* if the absolute value of each function in M is also in M. Before considering subalgebras, we first show that, except for discrete S, not even linear sublattices inherit the property of having no extreme points on their unit sphere. Of course, since any finite-dimensional subspace of $C_0(S)$ has a compact unit sphere which then has an extreme point by the Krein-Milman theorem, we may consider only infinite-dimensional subspaces. The result in Theorem 1 for S discrete is due to Garling [3].

THEOREM 1. *If S is discrete, then every infinite-dimensional subspace of $C_0(S,R)$ and $C_0(S,C)$ has a unit sphere with no extreme points. If S is not discrete, then there is a closed infinite-dimensional linear sublattice of $C_0(S,R)$ or $C_0(S,C)$ whose unit sphere contains an extreme point.*

Proof. If M is an infinite-dimensional subspace of $C_0(S)$ with S discrete and f an element of norm 1, then the set $\{s: |f(s)| \geqslant \frac{1}{2}\}$ is compact and so is a finite set $\{s_1, \ldots, s_n\}$. Then there is a non-zero g in M with $g(s_1) = \ldots = g(s_n) = 0$, else the map $h \to (h(s_1), \ldots, h(s_n))$ is a one-to-one map of M into E^n. For this g, $\|f\pm g/(2\|g\|)\| = 1$ and f is not extreme in the unit sphere of M.

Suppose that S is not discrete and let K be an infinite compact subset of S. First assume that K contains a non-void perfect set. Then by the main theorem in [4], which we will have occasion to use again, there is a continuous map of K onto $[0,1]$ which, by the Tietze extension theorem, can be extended to a continuous map φ of all of S onto $[0,1]$. Define the linear operator T taking $C[0,1]$ into $BC(S)$, the bounded continuous functions on S, by $Tf = f\circ\varphi$. The map T is a linear isometry with $T|f| = |Tf|$ and so the image $B = T(C[0,1])$ is a closed infinite-dimensional sublattice of $BC(S)$ which contains 1 and has the following property:

$$(*) \qquad \|g\| = \sup\{|g(s)|: s \text{ in } K\} \quad \text{for } g \text{ in } B.$$

Since K is compact, there is a continuous function f vanishing at infinity with $0\leqslant f(s)\leqslant 1$ for all s and $f(s)=1$ for s in K. Let $A = \{fg: g \text{ in } B\}$. From

$$\|g\| = \|f\|\|g\| \geqslant \|fg\| \geqslant \sup|f(s)g(s)|: s \text{ in } K\} = \|g\|,$$

we see that the map $g\to fg$ is a linear isometry which preserves absolute value and so see that A is a closed infinite-dimensional linear sublattice of $C_0(S)$. The function f is an extreme point in the unit sphere of A, for if $\|f\pm h\|\leqslant 1$ with h in A, then $|1\pm h(s)|\leqslant 1$ for s in K, which implies that $h(s)=0$ and so by property $(*)$ h must be zero. Second, suppose that K contains no non-void perfect set. Then there is a sequence $\{s_i\}$ of distinct isolated points of K. The map φ defined by $\varphi(s_i) = 1/i$ and $\varphi(s)=0$ for s not in $\{s_i\}$ is a continuous map of S onto $\{0,1,\frac{1}{2},\frac{1}{3},\dots\}$ which takes K onto the same set. As above, when the image space was $[0,1]$, this will give rise to an infinite-dimensional closed linear sublattice of $C_0(S)$ which has a unit sphere with an extreme point.

THEOREM 2. *The space $C_0(S,R)$ contains an infinite-dimensional subalgebra whose unit sphere has an extreme point if and only if S contains an infinite compact open set.*

Proof. Let A be an infinite-dimensional subalgebra of $C_0(S,R)$ and let f be an extreme point of the unit sphere of A. By a result of Phelp's ([5], corollary 3.2, p. 271), f^2 and f^4 are also extreme points in the unit sphere of A. Since $\|f^4\pm(f^2-f^4)\|\leqslant 1$ and f^4 is extreme, $f^2=f^4$. Hence f^2 takes on only the values 0 and 1 and the set $K=\{s: f^2(s)=1\}$ is a compact open set in S. Any function g in A must vanish off K, for if $g(s_0)\neq 0$ and s_0 is not in K, then $g-f^2g = h$ is a non-zero element of A vanishing on K. But then $\|f\pm h/\|h\|\,\|=1$ and f is not extreme. Thus the open compact set K must be infinite as A is infinite-dimensional.

Conversely, if S contains an infinite compact open set K, then the set of all f vanishing on $S-K$ is an infinite-dimensional closed subalgebra of $C_0(S,R)$ whose unit sphere contains the characteristic function of K as an extreme point.

If A is a finite-dimensional subalgebra of $C_0(S,R)$ or $C_0(S,C)$ and f is in A, then the range of f must consist of only a finite number of scalars, for if not, the functions $f, f^2, f^3, \dots$ in A are linearly independent. So the inverse image under f of each non-zero point in its range is an open compact set in S. Then it is not hard to see that A is the span of a finite number of characteristic functions of compact open subsets of S. In particular, when S is connected, $C_0(S,R)$ can contain no subalgebra whose unit sphere has an extreme point, for it contains no such infinite-dimensional subalgebras by theorem 2 and contains no finite-dimensional subalgebras.

That the subalgebra A of theorem 2 need not be closed, raises the question of the relation of the extreme points of the unit sphere of A to the closure of A. As we will see, this relation differs for subalgebras of $C_0(S,R)$ and $C_0(S,C)$.

THEOREM 3. *If A is a subalgebra or a linear sublattice of $C_0(S,R)$ and f is an extreme point of the unit sphere of A, then f is an extreme point of the unit sphere of the closure of A. If S is not discrete there is a subspace M of $C_0(S,R)$ (or $C_0(S,C)$) and a point f extreme in the unit sphere of M but not extreme in the unit sphere of the closure of M.*

Proof. Let A be a subalgebra of $C_0(S,R)$ and f an extreme point in the unit sphere of A. As we saw in theorem 2, $K=f^{-1}(1)\cup f^{-1}(-1)$ is a compact open set in S and each function in A vanishes on $S-K$. Hence each function in the closure of A also vanishes on $S-K$ and from this it follows that f is extreme in the unit sphere of the closure of A.

Let A be a linear sublattice of $C_0(S,R)$ and f an extreme point in the unit sphere of A. If $\||f|\pm g\|\leqslant 1$ for g in A, then $\|f\pm g\|\leqslant 1$ and $g=0$. So $|f|$ is extreme in the unit sphere of A. Let g in A have $\|g\|\leqslant 1$ and set $h=\max(|f|,|g|)$, an element of A. From the inequality

$$\||f|\pm(h-|f|)\|\leqslant\|h\|\leqslant 1$$

and the fact that $|f|$ is extreme, we see that $h=|f|$. Consequently, $|g(s)|\leqslant|f(s)|$ for each g in A with $\|g\|\leqslant 1$, and from this it follows that whenever a non-zero function in A attains its norm at a point s_0, the points s_0 must be in the compact set $F=\{s: |f(s)|=1\}$. Assume that g_0 is a function in the closure of A with $\|f\pm g_0\|\leqslant 1$. There is a sequence $\{g_n\}$ of functions in A converging to g_0. Since $\|f\pm g_n\|\to 1$, $g_n(s)\to 0$ uniformly for s in F. As each g_n attains its norm on the set F, $\|g_n\|\to 0$ and g_0 must be 0. Hence f is an extreme point of the unit sphere of the closure of A.

When S is discrete, from theorem 1 we see that we cannot construct an example of a subspace whose unit sphere has an extreme point but which is not extreme for the closure. Supposing that S is not discrete, there is a point with compact neighborhood K whose interior is infinite. We can find an infinite collection $\{U_n\}$ of disjoint open subsets of K with points s_n in U_n. By Tietze's theorem there are continuous functions f_n, $0 \leqslant f_n(s) \leqslant 1$ for all s, $f_n(s_n) = 1$ and $f_n(s) = 0$ for s not in U_n and there is a continuous function g in $C_0(S, R)$ with $0 \leqslant g(s) \leqslant 1$ for all s and $g(s) = 1$ for s in K. For each element $\{a_i\}$ in the Banach space c of covergent sequences of scalars, define

$$T\{a_i\} = \sum_{i=1}^{\infty} (a_i - \lim a_j) f_i + \lim a_j g.$$

It is easy to see that T is a linear isometry of c into $C_0(S)$. In this way we see that $C_0(S)$ contains a subspace isometric to c. To complete the proof, it will then be enough to give an example of a subspace M of c with a point x extreme in the unit sphere of M but not extreme in the unit sphere of the closure of M. To do this, regard c as a collection of functions defined on $N = \{1, 2, 3, \ldots\}$ and let $\{N_i\}$ be a sequence of subsets of N with the following properties:

(a) 1 is in N_i for all i and 2 is in $N - N_i$ for all i,

(b) N_i properly contains N_{i+1},

(c) if $p \neq 1$ is in N_i, then $p \geqslant i$.

Let y_i be the product $C_{N_i}(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, where C_{N_i} is the characteristic function of the set N_i, let $x = (\frac{1}{2}, 1, 1, 1, \ldots)$, and let M be the subspace of c spanned by $x, y_1, y_2, \ldots$ To show that x is extreme in the unit sphere of M suppose that

$$\Big\| x \pm \Big(a_0 x + \sum_{1}^{p} a_i y_i \Big) \Big\| \leqslant 1.$$

Evaluating this sequence at 2, $|1 \pm a_0| \leqslant 1$ and a_0 must be 0. Then successively evaluating at the points n_i in $N_i - N_{i+1}$ for $i = 1, 2, \ldots, p$, we get $a_1 = \ldots = a_p = 0$. But x is not extreme in the unit sphere of the closure of M, for

$$\|x \pm (\tfrac{1}{2}) \lim y_i\| = 1.$$

We now consider the case of complex-valued functions. First we show that we may restrict our attention to closed subalgebras which are not self-adjoint. Recall that a subalgebra is *self-adjoint* if it contains the complex conjugate of each of its functions.

THEOREM 4. *If S is not discrete, then $C_0(S, C)$ contains an infinite-dimensional subalgebra (which is not necessarily closed) whose unit sphere has an extreme point. The space $C_0(S, C)$ contains an infinite-dimensional self-adjoint subalgebra whose unit sphere contains an extreme point if and only if S contains an infinite compact open set.*

Proof. Suppose that S is not discrete and let the functions $\{f_n\}$ and g be as in the proof of theorem 3. For the scalars $a_n = \exp(2\pi i/n)$, the function

$$f = \sum_{n=1}^{\infty} (a_n - 1) f_n + g$$

is in $C_0(S, C)$ and has $f(s_n) = a_n$. The subalgebra $A = \mathrm{span}(f, f^2, f^3, \ldots)$ spanned by the powers of f is infinite-dimensional since the range of f contains infinitely many scalars. We claim that f is an extreme point of the unit sphere of A. For suppose that

$$\Big\| f \pm \sum_{n=1}^{p} b_n f^n \Big\| \leqslant 1;$$

then, evaluating this function at the point s_j,

$$\Big| a_j \pm \sum_{n=1}^{p} b_n a_j^n \Big| \leqslant 1$$

which, since a_j is a scalar of modulus 1, implies that

$$\sum_{n=1}^{p} b_n a_j^n = 0.$$

As this is true for the p non-zero distinct values $a_1, a_2, \ldots, a_p$, $b_1 = b_2 = \ldots = b_p = 0$.

Suppose that A is an infinite-dimensional self-adjoint subalgebra of $C_0(S, C)$ whose unit sphere contains the extreme point f. The function $h = \bar{f} f = |f^2|$ is in the unit sphere of A. To show that it is extreme there, suppose that $\|h \pm g\| \leqslant 1$ with g in A. Then, as noted in [5] (lemma 3.1, p. 271), $h + |(g/4)^2| \leqslant 1$ and so $\|f^2 \pm (g/4)^2\| \leqslant 1$. The function f^2 is extreme in the unit sphere of A ([5], corollary 3.2, p. 271) which implies that $g = 0$. Because A is a self-adjoint subalgebra, the set $\mathrm{Re}\, A$ of real-valued functions in A, forms a subalgebra of $C_0(S, R)$, which is infinite-dimensional since A is, whose sphere contains the extreme point h. By theorem 2, S contains an infinite compact open set.

If S contains an infinite compact open set K, then the set of those functions in $C_0(S, C)$ which vanish on $S - K$ is an infinite-dimensional closed self-adjoint subalgebra whose unit sphere contains the characteristic function of K as an extreme point.

We now consider closed subalgebras of $C_0(S, C)$. As we saw in theorems 2 and 3, in $C_0(S, R)$ it does not matter whether the subalgebra under consideration is closed. That it does matter in $C_0(S, C)$ can be seen from theorems 4 and 5 below.

THEOREM 5. *The space $C_0(S, C)$ contains an infinite-dimensional closed subalgebra whose unit sphere has an extreme point if and only if either S contains an infinite compact open set or S contains a non-void compact perfect set.*

Proof. If S contains no compact perfect set, then for each compact set K in S and f in $C_0(S, C)$, $f(K)$ is countable ([6], theorem 2, p. 40). So the range of f must be countable since $f(S)-\{0\} = \bigcup f(K_n)$, where K_n is the compact set $\{s: |f(s)| \geqslant 1/n\}$. Arguing exactly as in the proof of theorem 3 of [6], we see that every closed subalgebra of $C_0(S, C)$ is self-adjoint. From theorem 4, if such a subalgebra is infinite-dimensional and has a unit sphere with an extreme point, then S contains a compact open infinite set. This completes the first half of the proof.

Suppose that S contains a non-void compact perfect set E. By the main theorem in [4], p. 214, there is a continuous map of E onto $[0, 1]$ which in turn can be mapped onto $\{a: |a| = 1\}$. Call this composite map f_0. Passing to the one-point compactification S^* of S, set $f_0(\infty) = 0$. This map f_0 of $E \cup \{\infty\}$ into $D = \{a: |a| \leqslant 1\}$ can be extended to a map f on all S^* with range still in D, since D is homeomorphic to the unit square and is thus an absolute retract for normal spaces. We claim that this function f on S is an extreme point in the unit sphere of the closed subalgebra A which it generates. Suppose that g is in A with $\|f \pm g\| \leqslant 1$. There are functions

$$g_n = \sum_{j=1}^{m_n} a_{jn} f^j \quad \text{with } \|g_n - g\| \to 0.$$

For $z = f(s)$ in the range of f, $p_n(z) = \Sigma a_{jn} z^j$ converges uniformly to $g(s)$. So the polynomials $\{p_n\}$ converge uniformly on $f(S)$ which contains the boundary ∂D of D. By the maximum modulus theorem, $\{p_n\}$ is a Cauchy sequence in $C(D)$ and so converges uniformly to a function g_0 on D which is analytic in the interior of D and for which $g_0(f(s)) = g(s)$. From the fact that $\|f \pm g\| \leqslant 1$, we see that $g(s) = 0$ when $|f(s)| = 1$ and so $g_0(\partial D) = 0$. Hence g_0, being analytic in the interior of D, must vanish there and $g(s)$ must then be zero for each s in S.

Of course, if S contains a compact infinite open set K, then the functions vanishing on $S-K$ form an infinite-dimensional closed subalgebra of A whose sphere has an extreme point.

To again contrast the case of real-valued functions with complex-valued functions, compare theorem 3 with theorem 6.

THEOREM 6. *If A is a linear sublattice of $C_0(S, C)$, then an extreme point of the unit sphere of A is also an extreme point of the unit sphere of the closure of A. There is a subalgebra A with a point extreme in the unit sphere of A but not in the unit sphere of the closure of A if and only if S is not discrete.*

Proof. Let A be a linear sublattice of $C_0(S, C)$ and f an extreme point of the unit sphere of A. First note that if $\||f|+ag\| \leqslant 1$ for g in A and all scalars a with $|a| = 1$, then $\|f \pm g\| \leqslant 1$ and so $g = 0$. For w an element of A with $\|w\| \leqslant 1$, as in the proof of theorem 3, let $h = \max(|f|, |w|)$. For each scalar a with $|a| = 1$, $\||f| + a(h-|f|)\| \leqslant \|h\| \leqslant 1$ and as we have seen this implies that $h = |f|$. Thus if g is in A and $|g(s_0)| = \|g\| \neq 0$, then s_0 belongs to the set $\{s: |f(s)| = 1\}$ and this implies that f is an extreme point in the unit sphere of the closure of A by arguing exactly as in theorem 3.

We suppose that S is not discrete.

First suppose, in addition, that S contains no compact perfect set and no infinite compact open set. Then the subalgebra A defined in the proof of theorem 4, which is spanned by f, has a unit sphere in which f is an extreme point. Since S contains no compact perfect set, then, as in the proof of theorem 5, the closure of A is a self-adjoint subalgebra of $C_0(S, C)$. If f were extreme in the unit sphere of the closure of A, then, by theorem 4, S would contain an infinite compact open set. Thus f is not an extreme point of the unit sphere of the closure of A.

Second, suppose, in addition, that S contains no compact perfect set but does contain an infinite compact open set K. Then K contains a sequence $\{s_n\}$ of isolated points. Let $a_n = \exp(2\pi i/n)$ and set

$$f = (-\tfrac{1}{2}) C_{\{s_1\}} + \sum_{n=2}^{\infty} (a_n - 1) C_{\{s_n\}} + C_K.$$

Then f is an extreme point of the unit sphere of the subalgebra A it generates because, as in theorem 4, the range of f contains infinitely many scalars of modulus 1. However, as in theorem 5, the closure of A is a self-adjoint subalgebra which then contains $|f^2| = (-\tfrac{3}{4}) C_{\{s_1\}} + C_K$ and thus contains $-f|f^2| + f = (\tfrac{3}{8}) C_{\{s_1\}}$ from which it is easy to see that f is not an extreme point in the unit sphere of the closure of A.

Third, and last, suppose that S contains a non-void perfect compact set K. Then there is a continuous map of E onto $[0, 1]$ ([4], p. 214) which can be extended to a continuous map f_1 of the one-point compactification S^* of S which takes infinity to zero and with range $[0, 1]$. Then f_1 is a map in $C_0(S)$ taking K onto $[0, 1]$. If f_2 is a continuous map of $[0, 1]$ onto

$$E = \{(x, y): y = 0 \text{ and } 0 \leqslant x \leqslant 1\} \cup \{(x, y): y = \sqrt{1-x^2} \text{ and } 0 \leqslant x \leqslant 1\}$$

with $f_2(0) = 0$, then the composition $f = f_2 f_1$ is in $C_0(S, C)$ and has range E. Let A be the algebra generated by f, i.e. the span of $f, f^2, \ldots$

As we saw in the proof of theorem 4, f is an extreme point in the unit sphere of A since the range of f contains infinitely many scalars of modulus 1. Now we need to show that f is not an extreme point in the unit sphere of the closure of A. The set E does not separate the plane and has no interior, consequently by Mergelyan's theorem any continuous function on E is the uniform limit of polynomials [1]. Given the function $h(x,y) = x(1-x)$ for $y=0$ and $h(x,y)=0$ for $y\neq 0$, there is a sequence $g_n(z) = \sum a_{mn} z^m$ converging uniformly to h on E and since $h(0,0)=0$, we may take $g_n(0)=0$ for all n. Then $g_n(f)$ is a Cauchy sequence in A and so converges to a function g in the closure of A which is not zero. Since

$$|f(s)\pm g(s)| = |f(s)\pm h(\operatorname{Re} f(s), \operatorname{Im} f(s))| \leqslant 1,$$

$\|f\pm g\| \leqslant 1$ and f is not extreme in the unit sphere of the closure of A.

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UNIVERSITY OF CALIFORNIA, IRVINE

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Lebesgue and Lipschitz spaces and integrals of the Marcinkiewicz type

by

RICHARD L. WHEEDEN* (New Brunswick, N. J.)

§ 1. Introduction. A theorem of Zygmund [16] states that for $1<p<\infty$ the L^p-norm of

$$(Mf)(x) = \left(\int_0^{2\pi} \left|\frac{F(x+t)+F(x-t)-2F(x)}{t}\right|^2 \frac{dt}{t}\right)^{1/2}$$

satisfies

$$\|Mf\|_p \leqslant A_p \|f\|_p$$

and, if $\int_0^{2\pi} f(x)\,dx = 0$,

$$\|f\|_p \leqslant A_p \|Mf\|_p,$$

where

$$F(x) = \int_0^x f(u)\,du.$$

The integral Mf is called the (*first*) *Marcinkiewicz integral* of F and is related in a rather natural way to the Hilbert transform of f. In fact, proceeding formally,

$$\int_{-\infty}^{\infty} f(x-t)\frac{dt}{t} = -\int_0^{\infty} [f(x+t)-f(x-t)]\frac{dt}{t}$$
$$= -\int_0^{\infty} \frac{d}{dt}[F(x+t)+F(x-t)-2F(x)]\frac{dt}{t}$$
$$= -\int_0^{\infty} \left(\frac{F(x+t)+F(x-t)-2F(x)}{t}\right)\frac{dt}{t}.$$

It was exactly this relation which led Stein in [9] to define an n-dimensional version of the Marcinkiewicz integral (1). Let $\Omega(z)$, $z\in E_n$,

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(1) For another generalization of Mf to E_n, see [11].