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Reflexivity and summability, II*

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In the preceding note with T. Nishiura [2] our principal interest was the question: Is reflexivity of a Banach space equivalent to a summability property? This question and the others considered arose from the classical theorem of Banach and Saks on $(C, 1)$-summability and L_p-spaces. We answered this question affirmatively by showing that a Banach space B is reflexive if and only if for every bounded sequence there is a regular essentially positive summability method T and a subsequence whose T-means converge (either weakly or strongly).

Singer has raised the problem of reducing the requirements on the summability method and has shown [4] that the requirement of essential positivity may be omitted. The result which we now present gives somewhat more information.

Modifying slightly our previous notation [2], we will say that a Banach space B has property $\mathscr{S}$ (w$\mathscr{S}$) if for every bounded sequence in B there is a summability method T and a subsequence such that the T-means of the subsequence converge strongly (weakly).

Letting $T = (c_{mn})$, the class of convergence-preserving methods, i.e., those methods which sum every convergent sequence, is characterized by the following conditions:

(i) $\sum_{n=1}^{\infty} |c_{mn}| < H < \infty$ for every m;

(ii) $\sum_{n=1}^{\infty} c_{mn} \to c$ as $m \to \infty$;

(iii) $c_{mn} \to c_n$ as $m \to \infty$ for every n.

Here, of course, c and c_n are finite. A convergence preserving method is regular, i.e., it sums every convergent sequence to its ordinary sum, if and only if $c = 1$ and $c_n = 0$ for all n.

A method will be called *almost regular** if it satisfies (ii), (iii), and

(iv) $c \neq \sum_{n=1}^{\infty} c_n$,

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the latter sum being supposed convergent. The *regular** methods (T^* in the notation of Zygmund [6], p. 202-205), are almost regular* with $c = 1$ and $c_n = 0$ for all n. Although these methods need not preserve convergence, they are of considerable analytic interest. The most familiar example of a method which is regular*, but not regular, is Lebesgue summability ([1], p. 15-18).

We prove the following

THEOREM. *In a Banach space B, property w $\mathscr{S}$ with almost regular* T implies reflexivity, and reflexivity implies $\mathscr{S}$ with positive row-finite column finite regular T.*

Proof. The second part of the result was proved in our first paper [2], p. 55. Turning to the first part, we note that inserting columns of zeros into T does not affect (ii)-(iv). Since we do not require T to be the same for all sequences, we may then omit the consideration of the subsequence from our definitions of $\mathscr{S}$ and w $\mathscr{S}$ and suppose instead that the T-means of the given sequence converge.

Let $\{\Phi_i\}$ be a Schauder basis for a subspace E of B with the property

$$\Big\|\sum_{i=1}^{n}\Phi_i\Big\| < M < \infty \quad \text{for all } n.$$

If, in addition, we had

$$\inf\|\Phi_i\| > 0,$$

this basis would be said to have property P ([5], p. 354).

According to our hypothesis, there is an almost regular* method $T = (c_{mn})$ and a point $x = \sum b_i\Phi_i$ such that the T-means of

$$x_n = \sum_{i=1}^{n}\Phi_i$$

converge weakly to x. We have, proceeding formally,

$$t_m = \sum_{n=1}^{\infty}c_{mn}x_n = \sum_{i=1}^{\infty}\Big(\sum_{n=i}^{\infty}c_{mn}\Big)\Phi_i.$$

To verify this last equality, we show that the series involved are strongly equi-convergent. We have

$$\Big\|\sum_{i=1}^{N}\Big(\sum_{n=i}^{\infty}c_{mn}\Big)\Phi_i - \sum_{n=1}^{N}c_{mn}\Big(\sum_{i=1}^{n}\Phi_i\Big)\Big\| = \Big\|\sum_{i=1}^{N}\Big(\sum_{n=i}^{\infty}c_{mn}\Big)\Phi_i - \sum_{i=1}^{N}\Big(\sum_{n=i}^{N}c_{mn}\Big)\Phi_i\Big\|$$
$$= \Big\|\sum_{i=1}^{N}\Phi_i\sum_{n=N+1}^{\infty}c_{mn}\Big\| < M\Big|\sum_{n=N+1}^{\infty}c_{mn}\Big| \to 0 \text{ as } N\to\infty.$$

Thus

$$\sum_{i=1}^{\infty}\Big(\sum_{n=i}^{\infty}c_{mn}-b_i\Big)\Phi_i \to 0 \quad \text{weakly as } m\to\infty,$$

implying that, for each i,

$$\lim_{m\to\infty}\Big(\sum_{n=i}^{\infty}c_{mn}-b_i\Big) = \Big(c-\sum_{n=1}^{i-1}c_n\Big)-b_i = 0.$$

Then (iv) implies

$$\liminf_{i\to\infty}|b_i| > 0.$$

Since $\sum b_i\Phi_i$ converges, we have

$$\inf_i\|\Phi_i\| = 0$$

and therefore $\{\Phi_i\}$ does not have property P. From a theorem of Singer [5], p. 362, we know then that E is reflexive. Pełczyński [3] has shown that the reflexivity of a Banach space is equivalent to the reflexivity of each of its subspaces which has a Schauder basis. Hence B is reflexive.

References

[1] N. K. Bary, *A treatise on trigonometric series*, Vol. II, New York 1964.

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[3] A. Pełczyński, *A note on the paper of I. Singer*, ibidem 21 (1962), p. 371-374.

[4] I. Singer, *A remark on reflexivity and summability*, ibidem 26 (1965), p. 113-114.

[5] — *Basic sequences and reflexivity of Banach spaces*, ibidem 21 (1962), p. 351-369.

[6] A. Zygmund, *Trigonometric series*, Vol. I, Cambridge 1959

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