Hypoelliptic and entire elliptic convolution equations
in subspaces of the space of distributions (II)

by

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In part I of this work (see [5]) we showed how to define in a general
manner hypoelliptic and entire elliptic convolution operators in sub-
spaces of the space of distributions. We also characterized hypoelliptic
and entire elliptic convolution operators in the space \( \mathcal{S}' \) of tempered
distributions.

The purpose of this paper is to study hypoelliptic convolution opera-
tors in the space \( \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \) of distributions of exponential growth
introduced by Sebastião e Silva [4] and Hasumi [1].

The space \( \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \) of convolution operators in \( \mathcal{K}' \) (which is
a space of distributions) was characterized in [1] and its topological
properties were investigated in [6].

Using the notation of [5] we define \( \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \) to be the set of all \( C^\alpha \)-func-
tions \( f \in \mathcal{K}' \) such that, for every \( S \in C^\alpha(\mathcal{K}'; \mathcal{K}) \), the convolution \( S \ast f \)
is a \( C^\alpha \)-function and \( S \ast \delta \) is a continuous mapping from \( C^\alpha(\mathcal{K}'; \mathcal{K}) \)
into the space \( \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \) of distributions. Then a distribution \( \delta \ast \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \)
is said to be hypoelliptic in \( \mathcal{K}' \), if every solution \( U \in \mathcal{K}' \) of the con-
volution equation

\[
S \ast U = F
\]

(1)

is in \( \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \), when \( F \in \mathcal{E}'(\mathcal{K}'; \mathcal{K}) \); in that case equation (1) is also called hypoe-
lliptic in \( \mathcal{K}' \).

As a supplement of the standard notation (see [3] and [5]) we use \( \mathbb{R}^n \) as the set of all points in \( \mathbb{R}^n \), whose coordinates are non-negative integers; we write \( N \) and \( E \) instead of \( \mathbb{N} \) and \( \mathbb{R} \) respectively. Further-
more, we denote by \( P^n(Q^n) \) resp. the set of all points \( p = (p_1, \ldots, p_n) \)
\( (q = (q_1, \ldots, q_n) \) resp.) such that \( p_1 = 1 \) or \( -1 \) (\( q_1 = 1 \) or \( 0 \) resp.). In
particular, \( Q^n \) contains the points \( l = (1, 1, \ldots, 1) \) and \( 0 = (0, 0, \ldots, 0) \).

For a point \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we sometimes write \( x = (x', x_n) \),
where \( x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \). Also, for \( x = (a_1, \ldots, a_n) \) and \( \xi = (\xi_1, \ldots, \xi_n) \) in \( \mathbb{R}^n \) we use the product \( x \xi = (a_1 \xi_1, \ldots, a_n \xi_n) \) beside the scalar
product \( x \cdot z = x_1 z_1 + \ldots + x_n z_n \). The same notation applies to points in \( C^a \), which are denoted by \( z = x + iy \) or \( z = x + iy, x, y, z \in \mathbb{R}^a \).

Given an \( a \in R_1, a > 0, I_a \) stands for the open cube in \( \mathbb{K} \) with center at the origine and side \( 2a \), i.e.

\[
I_a = \{ z = (x_1, \ldots, x_n) \in \mathbb{K}^a : |x_j| < a, j = 1, \ldots, n \}.
\]

\( \bar{I}_a \) is the closure of \( I_a \).

A horizontal strip in \( C^a \), width \( b > 0 \) is defined as

\[
Y_b = \{ z = (z_1, \ldots, z_n) \in C^a : |z_j| < b, j = 1, \ldots, n \}.
\]

We constantly make use of the function

\[
s_b(z) = \sum_{j=0}^{b} + \prod_{j=0}^{n} (e^{x_j} + e^{-x_j}),
\]

where \( z = (z_1, \ldots, z_n) \in C^a \) and \( b \in R \).

1. The basic spaces. For the convenience of the reader we characterize briefly the basic spaces used in this paper.

\( \mathfrak{X}_k \) is the space of all \( C^a \)-functions \( \psi \) in \( \mathbb{K}^a \) such that \( s_b(x) \mathfrak{D}^k \psi(x) \) is bounded in \( \mathbb{K}^a \), for every \( k \in \mathbb{N} \) and \( r \in \mathbb{N} \). The topology in \( \mathfrak{X}_k \) is defined by the system of semi-norms

\[
v_k(x) = \sup_{x \in B^a, \mu \in \mathbb{N}} s_b(x) \mathfrak{D}^k \psi(x), \quad k = 0, 1, \ldots
\]

Then \( \mathfrak{X}_k \) is a Frechet nuclear space [11, proposition 1].

The dual \( \mathfrak{X}_k' \) of \( \mathfrak{X}_k \) is the space of distributions of exponential growth. A distribution \( T \) in \( \mathfrak{X}_k \) if and only if \( T \) can be represented in the form

\[
T = \mathfrak{D} [s_b(x) \psi(x)],
\]

where \( r \in \mathbb{N} \), \( \mu \in \mathbb{N} \) and \( f \) is a bounded, continuous function on \( \mathbb{K}^a \) [11, proposition 3]. Under the strong topology \( \mathfrak{X}_k \) is a complete Montel space.

The space \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) of convolution operators in \( \mathfrak{X}_k \) can be characterized as follows [11, proposition 9]. A distribution \( S \) in \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) if and only if, for every \( k \in \mathbb{N} \), \( S \) can be represented as a finite sum of derivatives of continuous functions, whose products with \( s_b(x) \) are bounded in \( \mathbb{K}^a \). The topology of \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) is that induced in \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) by the space \( \mathfrak{K}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \); it makes \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) into a complete Montel space (see [6]).

Note that the convolution \( S \ast T \) can be defined even if neither \( S \) nor \( T \) is in \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \). If e.g. for \( \mu < \mu, s_b \) and \( s_b - T \) are bounded distributions, then one can find continuous functions \( F_\mu, r \in \mathbb{N}, |r| < k \), and \( G \) such that

\[
S = \sum_{|r| < k} \mathfrak{D}^r F_\mu, \quad T = \mathfrak{D}^r G
\]

and the convolutions \( F_\mu \ast G \) exist in the usual sense. Then we set

\[
S \ast T = \sum_{|r| < k} \mathfrak{D}^r (F_\mu \ast G).
\]

One can show that the convolution \( S \ast T \) so defined does not depend on the representation (2).

The set \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) can be identified with the dual \( \mathfrak{L}_k(\mathfrak{X}_k; \mathfrak{X}_k') \) of \( \mathfrak{L}_k(\mathfrak{X}_k; \mathfrak{X}_k) \) similarly as in the case of the set \( \mathfrak{L}^k \) (see [3], p. 322). Thus \( \mathfrak{E}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) consists of all \( \mathfrak{C}^a \)-functions \( f \) such that one can find a \( k \in \mathbb{N} \) satisfying the condition

\[
\mathfrak{D}^k f(x) = O(s_b(x))
\]

as \( |x| \to \infty \), for all \( r \in \mathbb{N} \) ([6], theorem 10).

For a function \( \varphi \in \mathfrak{X}_k \), its Fourier transform

\[
\hat{\varphi} \xi = \int_{\mathbb{R}^a} e^{2\pi i x \cdot \xi} \hat{\varphi}(\xi) d\xi
\]

can be extended over \( \mathbb{R}^a \) as an entire function such that

\[
|\hat{\varphi}(\xi)| = O(1 + |\xi|^k)\hat{\varphi}(\xi) < \infty, \quad k = 1, 2, \ldots
\]

The space \( \mathfrak{K}_k \) of all entire functions with the latter property corresponds to \( \mathfrak{X}_k \) under the Fourier transform. If the topology in \( \mathfrak{X}_k \) is defined by the system of semi-norms \( s_b(x) \), \( k = 1, 2, \ldots \), then the Fourier transform is a topological isomorphism of \( \mathfrak{X}_k \) onto \( \mathfrak{K}_k \) ([11], proposition 4).

The dual \( \mathfrak{L}_k \) of \( \mathfrak{K}_k \) is the space of Fourier transforms of distributions from \( \mathfrak{X}_k \). For a distribution \( T \in \mathfrak{X}_k \), its Fourier transform \( \hat{T} \) is defined by the Parseval equation

\[
\hat{T} \varphi(\xi) = T \varphi(-\xi),
\]

\( \mathfrak{K}_k \) is provided with the strong topology. Then the Fourier transform is a topological isomorphism of \( \mathfrak{X}_k \) onto \( \mathfrak{K}_k \).

The Fourier transform \( \mathfrak{S} \) of a distribution \( \mathfrak{S} \in \mathfrak{L}_k(\mathfrak{X}_k' ; \mathfrak{X}_k') \) is a \( \mathfrak{C}^a \) -function extendable over \( \mathbb{R}^a \) as an entire function; moreover, for every \( k \in \mathbb{N} \) there exists an \( k \) in \( \mathbb{N} \) such that

\[
\sup_{(1 + |\xi|^k)} s_b(x) < \infty
\]
(see [13], propositions 8 and 9, or [6], theorem 3). Also, for \( \hat{S} \ast \phi(\mathcal{X}_1 : \mathcal{X}_f) \) and \( T \in \mathcal{X}_1 \) we have the formula

\[
\hat{S} \ast T = \hat{S} \ast T,
\]

where the product on the right-hand side is well defined in \( K_1 \).

2. Hypoelliptic operators in \( \mathcal{X}_1 \). Necessary condition. We prove a necessary condition for a convolution operator \( S \ast \phi(\mathcal{X}_1 : \mathcal{X}_f) \) to be hypoelliptic in \( \mathcal{X}_1 \). The proof is based on an idea similar to that used in [5] for convolution operators in \( \mathcal{X}_1 \). We begin with a lemma.

Lemma 1. Let \( T \) be a distribution, whose Fourier transform \( \hat{T} \) is of the form

\[
\hat{T} = \sum_{j \in \mathbb{Z}} a_j \delta_{|\zeta|^{2j}},
\]

where the \( \zeta = \xi + i\eta \ast \phi^{(n)} \) satisfy conditions

\[
|\zeta| > 2|\zeta| > 2^j, \quad |\eta| \leq B,
\]

and \( a_j \) are complex numbers such that

\[
a_j = O(|\zeta|^{-j}),
\]

for some \( \mu \in \mathbb{N} \); then the series in (3) converges in \( K_1 \). We assert that \( T \in \mathcal{E}_1 \), if and only if

\[
a_j = o(|\zeta|^{-j}),
\]

for every \( \zeta \in \mathcal{X}_1 \).

Proof. By virtue of equality (3) and condition (5),

\[
T = \sum_{j = -\infty}^{\infty} a_j e^{i\zeta \omega_j},
\]

where the series converges in \( \mathcal{X}_1 \). If the coefficients \( a_j \) satisfy condition (6), then the last series and all its term-by-term derivatives converge uniformly in \( B^n \) on dividing by \( \phi^{(n)} \). Consequently, \( T \) is in \( \mathcal{E}_1 \).

Conversely, assume that \( T \) is a function from \( \mathcal{E}_1 \). Then, for every \( \zeta \in \mathcal{X}_1 \),

\[
\sqrt{\zeta} e^{i\zeta \omega_j} A T \phi(\zeta) \rightarrow 0,
\]

as \( |u| \rightarrow \infty, u \in \phi^{(n)}, |\mathcal{X}| \leq B \); \( A \) is the iterated Laplace operator. Hence, passing to the Fourier transform, we see that

\[
T_n(\zeta) \ast \hat{\phi}(\zeta) = \sum_{j = -\infty}^{\infty} a_j e^{i\zeta \omega_j} \hat{\phi}(\zeta - u) \rightarrow 0,
\]

as \( |u| \rightarrow \infty, u \in \phi^{(n)}, |\mathcal{X}| \leq B \). We fix a function \( \phi \in \mathcal{X}_1 \) such that

\[
|\phi(0)| \geq 1.
\]

Suppose now that condition (6) is not satisfied. Then there is a \( \eta > 0 \) and a \( \nu \in \mathbb{N} \) such that

\[
|\zeta|^{2\nu}|a_j| \geq \eta
\]

for a subsequence of \( \{a_j\} \), which we may take as the whole sequence without loss of generality. Also, since \( \phi \ast K_1 \),

\[
\hat{\phi}(\zeta) = O(|\zeta|^{-2\nu - 1}),
\]

as \( |\zeta| \rightarrow \infty, \zeta = \xi + i\eta \ast \phi^{(n)}, |\eta| \leq B \).

We set now \( \bar{\mu} = \zeta \).

Making use of (4), (5) and (10) we obtain the estimation

\[
\sum_{j \in \mathbb{N}} a_j |\zeta|^{2\nu} |\phi(\zeta - u)| = O(2^{-\mu}).
\]

On the other hand, conditions (8) and (9) imply that, for sufficiently large \( \mu \),

\[
|a_j||\zeta|^{2\nu} |\phi(\zeta)| \geq \frac{\eta}{2}.
\]

This contradicts the convergence (7). Our assertion is thus established.

Remark. The above lemma is a generalization of lemma 1 in [5], which can be obtained by setting \( B = 0 \).

Theorem 1. If a distribution \( S \in \mathcal{E}_1(\mathcal{X}_1 : \mathcal{X}_f) \) is hypoelliptic in \( \mathcal{X}_1 \), then for every \( B \geq 0 \) there are constants \( A \) and \( A \) such that the Fourier transform \( \hat{S} \) of \( S \) satisfies the condition

\[
|\hat{S}(\zeta)| \geq |\zeta|^{-A}, \quad \zeta = \xi + i\eta \ast \phi^{(n)}, |\eta| \leq B, |\xi| > A.
\]

Proof. Suppose that condition (11) is not satisfied. Then there exists a \( B \geq 0 \) and a sequence of points \( \zeta = \xi + i\eta \ast \phi^{(n)} \), defined as in lemma 1, such that

\[
|\hat{S}(\zeta)| \leq |\zeta|^{-A}.
\]

The series

\[
\sum_{j = -\infty}^{\infty} |a_j| \psi(\zeta)
\]

converges in \( K_1 \) to \( \hat{U} \), say. Hence \( U \in \mathcal{X}_1 \), and, by lemma 1, \( U \) is not in \( \mathcal{E}_1 \). But the convolution \( \hat{S} \ast U \) can be transformed according to the formula

\[
\hat{S} \ast U = \hat{S} \ast U = \sum_{j = -\infty}^{\infty} |a_j| \psi(\zeta).
\]
Applying now inequality (12) and once more lemma 1 we conclude that $S \ast U$ is in $\mathcal{E}'_1$. Thus $S$ is not hypoelliptic in $\mathcal{E}'_1$, q.e.d.

If a partial differential operator with constant coefficients, i.e. an operator of the form

$$S = P(D)\delta,$$

where $P(D)$ denotes a polynomial of derivation and $\delta$ the Dirac measure, is hypoelliptic in $\mathcal{E}'_1$, then it is hypoelliptic in $\mathcal{D}'$. This follows from theorem 1 and a theorem of Hörmander ([2], p. 99, theorem 4.1.3).

3. Two lemmas. The following two lemmas are necessary for our investigations in the next section.

Lemma 2. Let $\gamma(\zeta)$ be a function defined in the horizontal strip $V_n$ as

$$\gamma(\zeta) = \begin{cases} 0 & \text{for } \zeta = \Re \zeta e^{i\eta}, \\ 1 & \text{otherwise.} \end{cases}$$

Then, for every $p \in \mathcal{P}$,

$$\int_{t_p} \gamma(\zeta) e^{\xi(\zeta)x} d\zeta = \int_{t_p} \frac{1 - e^{-2\pi i \rho p}}{(2\pi i)^n} \int_{t_1}^{t_n} (e^{\xi(\zeta)x} - e^{-\xi(\zeta)x}) d\zeta,$$

where $t_p = t_1 \times t_2 \times \ldots \times t_n$ and $t_1$ consists of three line segments: from $-a$ to $-a+i\beta_1$, from $-a+i\beta_1$ to $a+i\beta_2$, and from $a+i\beta_2$ to $a$.

Proof. We use the contours $l_1 = l_1 \times \ldots \times l_n$ and $t_p = l_1 \times \ldots \times l_n$, where $l_1$ is the line segment from $-a$ to $-a+i\beta_1$ to $a+i\beta_2$ and $l_n$ consists of two line segments from $-a$ to $-a+i\beta_2$ and from $a+i\beta_2$ to $a$.

The lemma will be proved by induction on the number of variables $n$. For $n = 1$, let $\gamma_1$ be the function of one variable, which corresponds to $\gamma$. Then we have

$$\int_{t_1} \gamma(\zeta) e^{\xi(\zeta)x} d\zeta = 0$$

and

$$\int_{t_1} \gamma(\zeta) e^{\xi(\zeta)x} d\zeta = \frac{1 - e^{-2\pi i \rho p}}{(2\pi i)^n} (e^{\xi(\zeta)x} - e^{-\xi(\zeta)x}),$$

where $p = 1$ or $-1$. Thus equality (13) is satisfied in case of one variable.

In order to perform the induction step we use the points $x', x'', x''', x'''$ as defined in the introduction. We also write e.g. $l_p = l_1 \times l_2 \times l_3 \times l_4 \times l_{n-1}$ and denote by $\gamma_{n-1}$ the function of $n-1$ variables corresponding to $\gamma$. Then one can easily verify that, for every $p \in \mathcal{P}$,

$$\int_{t_p} \gamma(\zeta) e^{\xi(\zeta)x} d\zeta = \int_{t_p} \gamma_{n-1}(\zeta) e^{\xi(\zeta)x} d\zeta = \int_{t_p} \gamma_{n-1}(\zeta) e^{\xi(\zeta)x} d\zeta,$$

Assume now that equality (13) is true for $n-1$ variables, i.e.

$$\int_{t_p} \gamma_{n-1}(\zeta) e^{\xi(\zeta)x} d\zeta = \frac{1 - e^{-2\pi i \rho p}}{(2\pi i)^{n-1}(\xi)^p} \int_{t_1}^{t_{n-1}} (e^{\xi(\zeta)x} - e^{-\xi(\zeta)x}).$$

Then the right-hand side of (14) can be transformed into the form

$$\int_{t_p} \gamma_{n-1}(\zeta) e^{\xi(\zeta)x} d\zeta = \frac{1 - e^{-2\pi i \rho p}}{(2\pi i)^{n-1}(\xi)^p} \int_{t_1}^{t_{n-1}} (e^{\xi(\zeta)x} - e^{-\xi(\zeta)x}),$$

which shows that equality (13) holds also for $n$ variables, q.e.d.

Lemma 3. Let $f(\zeta)$ be a function defined for $\zeta = x + iy \in V_n$, which is analytic for $\xi \in \mathcal{D}' \setminus I_2$, continuous for $\xi \in \mathcal{D}' \setminus I_1$ and vanishes for $\xi \in I_1$. Furthermore, let $l_p, p \in \mathcal{P}$, be the contours from lemma 2. Then consider the function

$$v(x, t) = \sum_{p \in \mathcal{P}} \left[ \frac{e^{\theta p (x - t)}}{\sigma_{2n}(x - t)} - \frac{e^{\theta p t}}{\sigma_{2n}(t)} \right] \int_{t_p} f(\zeta) e^{\xi(\zeta)x} d\zeta,$$

which is analytic for $x \in V_1, c < 1/4b$, and $t \in \mathbb{R}^n$. We assert that

$$v(x, t) = \frac{1}{\sigma_{2n}(x)} \sum_{p \in \mathcal{P}} \sum_{\phi \in \mathcal{O}_n} \phi(\theta p) \times \left[ e^{\theta p (x - t)} - e^{\theta p t} \right] \int_{l_p} f(\zeta + iy \phi) e^{\xi(\zeta)x} d\zeta.$$
where $b_{0j} = b_{0j_1} \times \cdots \times b_{0j_n}$ and $b_{ij}$ is either $b_i$ or the segment of the $x_i$-axis from $-a$ to $a$, depending on whether $q_i = 0$ or $q_i = 1$.

Proof. Let $d_i$ denote the segment of the $x_i$-axis from $-a$ to $a$ and $d = d_1 \times \cdots \times d_n$. We also write $d' = d_1 \times \cdots \times d_{n-1} \times b_{i_n} \times b_{ij}$, etc.

The lemma will be proved again by induction on $n$. In case $n = 1$ we obtain

$$u(z, t) = \frac{1}{\sigma_{0a}(z) \sigma_{0a}(z - t)} \sum_{p, \mu} \left[ e^{-\mu \rho^p_{0a}z} - e^{-\mu \rho^p_{0a}z} \right] \int f(z) e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu,$$

which is the desired formula (13), since $q = 0$.

For the general case of $n$ variables we first observe that

$$u(z, t) = \frac{1}{\sigma_{0a}(z) \sigma_{0a}(z - t)} \times \sum_{p, \mu} \left[ e^{\mu \rho^p_{0a}z} \left( \frac{\sigma_{0a}(z')}{\sigma_{0a}(z)} - \frac{\sigma_{0a}(z')}{\sigma_{0a}(z - t)} \right) \right] \times \times \int f(z') e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu \times \times \left( e^{-\mu \rho^p_{0a}z} - e^{-\mu \rho^p_{0a}z} \right) \int f(z) e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu,$$

identically in $z$ and $t$. Hence we infer that

$$\sum_{p, \mu} \left[ \frac{\sigma_{0a}(z')}{\sigma_{0a}(z)} - \frac{\sigma_{0a}(z')}{\sigma_{0a}(z - t)} \right] \times \times \int f(z') e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu = 0,$$

and consequently

$$\sum_{p, \mu} \left[ e^{\mu \rho^p_{0a}z} \left( \frac{\sigma_{0a}(z')}{\sigma_{0a}(z)} - \frac{\sigma_{0a}(z')}{\sigma_{0a}(z - t)} \right) \right] \times \times \int f(z') e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu = 0.$$

Formula (16) follows immediately by application of the latter equality.

Suppose now that equality (15) is true for $n - 1$ variables. Then we obtain

$$\sum_{p, \mu} \left[ e^{\mu \rho^p_{0a}z} \left( \frac{\sigma_{0a}(z')}{\sigma_{0a}(z)} - \frac{\sigma_{0a}(z')}{\sigma_{0a}(z - t)} \right) \right] \times \times \int f(z') e^{\mu \rho^p_{0a}z_\mu} \, dz_\mu = 0,$$

and therefore

$$\sum_{p, \mu} \left[ e^{-\mu \rho^p_{0a}z} \left( \frac{\sigma_{0a}(z')}{\sigma_{0a}(z)} - \frac{\sigma_{0a}(z')}{\sigma_{0a}(z - t)} \right) \right] \times \times \int f(z') e^{-\mu \rho^p_{0a}z_\mu} \, dz_\mu = 0.$$

Combining (16) with (17) and (18) we conclude that equality (15) holds also for $n$ variables, q.e.d.

Corollary. For every $r, s \in \mathbb{N}^n, c, (s, t)$ satisfies the growth condition

$$\sup_{z \in V, \epsilon} \frac{\sigma_{0a}(z) D_0 D_1 e^{(z, c)}}{\sigma_{0a}(z)} < \infty,$$

where the supremum is taken over all $z \in V$, $\epsilon < 1/|b|$ and $1 \epsilon^c$. 
Condition (19) can be proved by estimating the derivatives of each term of the sum in (15). For example, if \( r = s = 0 \), it is sufficient to show that
\[
\frac{1}{\sigma_{n+1}(r - q^2 - q^2)} \int_{\{z\}} f(z + ibp) e^{i\alpha(z - \alpha)} d\zeta
\]
is bounded for every \( p \in \mathbb{P} \), \( q \in \mathbb{Q} \), and to apply the inequality
\[
\left| \frac{e^{i\alpha(z - \alpha)}}{\sigma_{n+1}(r - q^2)} \right| \leq \sigma_{n+1}(q).
\]

The same argument can be used for arbitrary \( r, s \in \mathbb{N} \). We omit the details of the proof.

4. Hypoelliptic operators in \( \mathcal{X} \). Sufficient condition. We now prove that condition (11) of theorem 1 is also sufficient for a distribution \( S \in \mathcal{C}^{\prime}_{\varepsilon}(\mathcal{X}'; \mathcal{X}') \) to be hypoelliptic in \( \mathcal{X} \). For this purpose we need an appropriate family of parametrix for \( \mathcal{S} \) which we define as follows. Given any \( b > 0 \), we say that \( \mathcal{P} \) is a \( \beta \)-parametrix for \( \mathcal{S} \), if the product \( \sigma_{n+1}\mathcal{P} \) is a bounded distribution and
\[
\mathcal{S} \ast \mathcal{P} = \delta - \mathcal{W},
\]
where \( \mathcal{W} \) is a \( \mathcal{C}^{\mathbb{N}} \)-function such that
\[
\sup_{x \in \mathbb{R}^{n}} \sigma_{n+1}(x)|\mathcal{W}(x)| < \infty
\]
for all \( r \in \mathbb{N} \).

**Theorem 2.** If \( \mathcal{S} \in \mathcal{C}^{\prime}_{\varepsilon}(\mathcal{X}'; \mathcal{X}') \) satisfies condition (11), then for every \( b > 0 \) there exists a \( \beta \)-parametrix for \( \mathcal{S} \).

**Proof.** By assumption, for every \( b > 0 \) there is an \( a > 0 \) and an \( \alpha \in \mathbb{R} \) such that
\[
|S(\zeta)| \geq |\zeta|^\alpha,
\]
when \( \zeta = \xi + i\eta \in \mathcal{V} \) and \( \xi \in \mathbb{R}^n \setminus I_\alpha \). We define the function \( f \) in \( \mathcal{V} \) by the formula
\[
f(\zeta) = \begin{cases} 
0 & \text{for } \xi \in I_\alpha, \\
1 & \text{for } \xi \in \mathbb{R}^n \setminus I_\alpha,
\end{cases}
\]
where \( \mu \in \mathbb{N} \) is chosen so large that
\[
|f(\zeta)| \leq M|\zeta|^{-\mu-1}
\]
for some constant \( M \). Condition (22) guarantees that such a \( \mu \) exists. Then the function
\[
\tilde{g}(\xi) = \sum_{\mathbb{P}} f(\xi + ibp)
\]
is integrable over \( \mathbb{R}^n \). Its inverse Fourier transform \( \tilde{g}(x) \) is given by the formula
\[
\tilde{g}(x) = \sum_{\mathbb{P}} \int_{\mathbb{R}^n} f(\zeta) e^{i\alpha(x - \alpha)} d\zeta;
\]
\( \tilde{g}(x) \) is continuous and bounded in \( \mathbb{R}^n \).

But \( f(\zeta) \) is analytic for \( \xi \in \mathbb{R}^n \setminus I_\alpha \), continuous for \( \xi \in \mathbb{R}^n \setminus I_\alpha \) and satisfies condition (24). Therefore, by repeated application of Cauchy's integral theorem, integration in (23) along the lines \( \xi + ibp, \xi \neq \xi_j \in \mathbb{R}^n \setminus I_\alpha \), can be replaced by integration along the real lines and the quadrangles with vertices at \( -a, a \) and \( a + ibp, a \) in the indicated direction. It also has to be observed that, except for the integral over \( \mathbb{R}^n \), integration along a real line can be reduced to the segment from \( -a \) to \( a \), again by Cauchy's integral theorem. This procedure leads to the formula
\[
\frac{\tilde{g}(x)}{\sigma_{n+1}(x)} = \int_{\mathbb{R}^n} f(\zeta) e^{i\alpha(x - \alpha)} d\zeta + \sum_{\mathbb{P}} \frac{\sigma_{n+1}(x)}{\sigma_{n+1}(\xi)} \int_{\mathbb{R}^n} f(\zeta) e^{i\alpha(x - \alpha)} d\zeta,
\]
where the contours \( \mathbb{P} \) are those defined in lemma 2.

We assert that
\[
P = \left( -\frac{d}{4\pi^2} \right)^{\gamma} \tilde{g}(x)
\]
is a \( \beta \)-parametrix for \( \mathcal{S} \). In fact, \( P \) satisfies the growth condition for a \( \beta \)-parametrix, i.e. \( \sigma_{n+1}P \) is a bounded distribution. Furthermore, by virtue of (26), \( P \) is a sum of the distribution

\[
P_1 = \left( -\frac{d}{4\pi^2} \right)^{\gamma} \tilde{f},
\]
where \( \tilde{f} \) is the inverse Fourier transform of \( f \), and the function
\[
P_1(x) = \left( -\frac{d}{4\pi^2} \right)^{\gamma} \int_{\mathbb{R}^n} \frac{\sigma_{n+1}(x)}{\sigma_{n+1}(\xi)} \int_{\mathbb{R}^n} f(\zeta) e^{i\alpha(x - \alpha)} d\zeta,
\]
which belongs to \( \mathcal{S} \mathcal{X} \).

Now, in view of (23) and the definition of \( \gamma(\zeta) \) in lemma 2,
\[
(\mathcal{S} \ast P_1)(\zeta) = \mathcal{S}(\zeta + \xi) f(\xi) = \gamma(\zeta),
\]
and so

$$S \ast P = \delta - W_t,$$

where

$$W_t(x) = \int_{\mathbb{R}^n} e^{\alpha x, \zeta} d\zeta = \frac{1}{(2\pi)^{n/2}} \prod_{j=1}^{n} (e^{\omega_{-i\alpha_j}} - e^{\omega_{i\alpha_j}}).$$

Next we define the function $h(x, t)$ on $\mathbb{R}^n$ as

$$h(x, t) = \sum_{j \in \mathbb{N}} \frac{e^{\alpha x, \zeta}}{\omega_{-i\alpha_j}} \int_{\mathbb{R}} f(\zeta) e^{\alpha \omega j - \omega x} d\zeta.$$ 

For any fixed $x \in \mathbb{R}^n$, $h(x, t)$ is in $\mathcal{S}'(\mathbb{R})$ as a function of $t$. Moreover,

$$-\frac{1}{4\pi^2} \frac{d}{dt} h(x, t) = \sum_{j \in \mathbb{N}} \frac{e^{\alpha x, \zeta}}{\omega_{-i\alpha_j}} \int_{\mathbb{R}} \gamma(\zeta) e^{\alpha \omega j - \omega x} d\zeta$$

$$= W_t(x) - \frac{1}{(2\pi)^{n/2} \omega_{-i\alpha_j}} \prod_{j=1}^{n} (e^{\omega_{-i\alpha_j}} - e^{\omega_{i\alpha_j}}),$$

by equality (23) and lemma 2.

On the other hand,

$$S \ast P(x) = \left( -\frac{1}{4\pi^2} \frac{d}{dt} h(x, t) + v(x, t) \right),$$

where $v(x, t)$ is the function from lemma 3.

But

$$W_t(x) = \left( -\frac{1}{4\pi^2} \frac{d}{dt} h(x, t) \right),$$

is a $C^\infty$-function, which satisfies condition (21), by the corollary following lemma 3. Thus from (28), (29) and (30) we conclude that $P$ satisfies equation (20) with the function

$$W(x) = W_t(x) - \frac{1}{(2\pi)^{n/2} \omega_{-i\alpha_j}} \prod_{j=1}^{n} (e^{\omega_{-i\alpha_j}} - e^{\omega_{i\alpha_j}}),$$

which has the desired properties.

**Theorem 3.** If $S \ast \epsilon_0(\mathcal{F}', \mathcal{F}')$ and, for every $b > 0$, there exists a $b$-parametrix for $S$ then $S$ is hypoelliptic in $\mathcal{F}'$.

**Proof.** Assume that $U$ is a solution in $\mathcal{F}'$ of the equation

$$S \ast U = U',$$

where $U \in \mathcal{S}'$. Then there exists a $k \in \mathbb{N}$ such that $\frac{1}{2\pi^k} U$ is a bounded distribution and

$$\sup_{x \in \mathbb{R}^n} \frac{1}{2\pi^k} |U(x)| < \infty$$

for every $x \in \mathbb{R}^n$.

Let now $P$ be a $b$-parametrix for $S$, $b > h$, and $W$ the corresponding function in (20). Note that $P$ and $W$ may not be in $\mathcal{S}'(\mathcal{F}', \mathcal{F}')$.

Still we can write

$$U = U \ast \delta = U \ast (S \ast P) + U \ast W,$$

where the convolutions with $U$ on the right-hand side are well defined (see section 1). Moreover,

$$U \ast (S \ast P) = (U \ast S) \ast P = P \ast P$$

and the last term belongs to $\mathcal{S}'$. Also $U \ast W$ is obviously in $\mathcal{S}'$. Thus $U$ is, in fact, in $\mathcal{S}'$, q.e.d.

Combining theorem 2 and theorem 3 we obtain

**Theorem 4.** A distribution $S \ast \epsilon_0(\mathcal{F}', \mathcal{F}')$ satisfying condition (11)

is hypoelliptic in $\mathcal{F}'$.

In view of theorem 1 we can now state the following corollary:

**Corollary.** Condition (11) is necessary and sufficient for a distribution $S \ast \epsilon_0(\mathcal{F}', \mathcal{F}')$ to be hypoelliptic in $\mathcal{F}'$.

References


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