[5] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canadian J. Math. 1953, p. 129-173.

[6] R. S. Phillips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), p. 516-541.

[7] H. Rosenthal, *Projections onto translation invariant subspaces of $L^p(G)$*, Memoirs Amer. Math. Soc. 63 (1966).

[8] W. Rudin, *Fourier analysis on groups*, 1962.

[9] G. Šilov, *Homogeneous rings of functions*, A. M. S. Translation 92 (1953).

[10] B. Wells, *Weak compactness of measures*, Proc. Amer. Math. Soc. (to appear).

UNIVERSITY OF CALIFORNIA, BERKELEY
UNIVERSITY OF OREGON, EUGENE

Reçu par la Rédaction le 28. 12. 1967

STUDIA MATHEMATICA, T. XXXII. (1969)

Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (II)

by

Z. ZIELEŹNY (Wrocław)

In part I of this work (see [5]) we showed how to define in a general manner hypoelliptic and entire elliptic convolution operators in subspaces of the space of distributions. We also characterized hypoelliptic and entire elliptic convolution operators in the space $\mathscr{S}'$ of tempered distributions.

The purpose of this paper is to study hypoelliptic convolution operators in the space $\mathscr{K}_1'$ $(=\Lambda_\infty)$ of distributions of exponential growth introduced by Sebastião e Silva [4] and Hasumi [1].

The space $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ of convolution operators in $\mathscr{K}_1'$ (which is a space of distributions) was characterized in [1] and its topological properties were investigated in [6].

Using the notation of [5] we define $\mathscr{E}\mathscr{K}_1'$ to be the set of all C^∞-functions $f\in\mathscr{K}_1'$ such that, for every $S\in\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$, the convolution $S*f$ is a C^∞-function and $S\to S*f$ is a continuous mapping from $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ into the space $\mathscr{E}$ of all C^∞-functions in R^n. Then a distribution $S\in\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ is said to be *hypoelliptic* in $\mathscr{K}_1'$, if every solution $U\in\mathscr{K}_1'$ of the convolution equation

$$S*U = F \tag{1}$$

is in $\mathscr{E}\mathscr{K}_1'$, when $F\in\mathscr{E}\mathscr{K}_1'$; in that case equation (1) is also called *hypoelliptic* in $\mathscr{K}_1'$.

As a supplement of the standard notation (see [3] and [5]) we use N^n as the set of all points in R^n, whose coordinates are non-negative integers; we write N and R instead of N^1 and R^1 respectively. Furthermore, we denote by P^n (Q^n resp.) the set of all points $p=(p_1,\ldots,p_n)$ ($q=(q_1,\ldots,q_n)$ resp.) such that $p_j=1$ or -1 ($q_j=1$ or 0 resp.). In particular, Q^n contains the points $\mathbf{1}=(1,1,\ldots,1)$ and $\mathbf{0}=(0,0,\ldots,0)$.

For a point $x=(x_1,\ldots,x_n)\in R^n$ we sometimes write $x=(x',x_n)$, where $x'=(x_1,\ldots,x_{n-1})\in R^{n-1}$. Also, for $x=(x_1,\ldots,x_n)$ and $\xi=(\xi_1,\ldots,\xi_n)$ in R^n we use the product $x\xi=(x_1\xi_1,\ldots,x_n\xi_n)$ beside the scalar product $x\cdot\xi = x_1\xi_1+\ldots+x_n\xi_n$. The same notation applies to points in C^n, which are denoted by $z = x+iy$ or $\zeta = \xi+i\eta$, $x, y, \xi, \eta\in R^n$.

Given an $a\in R$, $a>0$, I_a stands for the open cube in R^n with center at the origine and side $2a$, i.e.

$$I_a = \{x = (x_1, \ldots, x_n)\in R^n\colon |x_j| < a,\ j = 1, \ldots, n\};$$

$\bar{I}_a$ is the closure of I_a.

A horizontal strip in C^n around R^n of width $b>0$ is defined as

$$V_b = \{z = (z_1, \ldots, z_n)\in C^n : |\mathscr{I} z_j| < b,\ j = 1, \ldots, n\}.$$

We constantly make use of the function

$$\sigma_b(z) = \sum_{p\in P^n} e^{bp.z} = \prod_{j=1}^{n} (e^{bz_j}+e^{-bz_j}),$$

where $z = (z_1, \ldots, z_n)\in C^n$ *and* $b\in R$.

1. The basic spaces. For the convenience of the reader we characterize briefly the basic spaces used in this paper.

$\mathscr{K}_1$ is the space of all C^∞-functions φ in R^n such that $\sigma_k(x)D^r\varphi(x)$ is bounded in R^n, for every $k\in N$ and $r\in N^n$. The topology in $\mathscr{K}_1$ is defined by the system of semi-norms

$$v_k(\varphi) = \sup_{x\in R^n, |r|\leqslant k} \sigma_k(x)|D^r\varphi(x)|, \qquad k = 0, 1, \ldots$$

Then $\mathscr{K}_1$ is a Fréchet nuclear space ([1], proposition 1).

The dual $\mathscr{K}_1'$ of $\mathscr{K}_1$ is the space of distributions of exponential growth. A distribution T is in $\mathscr{K}_1'$ if and only if T can be represented in the form

$$T = D^r[\sigma_\mu(x)f(x)],$$

where $r\in N^n$, $\mu\in R$ and f is a bounded, continuous function on R^n ([1], proposition 3). Under the strong topology $\mathscr{K}_1'$ is a complete Montel space.

The space $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ of convolution operators in $\mathscr{K}_1'$ can be characterized as follows ([1], proposition 9). A distribution S is in $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ if and only if, for every $k\in N$, S can be represented as a finite sum of derivatives of continuous functions, whose products with $\sigma_k(x)$ are bounded in R^n. The topology of $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is that induced in $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ by the space $\mathscr{L}_b(\mathscr{K}_1', \mathscr{K}_1')$; it makes $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ into a complete Montel space (see [6]).

Note that the convolution $S*T$ can be defined even if neither S nor T is in $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$. If e.g. for $\mu<\nu$, $\sigma_\nu S$ and $\frac{1}{\sigma_\mu}T$ are bounded distri-

butions, then one can find continuous functions F_r, $r\in N^n$, $|r|\leqslant k$, and G such that

$$S = \sum_{|r|\leqslant k} D^r F_r, \qquad T = D^s G \tag{2}$$

and the convolutions $F_r * G$ exist in the usual sense. Then we set

$$S*T = \sum_{|r|\leqslant k} D^{r+s}(F_r*G).$$

One can show that the convolution $S*T$ so defined does not depend on the representation (2).

The set $\mathscr{E}\mathscr{K}_1'$ can be identified with the dua l$\mathscr{O}_c(\mathscr{K}_1' : \mathscr{K}_1')$ of $\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ similarly as in the case of the set $\mathscr{E}\mathscr{S}'$ (see [5], p. 322). Thus $\mathscr{E}\mathscr{K}_1'$ consists of all C^∞-functions f such that one can find a $k\in N$ satisying the condition

$$D^r f(x) = O\big(\sigma_k(x)\big)$$

as $|x|\to\infty$, for all $r\in N^n$ ([6], theorem 10).

For a function $\varphi\in\mathscr{K}_1$, its Fourier transform

$$\hat{\varphi}(\xi) = \int_{R^n} e^{-2\pi i\xi.x}\varphi(x)\,dx$$

can be extended over C^n as an entire function such that

$$w_k(\hat{\varphi}) = \sup_{\zeta\in V_k}(1+|\zeta|)^k|\hat{\varphi}(\zeta)| < \infty, \qquad k = 1, 2, \ldots$$

The space K_1 of all entire functions with the latter property corresponds to $\mathscr{K}_1$ under the Fourier transform. If the topology in K_1 is defined by the system of semi-norms w_k, $k = 1, 2, \ldots$, then the Fourier transform is a topological isomorphism of $\mathscr{K}_1$ onto K_1 ([1], proposition 4).

The dual K_1' of K_1 is the space of Fourier transforms of distributions from $\mathscr{K}_1'$. For a distribution $T\in\mathscr{K}_1'$ its Fourier transform $\hat{T}$ is defined by the Parseval equation

$$\hat{T}_\xi\cdot\varphi(\xi) = T_x\cdot\varphi(-x).$$

K_1' is provided with the strong topology. Then the Fourier transform is a topological isomorphism of $\mathscr{K}_1'$ onto K_1'.

The Fourier transform $\hat{S}$ of a distribution $S\in\mathscr{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ is a C^∞-function extendable over C^n as an entire function; moreover, for every $k\in N$ there exists an $l\in N$ such that

$$\sup_{\zeta\in V_k}\frac{|\hat{S}(\zeta)|}{(1+|\zeta|)^l} < \infty$$

(see [1], propositions 8 and 9, or [6], theorem 3). Also, for $S \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ and $T \in \mathcal{K}'_1$ we have the formula

$$\widehat{S * T} = \hat{S}\hat{T},$$

where the product on the right-hand side is well defined in K'_1.

2. Hypoelliptic operators in $\mathcal{K}'_1$. Necessary condition. We prove a necessary condition for a convolution operator $S \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ to be hypoelliptic in $\mathcal{K}'_1$. The proof is based on an idea similar to that used in [5] for convolution operators in $\mathcal{S}'$. We begin with a lemma.

LEMMA 1. *Let T be a distribution, whose Fourier transform $\hat{T}$ is of the form*

$$\hat{T} = \sum_{j=1}^{\infty} a_j \delta_{({}_j\zeta)}, \tag{3}$$

where the ${}_j\zeta = {}_j\xi + {}_j\eta \in C^n$ satisfy conditions

$$|{}_j\zeta| > 2|{}_{j-1}\zeta| > 2^j, \qquad |{}_j\eta| \leqslant B, \tag{4}$$

and a_j are complex numbers such that

$$a_j = O(|{}_j\zeta|^{\mu}) \tag{5}$$

for some $\mu \in N$; then the series in (3) *converges in K'_1. We assert that $T \in \mathcal{E}\mathcal{K}'_1$ if and only if*

$$a_j = o(|{}_j\zeta|^{-\nu}), \tag{6}$$

for every $\nu \in N$.

Proof. By virtue of equality (3) and condition (5),

$$T = \sum_{j=1}^{\infty} a_j e^{2\pi i x \cdot {}_j\zeta},$$

where the series converges in $\mathcal{K}'_1$. If the coefficients a_j satisfy condition (6), then the last series and all its term-by-term derivatives converge uniformly in R^n on dividing by $e^{B|x|}$. Consequently T is in $\mathcal{E}\mathcal{K}'_1$.

Conversely, assume that T is a function from $\mathcal{E}\mathcal{K}'_1$. Then, for every $\nu \in N$ and every $\varphi \in \mathcal{K}_1$,

$$e^{2\pi i u \cdot x} \Delta^{\nu} T_x \cdot \varphi(-x) \to 0,$$

as $|u| \to \infty$, $u \in C^n$, $|\mathcal{I} u| \leqslant B$; Δ^{ν} is the iterated Laplace operator. Hence, passing to the Fourier transform, we see that

$$\tau_u[(\zeta\cdot\zeta)^{\nu}\hat{T}_{\zeta}]\cdot\hat{\varphi}(\zeta) = \sum_{j=1}^{\infty} a_j ({}_j\zeta\cdot{}_j\zeta)^{\nu}\hat{\varphi}({}_j\zeta - u) \to 0, \tag{7}$$

as $|u| \to \infty$, $u \in C^n$, $|\mathcal{I} u| \leqslant B$. We fix a function $\varphi \in \mathcal{K}_1$ such that

$$|\hat{\varphi}(0)| \geqslant 1. \tag{8}$$

Suppose now that condition (6) is not satisfied. Then there is a $\varrho > 0$ and a $\nu_0 \in N$ such that

$$|{}_j\zeta|^{2\nu_0}|a_j| \geqslant \varrho \tag{9}$$

for a subsequence of $\{a_j\}$, which we may take as the whole sequence without loss of generality. Also, since $\hat{\varphi} \in K_1$,

$$\hat{\varphi}(\zeta) = O(|\zeta|^{-\mu-2\nu_0-1}), \tag{10}$$

as $|\zeta| \to \infty$, $\zeta = \xi + i\eta \in C^n$, $|\eta| \leqslant B$.

We set now ${}_ju = {}_j\zeta$. Making use of (4), (5) and (10) we obtain the estimation

$$\sum_{\substack{j=1\\ j\neq k}}^{\infty} a_j ({}_j\zeta\cdot{}_j\zeta)^{\nu_0}\hat{\varphi}({}_j\zeta - {}_ku) = O(2^{-k}).$$

On the other hand, conditions (8) and (9) imply that, for sufficiently large k,

$$|a_k|\,|{}_k\zeta\cdot{}_k\zeta|^{\nu_0}|\hat{\varphi}(0)| \geqslant \frac{\varrho}{2}.$$

This contradicts the convergence (7). Our assertion is thus established.

Remark. The above lemma is a generalization of lemma 1 in [5], which can be obtained by setting $B = 0$.

THEOREM 1. *If a distribution $S \in \mathcal{O}'_c(\mathcal{K}'_1 : \mathcal{K}'_1)$ is hypoelliptic in $\mathcal{K}'_1$, then for every $B \geqslant 0$ there are constants a and A such that the Fourier transform $\hat{S}$ of S satisfies the condition*

$$|\hat{S}(\zeta)| \geqslant |\zeta|^{a} \quad \textit{for } \zeta = \xi + i\eta \in C^n,\ |\eta| \leqslant B,\ |\xi| \geqslant A. \tag{11}$$

Proof. Suppose that condition (11) is not satisfied. Then there exists a $B \geqslant 0$ and a sequence of points ${}_j\zeta = {}_j\xi + i_j\eta \in C^n$, defined as in lemma 1, such that

$$|\hat{S}({}_j\zeta)| < |{}_j\zeta|^{-j}. \tag{12}$$

The series

$$\sum_{j=1}^{\infty} \delta_{({}_j\zeta)}$$

converges in K'_1 to $\hat{U}$, say. Hence $U \in \mathcal{K}'_1$ and, by lemma 1, U is not in $\mathcal{E}\mathcal{K}'_1$. But the convolution $S * U$ can be transformed according to the formula

$$\widehat{S * U} = \hat{S}\hat{U} = \sum_{j=1}^{\infty} \hat{S}({}_j\zeta)\,\delta_{({}_j\zeta)}.$$

Applying now inequality (12) and once more lemma 1 we conclude that $S*U$ is in $\mathcal{E}\mathcal{K}'_1$. Thus S is not hypoelliptic in $\mathcal{K}'_1$, q.e.d.

If a partial differential operator with constant coefficients, i.e. an operator of the form

$$S = P(D)\delta,$$

where $P(D)$ denotes a polynomial of derivation and δ the Dirac measure, is hypoelliptic in $\mathcal{K}'_1$, then it is hypoelliptic in $\mathcal{D}'$. This follows from theorem 1 and a theorem of Hörmander ([2], p. 99, theorem 4.1.3).

3. Two lemmas. The following two lemmas are necessary for our investigations in the next section.

LEMMA 2. *Let $\gamma(\zeta)$ be a function defined in the horizontal strip V_b as*

$$\gamma(\zeta) = \begin{cases} 0 & \text{for } \xi = \Re\zeta \in I_a, \\ 1 & \text{otherwise}. \end{cases}$$

Then, for every $p \in P^n$,

$$\int_{l_p} \gamma(\zeta) e^{2\pi i\zeta\cdot x} d\zeta = \frac{1-e^{-2\pi bp\cdot x}}{(2\pi i)^n x^1} \prod_{j=1}^{n} (e^{2\pi iax_j} - e^{-2\pi iax_j}), \tag{13}$$

where $l_p = l_{p_1}\times l_{p_2}\times\ldots\times l_{p_n}$ and l_{p_j} consists of three line segments: from $-a$ to $-a+ibp_j$, from $-a+ibp_j$ to $a+ibp_j$ and from $a+ibp_j$ to a.

Proof. We use the contours $l_p^1 = l_{p_1}^1\times\ldots\times l_{p_n}^1$ and $l_p^2 = l_{p_1}^2\times\ldots\times l_{p_j}^2$, where $l_{p_n}^1$ is the line segment from $-a+ibp_j$ to $a+ibp_j$ and $l_{p_j}^2$ consists of two line segments from $-a$ to $-a+ibp_j$ and from $a+ibp_j$ to a.

The lemma will be proved by induction on the number of variables n. For $n = 1$, let γ_1 be the function of one variable, which corresponds to γ. Then we have

$$\int_{l_p^1} \gamma_1(\zeta) e^{2\pi i\zeta\cdot x} d\zeta = 0$$

and

$$\int_{l_p^2} \gamma_1(\zeta) e^{2\pi i\zeta\cdot x} d\zeta = \frac{1-e^{-2\pi bp\cdot x}}{2\pi ix} (e^{2\pi iax} - e^{-2\pi iax}),$$

where $p = 1$ or -1. Thus equality (13) is satisfied in case of one variable.

In order to perform the induction step we use the points x', ζ', p', $\mathbf{1}' \in C^{n-1}$ as defined in the introduction. We also write e.g. $l_{p'} = l_{p_1}\times l_{p_2}\times\ldots\times l_{p_{n-1}}$ and denote by γ_{n-1} the function of $n-1$ variables corresponding to γ. Then one can easily verify that, for every $p \in P^n$,

$$\int_{l_p} \gamma(\zeta) e^{2\pi i\zeta\cdot x} d\zeta = \int_{l_{p'}} \gamma_{n-1}(\zeta') e^{2\pi i\zeta'\cdot x'} d\zeta' \int_{l_{p_n}^1} e^{2\pi i\zeta_n x_n} d\zeta_n + \int_{l_{p'}^1} e^{2\pi i\zeta'\cdot x'} d\zeta' \int_{l_{p_n}^2} e^{2\pi i\zeta_n x_n} d\zeta. \tag{14}$$

Assume now that equality (13) is true for $n-1$ variables, i.e.

$$\int_{l_{p'}} \gamma_{n-1}(\zeta') e^{2\pi i\zeta'\cdot x'} d\zeta' = \frac{1-e^{-2\pi bp'\cdot x'}}{(2\pi i)^{n-1}(x')^{1'}} \prod_{j=1}^{n-1} (e^{2\pi iax_j} - e^{-2\pi iax_j}).$$

Then the right-hand side of (14) can be transformed into the form

$$\frac{1-e^{-2\pi bp'\cdot x'}}{(2\pi i)^{n-1}(x')^{1'}} \prod_{j=1}^{n-1} (e^{2\pi iax_j} - e^{-2\pi iax_j}) \frac{e^{-2\pi bp_n x_n}}{2\pi ix_n} (e^{2\pi iax_n} - e^{-2\pi iax_n}) +$$

$$+ \frac{1}{(2\pi i)^{n-1}(x')^{1'}} \prod_{j=1}^{n-1} (e^{2\pi iax_j} - e^{-2\pi iax_j}) \frac{1-e^{-2\pi bp_n x_n}}{2\pi ix_n} (e^{2\pi iax_n} - e^{-2\pi iax_n})$$

$$= \frac{1-e^{-2\pi bp\cdot x}}{(2\pi i)^n x^1} \prod_{j=1}^{n} (e^{2\pi iax_j} - e^{-2\pi iax_j}),$$

which shows that equality (13) holds also for n variables, q.e.d.

LEMMA 3. *Let $f(\zeta)$ be a function defined for $\zeta = \xi+i\eta \in V_b$, which is analytic for $\xi \in R^n\setminus \bar{I}_a$, continuous for $\xi \in R^n\setminus I_a$ and vanishes for $\xi \in I_a$. Furthermore, let l_p, $p \in P^n$, be the contours from lemma 2. Then consider the function*

$$v(z,t) = \sum_{p\in P^n} \left[\frac{e^{2\pi bp\cdot(z-t)}}{\sigma_{2\pi b}(z-t)} - \frac{e^{2\pi bp\cdot z}}{\sigma_{2\pi b}(z)}\right] \int_{l_p} f(\zeta) e^{2\pi i\zeta\cdot(z-t)} d\zeta,$$

which is analytic for $z \in V_c$, $c < 1/4b$, and $t \in R^n$. We assert that

$$v(z,t) = \frac{1}{\sigma_{2\pi b}(z)\sigma_{2\pi b}(z-t)} \sum_{p\in P^n} \sum_{q\in Q^n\setminus\{1\}} e^{2\pi bpq\cdot z} \times \tag{15}$$

$$\times [e^{-2\pi bp(1-q)\cdot t} - e^{2\pi bp(1-q)\cdot t}] \int_{l_{p,q}} f(\zeta+ibpq) e^{2\pi i\zeta\cdot(z-t)} d\zeta;$$

$l_{p,q} = l_{p_1,q_1} \times \ldots \times l_{p_n,q_n}$ and l_{p_j,q_j} is either l_{p_j} or the segment of the x_j-axis from $-a$ to a, depending on whether $q_j = 0$ or $q_j = 1$.

Proof. Let d_j denote the segment of the x_j-axis from $-a$ to a and $d = d_1 \times \ldots \times d_n$. We also write $d' = d_1 \times \ldots \times d_{n-1}$, $l_{p',q'} = l_{p_1,q_1} \times \ldots \times l_{p_{n-1},q_{n-1}}$, etc.

The lemma will be proved again by induction on n. In case $n = 1$ we obtain

$$v(z,t) = \frac{1}{\sigma_{2\pi b}(z)\,\sigma_{2\pi b}(z-t)} \sum_{p=\pm 1} [e^{-2\pi b p\cdot t} - e^{2\pi b p\cdot t}] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta,$$

which is the desired formula (15), since $q = 0$.

For the general case of n variables we first observe that

$$(16)\quad v(z,t) = \frac{2^{2n-1}}{\sigma_{2\pi b}(0', z_n)\,\sigma_{2\pi b}(0', z_n - t_n)} \times$$
$$\times \sum_{p\in P^n} \Big\{ e^{2\pi b p_n z_n} \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \times$$
$$\times \int_{l_{p'}\times d_n} f(\zeta', \zeta_n + ibp_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta + e^{-2\pi b p_n t_n} \times$$
$$\times \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta + \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \times$$
$$\times [e^{-2\pi b p_n t_n} - e^{2\pi b p_n t_n}] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta \Big\}.$$

In fact, by Cauchy's integral theorem,

$$\int_{l_{p'}\times l^2_{p_n}} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta = \int_{d'\times l^2_{p_n}} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta$$

identically in z and t. Hence we infer that

$$\sum_{p\in P^n} \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \int_{l_{p'}\times l^2_{p_n}} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta \equiv 0$$

and consequently

$$\sum_{p\in P^n} e^{4\pi b p_n z_n - 2\pi b p_n t_n} \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta$$
$$= \sum_{p\in P^n} e^{2\pi b p_n z_n} \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \int_{l_{p'}\times d_n} f(\zeta', \zeta_n + ibp_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta.$$

Formula (16) follows immediately by application of the latter equality.

Suppose now that equality (15) is true for $n-1$ variables. Then we obtain

$$(17)\quad \sum_{p\in P^n} \Big\{ e^{2\pi b p_n z_n} \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \times$$
$$\times \int_{l_{p'}\times d_n} f(\zeta', \zeta_n + ibp_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta + e^{-2\pi b p_n t_n} \times$$
$$\times \Big[\frac{e^{2\pi b p'\cdot(z'-t')}}{\sigma_{2\pi b}(z'-t', 0)} - \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} \Big] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta \Big\}$$
$$= \frac{2}{\sigma_{2\pi b}(z', 0)\,\sigma_{2\pi b}(z'-t', 0)} \sum_{p\in P^n} \sum_{q'\in Q^{n-1}\setminus\{1'\}} \{e^{2\pi b(p'q'\cdot z' + p_n z_n)} \times$$
$$\times [e^{-2\pi b p'(1'-q')\cdot t'} - e^{2\pi b p'(1'-q')\cdot t'}] \times$$
$$\times \int_{l_{p',q'}\times d_n} f(\zeta' + ibp'q', \zeta_n + ibp_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta +$$
$$+ e^{2\pi b(p'q'\cdot z' - p_n t_n)} [e^{-2\pi b p'(1'-q')\cdot t'} - e^{2\pi b p'(1'-q')\cdot t'}] \times$$
$$\times \int_{l_{p',q'}\times l_{p_n}} f(\zeta' + ibp'q', \zeta_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta.$$

Moreover,

$$\sigma_{2\pi b}(z'-t', 0)\, e^{2\pi b p'\cdot z'}$$
$$= 2 \sum_{q'\in Q^{n-1}} \exp\{2\pi b p'q'\cdot z' + 2\pi b p'q'\cdot(z'-t') + 2\pi b p'(1'-q')\cdot t'\}$$

and therefore

$$(18)\quad \sum_{p\in P^n} \frac{e^{2\pi b p'\cdot z'}}{\sigma_{2\pi b}(z', 0)} [e^{-2\pi b p_n t_n} - e^{2\pi b p_n t_n}] \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta$$
$$= \frac{2}{\sigma_{2\pi b}(z', 0)\,\sigma_{2\pi b}(z'-t', 0)} \sum_{p\in P^n} \sum_{q'\in Q^{n-1}} e^{2\pi b p'q'\cdot z' + 2\pi b p'(1'-q')\cdot t'} \times$$
$$\times [e^{-2\pi b p_n t_n} - e^{2\pi b p_n t_n}] \int_{l_{p',q'}\times l_{p_n}} f(\zeta' + ibp'q', \zeta_n)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta.$$

Combining (16) with (17) and (18) we conclude that equality (15) holds also for n variables, q.e.d.

COROLLARY. *For every $r, s \in N^n$, $v(z,t)$ satisfies the growth condition*

$$(19)\quad \sup \frac{|\sigma_{2\pi b}(z)\, D_z^r D_t^s v(z,t)|}{\sigma_{2\pi b}(t)} < \infty,$$

where the supremum is taken over all $z \in V_c$, $c < 1/4b$, and $t \in R^n$.

Condition (19) can be proved by estimating the derivatives of each term of the sum in (15). For example, if $r = s = 0$, it is sufficient to show that

$$\frac{1}{\sigma_{2\pi b}(z-qz-t+qt)} \int_{l_{p,q}} f(\zeta+ibpq)\, e^{2\pi i\zeta\cdot(z-t)}\, d\zeta$$

is bounded for every $p \in P^n$, $q \in Q^n$, and to apply the inequality

$$\left|\frac{e^{2\pi bpq\cdot z}}{\sigma_{2\pi b}(qz-qt)}\right| \leqslant \sigma_{2\pi b}(qt).$$

The same argument can be used for arbitrary $r, s \in N^n$. We omit the details of the proof.

4. Hypoelliptic operators in $\mathscr{K}_1'$. Sufficient condition. We now prove that condition (11) of theorem 1 is also sufficient for a distribution $S \in \mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ to be hypoelliptic in $\mathscr{K}_1'$. For this purpose we need an appropriate family of parametrices for S, which we define as follows. Given any $b > 0$, we say that P is a *b-parametrix* for S, if the product $\sigma_{2\pi b}P$ is a bounded distribution and

$$S*P = \delta - W, \tag{20}$$

where W is a C^∞-function such that

$$\sup_{\xi\in R^n} \sigma_{2\pi b}(\xi)\,|D^r W(\xi)| < \infty \tag{21}$$

for all $r \in N^n$.

THEOREM 2. *If $S \in \mathcal{O}_c'(\mathscr{K}_1' : \mathscr{K}_1')$ satisfies condition* (11), *then for every $b > 0$ there exists a b-parametrix for S.*

Proof. By assumption, for every $b > 0$ there is an $a > 0$ and an $a \in R$ such that

$$|\hat{S}(\zeta)| \geqslant |\zeta|^a, \tag{22}$$

when $\zeta = \xi + i\eta \in V_b$ and $\xi \in R^n \setminus I_a$. We define the function f in V_b by the formula

$$f(\zeta) = \begin{cases} 0 & \text{for } \xi \in I_a, \\ \dfrac{1}{\hat{S}(\zeta)(\zeta\cdot\zeta)^\mu} & \text{for } \xi \in R^n \setminus I_a, \end{cases} \tag{23}$$

where $\mu \in N$ is chosen so large that

$$|f(\zeta)| \leqslant M|\xi|^{-n-1} \tag{24}$$

for some constant M. Condition (22) guarantees that such a μ exists. Then the function

$$g(\xi) = \sum_{p\in P^n} f(\xi + ibp)$$

is integrable over R^n. Its inverse Fourier transform $\tilde{g}(x)$ is given by the formula

$$\tilde{g}(x) = \sum_{p\in P^n} e^{2\pi bp\cdot x} \int_{-\infty+ibp}^{\infty+ibp} f(\zeta)\, e^{2\pi i\zeta\cdot x}\, d\zeta; \tag{25}$$

$\tilde{g}(x)$ is continuous and bounded in R^n.

But $f(\zeta)$ is analytic for $\xi \in R^n \setminus \bar{I}_a$, continuous for $\xi \in R^n \setminus I_a$ and satisfies condition (24). Therefore, by repeated application of Cauchy's integral theorem, integration in (25) along the lines $\xi_j + ibp_j$, $-\infty < \xi_j < \infty$ $(j = 1, \ldots, n)$, can be replaced by integration along the real lines and the quadrangles with vertices at $-a$, $-a+ibp_j$, $a+ibp_j$, a, in the indicated direction. It also has to be observed that, except for the integral over R^n, integration along a real line can be reduced to the segment from $-a$ to a, again by Cauchy's integral theorem. This procedure leads to the formula

$$\frac{\tilde{g}(x)}{\sigma_{2\pi b}(x)} = \int_{R^n} f(\xi)\, e^{2\pi i\xi\cdot x}\, d\xi + \sum_{p\in P^n} \frac{e^{2\pi bp\cdot x}}{\sigma_{2\pi b}(x)} \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot x}\, d\zeta, \tag{26}$$

where the contours l_p are those defined in lemma 2.

We assert that

$$P = \left(-\frac{\Delta}{4\pi^2}\right)^\mu \left(\frac{\tilde{g}}{\sigma_{2\pi b}}\right) \tag{27}$$

is a b-parametrix for S. In fact, P satisfies the growth condition for a b-parametrix, i.e. $\sigma_{2\pi b}P$ is a bounded distribution. Furthermore, by virtue of (26), P is a sum of the distribution

$$P_1 = \left(-\frac{\Delta}{4\pi^2}\right)^\mu \tilde{f},$$

where $\tilde{f}$ is the inverse Fourier transform of f, and the function

$$P_2(x) = \left(-\frac{\Delta}{4\pi^2}\right)^\mu \sum_{p\in P^n} \frac{e^{2\pi bp\cdot x}}{\sigma_{2\pi b}(x)} \int_{l_p} f(\zeta)\, e^{2\pi i\zeta\cdot x}\, d\zeta,$$

which belongs to $\mathscr{E}\mathscr{K}_1'$.

Now, in view of (23) and the definition of $\gamma(\zeta)$ in lemma 2,

$$(\widehat{S*P_1})(\xi) = \hat{S}(\xi)(\xi\cdot\xi)^\mu f(\xi) = \gamma(\xi),$$

and so

(28) $$S*P_1 = \delta - W_1,$$

where

$$W_1(x) = \int_{I_a} e^{2\pi i\xi\cdot x}\,d\xi = \frac{1}{(2\pi i)^n x^1}\prod_{j=1}^{n}(e^{2\pi i a x_j} - e^{-2\pi i a x_j}).$$

Next we define the function $h(x,t)$ on R^{2n} as

$$h(x,t) = \sum_{p\in P^n}\frac{e^{2\pi b p\cdot x}}{\sigma_{2\pi b}(x)}\int_{I_p} f(\zeta)e^{2\pi i\zeta\cdot(x-t)}\,d\zeta.$$

For any fixed $x\in R^n$, $h(x,t)$ is in $\mathscr{E}\mathscr{K}_1'$ as a function of t. Moreover,

(29) $$\left(-\frac{\Delta_t}{4\pi^2}\right)^{\mu} S_t\cdot h(x,t) = \sum_{p\in P^n}\frac{e^{2\pi b p\cdot x}}{\sigma_{2\pi b}(x)}\int_{I_p}\gamma(\zeta)e^{2\pi i\zeta\cdot x}\,d\zeta$$

$$= W_1(x) - \frac{1}{(\pi i)^n x^1\sigma_{2\pi b}(x)}\prod_{j=1}^{n}(e^{2\pi i a x_j} - e^{-2\pi i a x_j}),$$

by equality (23) and lemma 2.

On the other hand,

(30) $$(S*P_2)(x) = \left(-\frac{\Delta_t}{4\pi^2}\right)^{\mu} S_t\cdot[h(x,t)+v(x,t)],$$

where $v(x,t)$ is the function from lemma 3.

But

$$W_2(x) = \left(-\frac{\Delta_t}{4\pi^2}\right)^{\mu} S_t\cdot v(x,t)$$

is a C^∞-function, which satisfies condition (21), by the corollary following lemma 3. Thus from (28), (29) and (30) we conclude that P satisfies equation (20) with the function

$$W(x) = W_2(x) - \frac{1}{(\pi i)^n x^1\sigma_{2\pi b}(x)}\prod_{j=1}^{n}(e^{2\pi i a x_j} - e^{-2\pi i a x_j}),$$

which has the desired properties.

THEOREM 3. *If $S\in\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ and, for every $b>0$, there exists a b-parametrix for S, then S is hypoelliptic in $\mathscr{K}_1'$.*

Proof. Assume that U is a solution in $\mathscr{K}_1'$ of the equation

$$S*U = F,$$

where $F\in\mathscr{E}\mathscr{K}_1'$. Then there exists a $k\in N$ such that $\frac{1}{\sigma_{2\pi k}}U$ is a bounded distribution and

$$\sup_{x\in R^n}\frac{1}{\sigma_{2\pi k}(x)}|D^r F(x)| < \infty$$

for every $r\in N^n$.

Let now P be a b-parametrix for S, $b>k$, and W the corresponding function in (20). Note that P and W may not be in $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$. Still we can write

(31) $$U = U*\delta = U*(S*P)+U*W,$$

where the convolutions with U on the right-hand side are well defined (see section 1). Moreover,

$$U*(S*P) = (U*S)*P = F*P$$

and the last term belongs to $\mathscr{E}\mathscr{K}_1'$. Also $U*W$ is obviously in $\mathscr{E}\mathscr{K}_1'$. Thus U is, in fact, in $\mathscr{E}\mathscr{K}_1'$, q.e.d.

Combining theorem 2 and theorem 3 we obtain

THEOREM 4. *A distribution $S\in\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ satisfying condition* (11) *is hypoelliptic in $\mathscr{K}_1'$.*

In view of theorem 1 we can now state the following corollary:

COROLLARY. *Condition* (11) *is necessary and sufficient for a distribution $S\in\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ to be hypoelliptic in $\mathscr{K}_1'$.*

References

[1] M. Hasumi, *Note on the n-dimentional tempered ultra-distributions*, Tôhoku Math. Journal 13 (1961), p. 94-104.

[2] L. Hörmander, *Linear partial differential operators*, Berlin-Göttingen-Heidelberg 1964.

[3] L. Schwartz, *Théorie des distributions I/II*, Paris 1957/1959.

[4] J. Sebastião e Silva, *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel*, Math. Ann. 136 (1958), p. 58-96.

[5] Z. Zieleźny, *Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (I)*, Studia Math. 28 (1967), p. 317-332.

[6] — *On the space of convolution operators in $\mathscr{K}_1'$*, Studia Math. 31 (1968), p. 219-232.

Reçu par la Rédaction le 2. 1. 1968