boundary problem with Cauchy's initial conditions. In fact, that problem
can be considered as the limit case of ours, viz., when the thickness \( \varepsilon \)
of \( I \) tends to 0. Note that, if we wish to admit, as solutions, functions \( f \)
which are not of class \( C^0 \), then the Cauchy conditions become inapplicable.
Still our formulation allows to extend the problem onto solutions \( f \)
which are arbitrary distributions (Theorems 5 and 6).

The problem of extending solutions of (1) was also considered by
Lojasiewicz in (3), but the purpose of his extension lemmas was quite
different. It can be noted that neither Theorems 2, 3, 5 and 6 of the
present paper nor the particular case, considered in this section can be
deduced from Lojasiewicz lemmas.

References

p. 163-169.
Matem. 25 (1961).
[4] K. Skórnik, Postać funkcji lokalizowanej, której w-1 pochodna
okala zero na przestrzeni uzupelnionej, Zeszyty Naukowe W. S. P. w Katowicach (to be printed).

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On orbits of elements

by

S. KOLEWICZ (Warszawa)

Let \( X \) be a linear metric space. Let \( A \) be a linear continuous operator
mapping \( X \) into itself. Let \( x \in X \). We shall write
\[
\mathcal{O}(A : x) = \{ A^n x : n = 0, 1, \ldots \}
\]
and we shall call \( \mathcal{O}(A : x) \) an orbit of \( x \) with respect to the operator \( A \).

It is well known that if \( X \) is a space of finite dimension, then there
are three possibilities:

1° \( \lim A^n x = 0 \),

2° \( \lim ||A^n x|| = + \infty \),

3° the closure of the orbit \( \mathcal{O}(A : x) \) is compact and 0 does not belong
to this closure.

This follows for instance from [4], lemma 1, p. 270.

The purpose of the present paper is to show that in the infinite-
dimensional case it is not so: it may happen that for some \( A \) and \( x \),
the orbit \( \mathcal{O}(A : x) \) is dense in the whole space \( X \). Examples of this situation
in concrete spaces are given. It is not clear whether it may take place
in an arbitrary infinite-dimensional separable Banach space (cf. Problem 1).
Some related questions are also discussed.

The basic terminology and notation are the same as in Banach's
we mean any complete linear metric space and by a \( R \)-space we mean
a locally convex \( F \)-space. The norm in the sense of [1] (i.e. a subadditive,
symmetric functional vanishing only at 0) is called in this paper an
\( F \)-norm; norms and pseudonorms used here are always homogeneous and
continuous.

Theorem 1. Let \( X \) be either \( L^p(1 \leq p < + \infty) \) or \( c_0 \). For any arbitrary
real \( a > 1 \), there are a linear continuous operator \( A \) and an element \( x_0 \)
such that the orbit \( \mathcal{O}(A : x_0) \) is dense in the whole of \( X \).
Proof. Let $S$ be the left shift operator

$$S((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots)$$

and $S'$ be the right shift operator

$$S'((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots).$$

Let $A = \alpha S$ and $B = S' \alpha$. Then $\|A\| = \alpha$, $\|B\| = 1/\alpha$, $AB = I$, where $I$, as usual, denotes the identity operator.

Now we shall start to construct the vector $x_0$. Let $x^\alpha$ be a sequence dense in $X$, with the property that all except finitely many coordinates of each element $x^\alpha$ are 0 (for instance, we may take finite rational combinations of unit vectors). Let $k(n)$ denote the greatest index of the coordinate of $x^\alpha$ that does not vanish.

Let $r(s)$ be a sequence of positive integers such that

\begin{equation}
(1) \quad r(s) > \max_{1 < i < n} k(i)
\end{equation}

and

\begin{equation}
(2) \quad \|B^{(r(k))} x^\alpha\| < \frac{1}{2^n}.
\end{equation}

Let

\begin{equation}
(3) \quad p(n) = \sum_{i=1}^{n} r(i).
\end{equation}

We write

\begin{equation}
(4) \quad x_0 = \sum_{n=1}^{\infty} B^{(p(n))} x^\alpha.
\end{equation}

Formula (2) implies that series (4) is convergent.

Condition (1) implies that

\begin{equation}
(5) \quad A^{(p(n))} x^\alpha = 0 \quad \text{for } i < n.
\end{equation}

Therefore

\begin{equation}
(6) \quad A^{(p(n))} x_0 = x_0 + \sum_{n=1}^{\infty} B^{(p(n))} x^\alpha.
\end{equation}

But

\begin{equation}
(7) \quad \left\| \sum_{n=1}^{\infty} B^{(p(n))} x^\alpha \right\| \leq \sum_{n=1}^{\infty} \left\| B^{(p(n))} x^\alpha \right\| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^n}.
\end{equation}

Therefore

\begin{equation}
(8) \quad \|A^{(p(n))} x_0 - x^\alpha\| \leq \frac{1}{2^n}.
\end{equation}

Let $x$ be an arbitrary element. Since the sequence $(x^\alpha)$ is dense in the space $X$, for each positive $\varepsilon$ there is an $x^\alpha$ such that $\|x - x^\alpha\| < \varepsilon$.

Without loss of generality we can assume that $1/2^n < \varepsilon$. Then

$$\|A^{(p(n))} x_0 - x\| \leq \|A^{(p(n))} x^\alpha - x^\alpha\| + \|x^\alpha - x\| \leq \frac{1}{2^n} + \frac{1}{2^n} < 2\varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we infer, that the orbit $\mathcal{O}(A : x_0)$ is dense in $X$.

Let us remark that in the proof of theorem 1 the fact that we have considered the space of scalar sequences does not play any role. We can replace in this proof the field of scalars by an arbitrary separable Banach space, i.e. we can prove theorem 1 for spaces $(X, \varphi)$ $(1 \leq p < + \infty)$ and $(X, \varphi)_0$ of all sequences $x = (x_n)$ of elements of a separable Banach space $X$ such that

$$x = \left( \sum_{n=1}^{\infty} \|x_n\|^{1\varphi} \right)^{1/\varphi} < + \infty$$

(respectively $\lim_{n=\infty} \|x_n\|_X = 0$)

with topology defined by the norm $\|\cdot\|$ (respectively by the norm $\|\cdot\|_X$). (We denote by $\|\cdot\|_X$ the norm in the space $X$.)

Therefore the following theorem holds:

**Theorem 1'.** Let $X$ be either $(X, \varphi)$ or $(X, \varphi)_0$, $X$ being an arbitrary separable Banach space. Then for each real number a greater than 1 there are in $X$ an element $x_0$ and a linear continuous operator $A$ of norm $a$ such that the orbit $\mathcal{O}(A : x_0)$ is dense in $X$.

**Corollary.** For an arbitrary greater than 1 there are in each space $C[0, 1]$, $L^p[0, 1]$, $1 \leq p < + \infty$, an operator $A$ of the norm $a$ and an element $x_0$ such that the orbit $\mathcal{O}(A : x_0)$ is dense in $X$.

**Proof.** The space $C[0, 1]$ is isomorphic to the space $C[0, 1]_0$ (respectively $L^p[0, 1]$).

**Problem 1.** Is it true that for every infinite-dimensional separable Banach space there are such a linear continuous operator $A$ and such an element $x$ that $\mathcal{O}(A : x)$ is dense in $X$?

**Problem 2.** Given a separable Banach space $X$. Characterize all continuous linear operators acting in $X$ which admit a vector $x$ such that $\mathcal{O}(A : x)$ is dense in $X$.

The operator $A$ considered in theorem 1 is not invertible, but in a similar way we can construct an invertible operator with the same property. For this purpose let us remark that the space $L^p[1, p < + \infty]$ and $C[0, 1]$ are isomorphic to the space $L^p(c_0)$ of all sequences $x = (x_n)$, $n = 0, 1, 2, \ldots$, such that

$$\|x\| = \left( \sum_{n=0}^{+ \infty} (\|x_n\|^{1/p})^{1/p} \right)^{1/p} < + \infty.$$
(respectively \( \lim_{n \to \infty} a^{-|a_n|} |x_n| = 0 \)) with the norm \( \|x\|_\infty = \sup_{n \to \infty} a^{-|a_n|} |x_n| \).

If \( a > 1 \), then the shift operator \( S \) is of norm \( a \) and has the desired property; moreover, we can construct such an element \( x_0 \) that the orbits \( C^0_{a_0} \) and \( C^{-1}_{a_0} \) are both dense in the space \( L^0_a \) (respectively \( C^0 \)). So far we have considered only powers of one operator. Similarly, we can construct a continuous semigroup \( (\cdot) \) (or even a group) of continuous linear operators \( T(s) \) such that there is an element \( x_0 \) such that the set

\[
C^0_{a_0} = \{ T(s)x_0 : s > 0 \}
\]

is dense in the whole space.

For example, let \( L^p_a(-\infty, +\infty) \), \( a > 1 \), \( 1 \leq p < +\infty \), denote the space of all measurable functions \( x \) such that

\[
\|x\|_p = \left( \int_{-\infty}^{\infty} |x(t)|^p dt \right)^{1/p} < +\infty
\]

with the norm \( \|x\|_p \). The space \( L^p_a(-\infty, +\infty) \) is obviously isomorphic to the space \( L^p[0,1] \).

The group of shift operators \( T(s) \) defined as

\[
T(s)x(t) = x(t-s)
\]

is a continuous group of continuous operators. In the same way as in the proof of theorem 1, we can construct such an element \( x_0 \) that \( C^0_{a_0} \) is dense in \( L^0_a(-\infty, +\infty) \).

A similar construction can also be performed in the space \( C^0_a(-\infty, +\infty) \) \((a > 1)\) of all continuous functions such that

\[
\lim_{t \to \infty} a^{-|a_t|} |x(t)| = 0
\]

with the norm

\[
\|x\| = \sup_{t} a^{-|a_t|} |x(t)|.
\]

The space \( C^0_a(-\infty, +\infty) \) is isomorphic to the space \( C[0,1] \).

If \( X \) is a Banach space and \( A \) is a continuous operator acting in \( X \), then there is an \( a > 0 \) (namely any \( 0 < a < \|A\|^{-1} \)) such that \( \lim_{n \to \infty} (aA)^n x = 0 \) for every \( x \in X \), and therefore the orbit of no element with respect to \( aA \) is dense in \( X \). If \( X \) is an F-space, then there can exist operators \( A \) such that

\[
(1) \text{ A semigroup } T(s) \text{ is called continuous if } \lim_{t \to s} T(s)T(t)x = T(s+t)x \text{ for all } x.
\]

(*) For every \( a > 0 \) there is an \( x_0 \) such that \( (aA) \) is dense in \( X \).

**Proposition 1.** Let \( X \) be an arbitrary F-space and let \( X = (\lambda) \) be the space of all sequences \( x = (x_n) \) with \( y_e X \) under the F-norm

\[
\|x\| = \sum \frac{1}{\lambda_n} |x_n|,
\]

where \( |x| \) denotes the F-norm in \( X \). Then the left shift operator \( S \) acting in \( X \) has the property (*).

**Proof.** The proof is on the same lines as the proof of theorem 1. It is only sufficient to remark that for all scalars \( x \) and \( x \in X \)

\[
(aS^\alpha x)^\beta < \frac{1}{2^\beta},
\]

where, as before, \( S' \) denotes the right shift operator.

**Remark.** If \( X \) is an F-space and \( A \) is a continuous linear operator acting in \( X \) which admits a pseudonorm \( \| \cdot \| \) such that \( A \) is continuous in \( \| \cdot \| \) i.e., \( \|Ax\| \leq b\|x\| \) for some \( b > 0 \) and all \( x \in X \), then \( A \) does not satisfy (*). In fact, all elements \( (a^{-1}A)^\alpha x_0 \) are in the set \( x \in X \) \( \|x\| < |a_n| \) and therefore the orbit of an element \( x_0 \) with respect to the operator \( a^{-1}A \) cannot be dense in \( X \).

**Problem 3.** Suppose that \( X \) is a separable \( B \)-space and \( A \) is a continuous linear operator acting in \( X \). Suppose that \( A \) does not satisfy (*). Does it imply that \( A \) is continuous with respect to some pseudonorm defined on \( X \)?

We will say that a \( B \)-space \( X \) is a space with a norm if there is such a pseudonorm \( \|\cdot\| \) in \( X \) that \( \|x\| = 0 \) implies \( x = 0 \). For example, the space \( C^0[0,1] \) of infinite-differentiable functions \( x(t) \) with topology given by a sequence of pseudonorms

\[
\|x\|_m = \sup_{t \in C[0,1]} \frac{d^m x}{dt^m}, \quad m = 0, 1, \ldots,
\]

is a space with a norm. In fact, if \( \|x\|_m = 0 \), then \( x = 0 \).

In paper [2] it is proved that a \( B \)-space \( X \) is a space with a norm if and only if \( X \) does not contain any subspace isomorphic to the space \( x \) of all sequences. Then obviously spaces \((\lambda)\) are not spaces with norms.

**Problem 4.** Does there exist a separable \( B \)-space \( X \) with a norm such that there are in \( X \) operators satisfying (*)?

**Problem 5.** Let \( X \) be a separable \( B \)-space with a norm. Let \( A \) be a continuous linear operator acting in \( X \). Does there exist a pseudonorm \( \|\cdot\| \) in which \( A \) is continuous?
An analytic approach to semiclassical potential theory

by

S. Kwapień (Warszawa)

§ 0. Introduction. The aim of this paper is to give a new non-probabilistic approach to the semiclassical potential theory. The method used here is, may be, less interesting but much simpler. The semiclassical potential theory was started in 1950 by M. Kao who, using probabilistic methods, derived an analytic formula for the capactiator potential. Then it was systematically developed by Z. Ciesielski who indicated analogies between classical and semiclassical potential theories. Such notions as balayage, thinness, Dirichlet problem and capacity have their corresponding ones in the semiclassical theory. The sets of Lebesgue measure zero play essentially the same role as the polar sets. A brief, non-probabilistic account of this theory is given in § 2. For detailed treatment of this subject the reader is referred to [2] and [3]. Improving Kac's technique, Stroock [7] has generalized the Kao formula on the strong balayage of an arbitrary superharmonic function. He has also obtained an analytic formula for the solution of the semiclassical Dirichlet problem. The method used in this article leads to the same formulas. We deal with this topic in § 3. § 4 is mainly devoted to non-probabilistic proofs of some Stroock's results (cf. [9]). In it a new method of solving the classical Dirichlet problem is established. The solution is obtained as a limit of solutions of some integral equations (cf. Corollaries 4.5 and 4.6). We finish this paper by suggesting some possible generalizations.

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§ 1. Some basic lemmas. In the following $U$ denotes a Greenian domain in the $k$-dimensional Euclidean space $R^k$ and $Q(x, y)$ the Green function for this domain. It will be convenient to employ the following notations:

\[ H^+ (U) \] — the class of all positive and superharmonic functions on $U$;
 \[ BH^+ (U) \] — the class of all bounded $f \in H^+ (U)$;
 \[ CH^+ (U) \] — the class of all continuous $f \in BH^+ (U)$.