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Uncomplemented function algebras

by

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1. Introduction. Eberlein [4] has shown that the space of almost periodic functions on a locally compact abelian group is a complemented subspace of the space of all weakly almost periodic functions. In this paper we specialize to the case of the real line and show that neither of the above two spaces is complemented when considered as a subspace of the bounded uniformly continuous functions. It is interesting that in both of our proofs we are able to employ a well known lemma of R. S. Phillips. The author is grateful to W. G. Bade for suggesting this investigation.

2. Preliminaries. The bounded finitely additive measures on a discrete set S will be denoted by $\mathrm{ba}(S)$.

LEMMA 2.1 (Phillips [6]). *Let* $\{\mu_n\} \in \mathrm{ba}(S)$. *Suppose*

$$\lim_n \mu_n(E) = 0 \quad \textit{for all } E \subset S.$$

Then

$$\lim_n \sum_{s\in S} |\mu_n(s)| = 0.$$

Remark. Grothendieck [5] has used this to show that every continuous linear map from m, the space of bounded sequences, to a separable space is weakly compact.

Definition 2.2. Let G be a locally compact group. A function $f \in \mathrm{BC}(G)$, the bounded continuous functions on G, is said to be *weakly almost periodic* (WAP) provided $\{f(x+t) \mid t \in G\}$ is a relatively weakly compact subset of $\mathrm{BC}(G)$; f is said to be *almost periodic* (AP) provided $\{f(x+t) \mid t \in G\}$ is a relatively compact subset of $\mathrm{BC}(G)$.

Thus any AP function is WAP. A WAP function is bounded and uniformly continuous. The sum, product, and scalar multiples of WAP (AP) functions are WAP (AP). Furthermore, the uniform limit of a sequence of WAP (AP) functions is a WAP (AP) function. Consequently, both

WAP(G) and AP(G) are closed subalgebras of BUC(G). A discussion of these facts for WAP functions may be found in Eberlein [3], and for AP functions in Bohr [2].

We now specialize to the case of $G = R$.

LEMMA 2.3 (Eberlein [3]). *The limit*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{s-T}^{s+T} f(t)\,dt = M(f)$$

exists for each $f\in$ WAP(R) independently of s and uniformly in s.

3. THEOREM 3.1. *There is no bounded projection from* BUC(R) *onto* WAP(R).

THEOREM 3.2. *There is no bounded projection from* BUC(R) *onto* AP(R).

Theorem 3.1 is directly within the grasp of Phillips' lemma due to the fact that if f is a continuous function vanishing at infinity, then f is a WAP function (see [3], p. 233), and $M(f) = 0$. However, no non-zero function vanishing at infinity is AP, and it seems necessary to resort to some harmonic analysis for the proof of Theorem 3.2. The proofs of both theorems may be generalized to give the same results for metrizable non-compact locally compact groups.

Proof of Theorem 3.1. Consider the mean value

$$M(f) = \lim_{T\to\infty}\frac{1}{2T}\int_{s-T}^{s+T} f(t)\,dt.$$

We may find an increasing sequence $\{s_n\}$ of real numbers such that

$$\lim_n \frac{s_{n+1}}{s_n} = \infty$$

(e.g. $s_n = n^n$). Then it is easy to verify that

$$\lim_{n\to\infty}\frac{1}{s_n - s_{n-1}}\int_{s_{n-1}}^{s_n} f(t)\,dt = M(f)$$

for each $f\in$ WAP(R).

We now define for each n a continuous function g_n having the following properties:

(1) $\|g_n\|_\infty = 1$;

(2) $\operatorname{supp} g_n \subseteq [s_{n-1}, s_n]$;

(3) $\dfrac{1}{s_n - s_{n-1}}\displaystyle\int_{s_{n-1}}^{s_n} g_n(t)\,dt > \dfrac{1}{2}$;

(4) $\sum_{n\in E} g_n$ is uniformly continuous for each subset E of positive integers.

For each $f\in$ WAP(R) define

$$\varphi_n(f) = M(f) - \frac{1}{s_n - s_{n-1}}\int_{s_{n-1}}^{s_n} f(t)\,dt.$$

Note that $\lim_n \varphi_n(f) = 0$ for each $f\in$ WAP(R).

Assume to the contrary that P is a bounded projection from BUC(R) onto WAP(R). For each subset E of positive integers define

$$\nu_n(E) = \varphi_n\Big(P\Big(\sum_{k\in E} g_k\Big)\Big).$$

For each n, ν_n is a bounded finitely additive measure on the positive integers, and for each subset E of positive integers $\lim_n \nu_n(E) = 0$. However,

$$|\nu_n(n)| = |\varphi_n(Pg_n)| = |\varphi_n(g_n)| > \frac{1}{2}.$$

This contradicts Lemma 2.1, q.e.d.

We require a preliminary discussion before proceeding to the proof of Theorem 3.2. When possible we use the notation of Rudin [8].

Let G be a locally compact group and let Γ be its dual. Let (x, γ) denote $\gamma(x)$, where $\gamma\in\Gamma$ and $x\in G$. If f is a function on G, then $\hat{f}$ denotes its Fourier transform.

Alternatively to Definition 2.2, AP(G) is defined as the uniform closure of all trigonometric polynomials on G, i.e. of all finite sums of the form

$$f(x) = \sum_{\nu=1}^{n} a_\nu (x, \gamma_\nu), \qquad x\in G,$$

where a_ν are complex numbers. Let $\bar{G}$ denote the Bohr compactification of G defined as the group dual to Γ with the discrete topology. Then AP(G) may be identified with $C(\bar{G})$.

If $f\in C(\bar{G})$, the *spectrum* of f, denoted by $\operatorname{sp} f$, is defined to be $\{\gamma\in\Gamma \mid \hat{f}(\gamma)\neq 0\}$. If Y is an invariant subspace of $C(\bar{G})$, letting $\operatorname{sp} Y$ denote $\bigcup_{f\in Y}\operatorname{sp} f$, the spectral synthesis theorem (see [9], p. 17) says that

$$Y = \{f\in C(\bar{G}) \mid \operatorname{sp} f \subseteq \operatorname{sp} Y\}.$$

Suppose now that Λ is a subgroup of Γ, the discrete dual of $\bar{G}$. Consider the closure of all trigonometric polynomials

$$f(x) = \sum_{\nu=1}^{n} a_\nu (x, \gamma_\nu) \quad \text{with } \gamma_\nu \in \Lambda.$$

Let this invariant subspace of $\mathrm{AP}(G)$ be denoted by $\mathrm{AP}_\Lambda(G)$. The following theorem has been communicated by H. P. Rosenthal:

THEOREM. *There is a bounded projection from* $\mathrm{AP}(G)$ *onto* $\mathrm{AP}_\Lambda(G)$.

Proof. By the above remarks

$$\mathrm{AP}_\Lambda(G) = \{f \in C(\bar{G}) \mid \operatorname{sp} f \subseteq \Lambda\}.$$

Let Λ denote $\{x \in \bar{G} \mid (x, \gamma) = 1 \text{ for all } \gamma \in \Lambda\}$. Set $H = \Lambda^{\perp}$. Then H is a compact subgroup of $\bar{G}$ and the orthogonality relation $H^{\perp} = \Lambda^{\perp\perp} = \Lambda$ holds. Let m_H denote normalized Haar measure on H, then

$$\hat{m}_H(\gamma) = \int_H (h, \gamma)\, dm_H(h)$$

is the Fourier-Stieltjes transform of m_H. If $\gamma \in \Lambda$ it follows that

$$\hat{m}_H(\gamma) = \int_H (h, \gamma)\, dm_H(h) = \int_H dm_H(h) = 1.$$

If $\gamma \notin \Lambda$, then $\hat{m}_H(\gamma) = 0$, since there exists an $h_0 \in H$ such that $(h_0, \gamma) \neq 1$ and

$$(h_0, \gamma) \int_H (h, \gamma)\, dm_H(h) = \int_H (h + h_0, \gamma)\, dm_H(h) = \int_H (h, \gamma)\, dm_H(h).$$

The required projection from $\mathrm{AP}(G)$ onto $\mathrm{AP}_\Lambda(G)$ is given by $P = m_H * f$. First note that

$$m_H * f(x) = \int_{\bar{G}} f(x - \tau)\, dm_H(\tau)$$

is a continuous function on $\bar{G}$ and that $m_H * f(\gamma) = \hat{m}_H(\gamma) \hat{f}(\gamma)$. Since $\hat{m}_H(\gamma) = 1$ if $\gamma \notin \Lambda$, we see that $m_H * f \in \mathrm{AP}_\Lambda(G)$. If $f \in \mathrm{AP}_\Lambda(G)$, then $m_H * \hat{f}(\gamma) = \hat{f}(\gamma)$ for every $\gamma \in \Gamma$, and by the uniqueness of Fourier transforms $f = m_H * f$, q.e.d.

APPLICATION 1. Let $G = R$, so that $\Gamma = R$ and let Λ be the subgroup of all rational multiples of π. Then there is a bounded projection from $\mathrm{AP}(R)$ onto $\mathrm{AP}_\Lambda(R)$.

APPLICATION 2. Let $G = Z$ (the integers) so that Γ is the circle T. Let $\tilde{\Lambda}$ be the subgroup of all points of T whose arguments are rational multiples of π. Then there is a bounded projection of $\mathrm{AP}(Z)$ onto $\mathrm{AP}_{\tilde{\Lambda}}(Z)$. The latter space is also the closure in the supremum norm of the linear space of periodic sequences.

Proof of Theorem 3.2. Let Λ be as above. It suffices to show that there is no bounded projection from $\mathrm{BUC}(R)$ onto $\mathrm{AP}_\Lambda(R)$, since if there were a bounded projection from $\mathrm{BUC}(R)$ onto $\mathrm{AP}(R)$, then by composing it with the projection from $\mathrm{AP}(R)$ onto $\mathrm{AP}_\Lambda(R)$ we would have a projection from $\mathrm{BUC}(R)$ onto $\mathrm{AP}_\Lambda(R)$.

Assume to the contrary that there is a bounded projection $P : \mathrm{BUC}(R) \to \mathrm{AP}_\Lambda(R)$. Let $g(x)$ be a continuous function with support contained in $[-\frac{1}{4}, \frac{1}{4}]$ and such that $g(0) = 1$. Define $g_n(x) = g(x - n)$ for $n = 0, \pm 1, \pm 2, \ldots$

Imbed m in $\mathrm{BUC}(R)$ as follows:

$$\{\xi_n\} \in m, \quad I : \{\xi_n\} \to \sum_{n=-\infty}^{\infty} \xi_n g_n(x).$$

Let Φ be the map from $\mathrm{AP}_\Lambda(R)$, $\Phi : f \to \{f(n)\}$.

It is claimed that the composition map $\Phi \circ P \circ I$ is a projection from m onto $\mathrm{AP}_{\tilde{\Lambda}}(Z)$. However, the latter space is separable, and any continuous linear map from m to a separable space is weakly compact ([5], p. 169). This will be the desired contradiction.

As noted before, the space of periodic sequences is dense in $\mathrm{AP}_{\tilde{\Lambda}}(Z)$. Let $\{\xi_n\}$ be a periodic sequence; then

$$I(\{\xi_n\}) = \sum_{n=-\infty}^{\infty} \xi_n g_n$$

is a periodic function belonging to $\mathrm{AP}_\Lambda(R)$. Thus $P \circ I(\{\xi_n\}) = I(\{\xi_n\})$ and $\Phi \circ P \circ I(\{\xi_n\}) = \Phi \circ I(\{\xi_n\}) = \{\xi_n\}$. Since $\Phi \circ P \circ I$ is continuous, we see that if $\{\xi_n\} \in \mathrm{AP}_{\tilde{\Lambda}}(Z)$, then $\Phi \circ P \circ I(\{\xi_n\}) = \{\xi_n\}$. Thus $\Phi \circ P \circ I$ is a projection, q.e.d.

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Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (II)

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In part I of this work (see [5]) we showed how to define in a general manner hypoelliptic and entire elliptic convolution operators in subspaces of the space of distributions. We also characterized hypoelliptic and entire elliptic convolution operators in the space $\mathscr{S}'$ of tempered distributions.

The purpose of this paper is to study hypoelliptic convolution operators in the space $\mathscr{K}_1'$ $(=\Lambda_\infty)$ of distributions of exponential growth introduced by Sebastião e Silva [4] and Hasumi [1].

The space $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ of convolution operators in $\mathscr{K}_1'$ (which is a space of distributions) was characterized in [1] and its topological properties were investigated in [6].

Using the notation of [5] we define $\mathscr{E}\mathscr{K}_1'$ to be the set of all C^∞-functions $f \in \mathscr{K}_1'$ such that, for every $S \in \mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$, the convolution $S*f$ is a C^∞-function and $S \to S*f$ is a continuous mapping from $\mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ into the space $\mathscr{E}$ of all C^∞-functions in R^n. Then a distribution $S \in \mathscr{O}_c'(\mathscr{K}_1':\mathscr{K}_1')$ is said to be *hypoelliptic* in $\mathscr{K}_1'$, if every solution $U \in \mathscr{K}_1'$ of the convolution equation

$$S*U = F \tag{1}$$

is in $\mathscr{E}\mathscr{K}_1'$, when $F \in \mathscr{E}\mathscr{K}_1'$; in that case equation (1) is also called *hypoelliptic* in $\mathscr{K}_1'$.

As a supplement of the standard notation (see [3] and [5]) we use N^n as the set of all points in R^n, whose coordinates are non-negative integers; we write N and R instead of N^1 and R^1 respectively. Furthermore, we denote by P^n (Q^n resp.) the set of all points $p = (p_1, \ldots, p_n)$ ($q = (q_1, \ldots, q_n)$ resp.) such that $p_j = 1$ or -1 ($q_j = 1$ or 0 resp.). In particular, Q^n contains the points $\mathbf{1} = (1, 1, \ldots, 1)$ and $\mathbf{0} = (0, 0, \ldots, 0)$.

For a point $x = (x_1, \ldots, x_n) \in R^n$ we sometimes write $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1}) \in R^{n-1}$. Also, for $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots$ $\ldots, \xi_n)$ in R^n we use the product $x\xi = (x_1\xi_1, \ldots, x_n\xi_n)$ beside the scalar