Also since $f$ verifies condition (3) of the theorem,

$$|f(a_n)| \leq a \Phi(|a_n|) \leq a \Phi(|a|),$$

and since $f(a_n) \to f(a)$ a.e., it follows by Lebesgue theorem on dominated convergence that

$$F(a) = \lim_{n \to \infty} F(a_n) = \lim_{n \to \infty} \int f(a_n) \, d\mu = \int f(a) \, d\mu,$$

thus completing the representation of $F$. The uniqueness of $f$ is verified as in theorem 2.

Conversely, if $f$ is a real-valued continuous function on $R$ satisfying conditions (1) and (3) of the theorem, then from Remark 2 it follows that the functional $F(a) = \int f(a) \, d\mu$ is well defined on $L_a$ and is additive. Next we verify that $F$ is continuous. Let $a_n$ be a sequence in $L_a$ converging to $a$. Thus by lemma 1 since $f$ is continuous, $f(a_n) \to f(a)$ converges in measure on sets of finite measure and further the inequality $\int f(a_n) \, d\mu \leq a \Phi(|a_n|) \leq a \Phi(|a|)$ implies that $\{f(a_n)\}_{n=1}^\infty$ are of uniformly absolutely continuous $L_a$-norms. Hence $\int f(a_n) \, d\mu \to \int f(a) \, d\mu$ and thus $F(a_n) \to F(a)$.

In conclusion it might be mentioned that the problem of representing additive functionals on Orlicz spaces $L_a$, where the space is not of absolutely continuous norm, is not considered here and it is conjectured that non-trivial continuous additive functionals do not exist in such spaces.

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References


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Banach spaces of functions

satisfying a modulus of continuity condition*

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1. Introduction and terminology. A function $\beta: [0, \infty) \to [0, \infty]$ will be called a modulus of continuity if it is monotone increasing, continuous at zero, and zero at zero only. Note that it need not be subadditive. For pseudometric spaces $(X, d)$ and $(Y, e)$, a function $f: (X, d) \to (Y, e)$ will be said to satisfy a modulus of continuity condition $\beta$ (locally) if there is some positive real $M$ (and some positive real $e$) such that $e(f(x), f(y)) \leq M \, d(x, y)$ (whenever $d(x, y) < e$) for all $x$ and $y$ in $X$. Obviously, such a function is uniformly continuous.

Let $P$ denote the real or complex numbers with the usual metric. For a pseudometric space $(X, d)$, let $\text{Lip}(X, \beta \circ \sigma)$ be the set of bounded $P$-valued functions on $X$ which satisfy a modulus of continuity condition $\beta$ locally. When $\beta(t) = t$, we will denote the set by $\text{Lip}(X, d)$. If only one metric is being considered on $X$, we will denote $\text{Lip}(X, \beta \circ \sigma)$ by $\text{Lip}(X, \beta)$, $\text{Lip}(X, \beta)$ is known that if $\beta$ is subadditive (so that $\beta \circ \sigma$ is a pseudometric) and the functions satisfy the modulus of continuity condition $\beta$ globally, then $\text{Lip}(X, \beta \circ \sigma)$ is a Banach space with a natural norm $[4]$. Let $(X, d), (X, d')$ and $(Y, e)$ be pseudometric spaces. If there exist $M, e > 0$ such that $d(x, y) \leq M \, d'(x, y)$ whenever $d'(x, y) < e$, we indicate it by writing $d \ll d'$. Then to say that $f: (X, d) \to (Y, e)$ satisfies a local Lipschitz condition can be denoted $e f \ll d$, where $e f (x, y) = (f(x), f(y))$. If $d \ll d'$ and $d' \ll d$, we say that $d$ and $d'$ are strongly equivalent (in contrast to topologically or uniformly equivalent) and denote it by $d \approx d'$.

We attempt to describe how the various spaces $\text{Lip}(X, \beta \circ \sigma)$ are related, if one considers different pseudometrics on $X$ or different moduli of continuity. In the first section, we give a natural norm for $\text{Lip}(X, \beta \circ \sigma)$, under which it is a Banach space. Then we show that $\text{Lip}(X, d)$ is con-


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continuously imbedded in Lip(X, d) iff d' ≪ d. In the second section, we show that if (X, d) is compact and \( \lim_{t \to 0} f(t) = 0 \), then the unit ball of Lip(X, d) is compact in Lip(X, β). Finally, we investigate Lip(X, β) for compact spaces which are "uniformly locally starlike" and show that
\[ \bigcap \{ \text{Lip}(X, \beta) \mid \lim_{t \to 0} \beta(t) = 0 \} = \text{Lip}(X, d). \]

2. The Banach space Lip(X, \beta ∘ d). As usual, we denote \( \sup \{|f(x)| : x \in X\} \) by \( \|f\|_\infty \). For \( f \in \text{Lip}(X, \beta ∘ d) \), set
\[ \|f\|_\beta = \sup \left\{ \frac{|f(x) - f(y)|}{\beta \circ d(x, y)} : d(x, y) > 0 \right\}. \]

2.1. Proposition. If \( f \in \text{Lip}(X, \beta ∘ d) \), then \( \|f\|_\beta < \infty \).

Proof. Let \( \varepsilon, M > 0 \) be such \( \|f(x) - f(y)\| < M \beta \circ d(x, y) \) whenever \( d(x, y) < \varepsilon \). Then
\[ \sup \left\{ \frac{|f(x) - f(y)|}{\beta \circ d(x, y)} : d(x, y) > 0 \right\} = \sup \left\{ \frac{|f(x) - f(y)|}{\beta \circ d(x, y)} : d(x, y) > 0 \right\} \leq \frac{2\|f\|_\infty}{\varepsilon} \leq M < \infty. \]

The standard arguments show that \( \|\cdot\|_\beta \) is a pseudonorm for Lip(X, β). Setting \( \|f\| = \|f\|_\infty \vee \|f\|_\beta \), we obtain a norm for Lip(X, β). It is well-known that Lip(X, β) is complete when β is subadditive. The proofs remain valid when β is not subadditive, so we omit them.

We shall be interested in a particular subspace of Lip(X, β), namely Lip(X, β) = \{ f \in Lip(X, β) : |f(x) - f(y)| \leq \beta \circ d(x, y) \} as \( d(x, y) \to 0 \).

The easily seen that lip(X, β) is a closed subspace of Lip(X, β).

2.2. Lemma. Let (X, d) be a pseudometric space and set \( d'(x, y) = d(x, y) \wedge 1 \). Then \( d' \approx d \) and
\[ d'(x, y) \leq \sup \{|f(x) - f(y)| \mid \|f\|_\beta \vee \|f\|_\infty \leq 1\}. \]

Proof. Define \( f_\varepsilon(x) = d(x, z) \wedge 1 \). Since \( |f_\varepsilon(x) - f_\varepsilon(y)| \leq d(x, y) \), we have \( \|f_\varepsilon\|_\beta \vee \|f_\varepsilon\|_\infty \leq 1 \). Now \( |f_\varepsilon(x) - f_\varepsilon(y)| = \|d(x, y) \wedge 1 - d(x, y) \wedge 1\| = d(x, y) \wedge 1 = d'(x, y) \). Thus \( \sup \{|f(x) - f(y)| \mid \|f\|_\beta \vee \|f\|_\infty \leq 1\} \leq d'(x, y) \).

That \( d' \approx d \) is obvious.

2.3. Theorem. Let X be a space with pseudometrics d and e. Then \( d \ll e \) iff \( \text{Id} : \text{Lip}(X, d) \to \text{Lip}(X, e) \) is a continuous imbedding.

Proof. Let \( d \ll e \). Then there exist \( K, \delta > 0 \) such that \( d(x, y) \leq Ke(x, y) \) whenever \( e(x, y) < \delta \). For \( f \in \text{Lip}(X, d) \),
\[ \|f\|_e = \sup \left\{ \frac{|f(x) - f(y)|}{e(x, y)} : e(x, y) > 0 \right\} = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : d(x, y) > 0 \right\} \leq \sup \left\{ \frac{|f(x) - f(y)|}{e(x, y)} : e(x, y) > 0 \right\} \leq \frac{\|f\|_\infty}{\delta} \leq (K + 2)\delta \|f\|_\beta \vee \|f\|_\infty \].

Hence \( f \in \text{Lip}(X, e) \), and the identity is continuous.

Conversely, suppose \( \text{Id} : \text{Lip}(X, d) \to \text{Lip}(X, e) \) is continuous. Let \( U \) denote the unit ball of Lip(X, d). By continuity, there is an \( M > 0 \) such that \( \|f\|_\beta \vee \|f\|_\infty \leq M \) for every \( f \in U \). In particular, \( f \in U \) implies that \( |f(x) - f(y)| \leq Me(x, y) \), setting \( d'(x, y) = d(x, y) \wedge 1 \), we apply (2.3) to see that \( d'(x, y) \leq \sup \{ |f(x) - f(y)| \mid f \in U \} \leq Me(x, y) \). Since \( d \approx d' \) and \( e \ll e \), we have \( d \ll e \).

2.4. Corollary. Let X be a space with pseudometrics d and e. Then \( d \approx e \) iff \( \text{Lip}(X, d) \to \text{Lip}(X, e) \) is a continuous imbedding.

Proof. Assume that \( \text{Lip}(X, d) \to \text{Lip}(X, e) \). Then we have \( \text{Lip}(X, d) \approx \text{Lip}(X, d) = \text{Lip}(X, e) \), and the norms of Lip(X, d) and Lip(X, e) are comparable with the norm of Lip(X, β ∘ d). We can apply the closed graph theorem to obtain that all three spaces are norm equivalent. Then apply (2.3).

The above result sharpens a result of Sherbert [6], p. 1392, by dropping the restriction that the metrics be bounded.

2.5. Corollary. Let (X, d) be a metric space and \( \beta, \gamma \) two moduli of continuity (with \( \beta \) subadditive). Then \( \text{Id} : \text{Lip}(X, \beta) \to \text{Lip}(X, \gamma) \) is a continuous imbedding iff (and only if) \( \limsup_{t \to 0} \beta(t)/\gamma(t) < \infty \).

Proof. The proof of (2.5) is obtained by substituting \( \beta \circ d \) for d and \( \gamma \circ d \) for e in (2.3) and in (2.5).

Note that the assertion of (2.5) is considerably weaker than that of (2.3). Unless subadditivity of \( \beta \) is assumed (or at least subadditivity in a neighborhood of zero), we are unable to prove the converse of (2.5).

The inequality \( \beta \circ d(x, y) - \beta \circ d(x, a) \leq \beta \circ d(y, z) \) seems to be necessary to construct non-constant functions which satisfy the modulus of continuity condition \( \beta \). We are unable to remove this restriction.
3. \text{Lip}(X, \beta) \text{ when } \lim_{t \to 0} \beta(t) = 0. \text{ Glicksberg [1], p. 91-97, has investigated } \text{Lip}(X, \beta) \text{ when } \lim_{t \to 0} \beta(t) = 0 \text{ for the special case of } X \text{ being a regular compact subset of } X^* \text{ and } \beta \text{ being subadditive. His proofs rely on sophisticated results from the theory of distributions and the fact that } \beta \circ \sigma \text{ is a metric if } \beta \text{ is subadditive.}

3.1. Theorem. Let \((X, d)\) be a compact metric space and \(\lim_{t \to 0} \beta(t) = 0.\) Then the unit ball of \(\text{Lip}(X, \beta)\) is compact in \(\text{Lip}(X, d).\)

Proof. Let \(f \in \text{Lip}(X, \beta)\) and \(\varepsilon > 0\) be given. Then

\[
\frac{|f(x) - f(y)|}{d(x, y)} = \frac{\beta \circ d(x, y)}{\beta(\varepsilon)} \cdot \frac{\beta \circ d(x, y)}{d(x, y)} < \frac{\|f\|_\beta \beta \circ d(x, y)}{\beta(\varepsilon)} \to 0
\]

as \(d(x, y) \to 0.\) Hence \(f \in \text{Lip}(X, d).\)

Let \(W = X \cup (X \times X - \Delta),\) endowed with the disjoint union topology. For each \(f \in \text{Lip}(X, \beta),\) define \(f^* : W \to W \) by \(f^*(x) = f(x)\) for \(x \neq X\) and \(f^*(x, y) = (f(x) - f(y))/d(x, y)\) for \((x, y) \in X \times X - \Delta.\) Each \(f \in \text{Lip}(X, d)\) can be extended continuously to \(W = X \cup X \times X\) by defining \(f^*(x, x) = 0\) for \(x \neq X\).

Since \(\text{Lip}(X, \beta) \approx \text{Lip}(X, \beta \wedge 1),\) we assume without loss of generality that \(\beta\) is bounded by one. Thus \(\beta(t)/t\) is a bounded function, so that

\[
\frac{|f(x) - f(y)|}{d(x, y)} \leq \frac{\|f\|_\beta \beta \circ d(x, y)}{d(x, y)} \leq \mathcal{K} \|f\|_\beta \text{ for some } \mathcal{K} > 0.
\]

We see then that \(\mathcal{U} = \{f^* \mid f \in \mathcal{F} \wedge \|f\|_\beta < 1\}\) is a set of uniformly bounded continuous functions on \(W.\) Restricted to \(X, \mathcal{F}^*\) in \(U\) clearly are an equicontinuous family. We will show that they are an equicontinuous family when restricted to \(X \times X.
\)

Let \(\varepsilon > 0\) be given. Choose \(\delta > 0\) so that \(d(x, y) < \Delta \text{ implies } \beta \circ d(x, y) / d(x, y) < \varepsilon/2.\) Choose \(0 < \delta' < \delta\) so that \(\delta' < \Delta/4(1 + \mathcal{K} \delta).\) We will show that if \(d(x, u) < \delta' \text{ and } d(y, v) < \delta',\) then \(|f^*(x, y) - f^*(u, v)| < \varepsilon\) for any \(f \in \mathcal{U}.\)

Case (i). \(d(x, y) < \delta\) (or \(d(u, v) < \delta\)). Then \(d(u, v) \leq d(u, x) + d(x, y) + d(y, v) < \Delta \delta.\) For \(f \in \mathcal{U},\)

\[
|f^*(x, y) - f^*(u, v)| \leq |f^*(x, y) - f^*(u, v)| + |f^*(u, v)|
\]

\[
\leq |f(x) - f(u)| / d(x, y) + |f(u) - f(v)| / d(u, v)
\]

\[
\leq \frac{\|f\|_\beta \beta(\varepsilon/2) + \|f\|_\beta \beta(\varepsilon/2)}{\varepsilon} < \varepsilon.
\]

Since \(\delta'\) does not depend on the particular \(f \in \mathcal{U},\) it is an equicontinuous family. Hence \(\mathcal{U}\) is totally bounded in \(C(W),\) the space of bounded continuous functions on \(W.\) But de Leeuw [3], p. 57, showed that \(\text{Lip}(X, \beta)\) is isometrically imbedded in \(C(W)\) under the mapping \(f \to f^*.\) Hence the unit ball of \(\text{Lip}(X, \beta)\) is precompact in \(\text{Lip}(X, d).\)

Let \(f_n\) be a sequence of functions from the unit ball of \(\text{Lip}(X, \beta)\) such that \(f_n \to f\) in \(\text{Lip}(X, d).\) Then

\[
|f(x) - f(y)| / d(x, y) \leq \lim_{n \to \infty} |f_n(x) - f_n(y)| / d(x, y) = 1
\]

for \((x, y) \in X \times X - \Delta.\) Hence \(f\) is in the unit ball of \(\text{Lip}(X, \beta),\) so the unit ball of \(\text{Lip}(X, \beta)\) is compact in \(\text{Lip}(X, d).\)

3. Corollary. Let \(X, \beta\) be as in (3.1). Then \(\text{Lip}(X, \beta) \to \text{Lip}(X, d)\) is a compact operator.

If \(X\) is a space with pseudometrics \(d\) and \(d',\) we will say that \(d'\) is \(o(d)\) if given \(\varepsilon > 0,\) there exists \(\delta > 0\) such that \(d(x, y) < \delta\) implies \(d'(x, y) < \varepsilon d(x, y).\) (Note the analogy to functions in \(\text{Lip}(X, d).\))

3.3. Corollary. Let \((X, d)\) be a compact metric space and \(d'\) a metric on \(X\) which is \(o(d).\) Then the unit ball of \(\text{Lip}(X, d')\) is compact in \(\text{Lip}(X, d).\)

Proof. The proof is the same as the proof of (3.1), except that \(d'(x, y)\) is substituted for \(\beta \circ d(x, y)\) wherever the latter appears.

3.4. Corollary. Let \((X, d)\) be a compact metric space and \(\beta, \gamma\) two modulus of continuity. If \(\gamma\) is subadditive in a neighborhood of zero and \(\lim_{t \to 0} \gamma(t)/t = 0,\) then \(\text{Lip}(X, \beta) \to \text{Lip}(X, \gamma)\) is a compact operator.
Proof. Suppose \( \gamma(t) \) is subadditive for \( t \leq \delta \). Letting \( a(t) = 2^\gamma t \wedge \delta \), \( a(t) \) is subadditive for all \( t \). By (2.5), \( \text{Lip}(X, a) \) is isomorphic under the identity to \( \text{Lip}(X, \gamma) \). Apply the proof of (3.1) again, substituting \( a \circ d \) for \( d \).

The proof of (3.1) relies heavily on Ascoli's theorem. Thus our results are only valid if \( X \) is compact. It would be interesting to know if that restriction can be dropped.

4. \( \cap \{ \text{Lip}(X, \gamma) : \text{lim sup} \beta(t)/t = \infty \} \). A metric space \( (X, d) \) is a starlike space from a point \( p \) in \( X \) if \( d(x, y) = d(p, y) \) for all \( x \in X \) with \( d(x, p) = d(p, x) \). A space \( (X, d) \) is uniformly locally starlike at \( x \) if there exists \( \delta > 0 \) such that the \( \delta \)-neighborhood of each point \( x \) are starlike from \( x \). Trivially, any convex metric space is uniformly locally starlike.

4.1. Lemma. Let \( (X, d) \) be a uniformly locally starlike metric space. Let \( f \) be an \( F \)-valued uniformly continuous function and \( \beta \) its modulus of continuity. Then \( \beta \) is subadditive in a neighborhood of \( 0 \), and if \( \beta(t) \) is not bounded in that neighborhood, then \( \lim \beta(t) = \infty \).

Proof. Let \( \delta > 0 \) such that the \( \delta \)-neighborhood of each point \( p \) is starlike from \( p \). If \( \delta_1 + \delta_2 < \delta \), then \( \beta(\delta_1 + \delta_2) = \sup \{ |f(x) - f(y)| : \|x - y\| < \delta_1 + \delta_2 \} \). When \( d(x, y) < \delta_1 + \delta_2 \), there exists a point \( z \) such that \( d(x, z) < \delta_1 \) and \( d(z, y) < \delta_2 \). Thus \( |f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \). Taking the sup over the left-hand side, we get \( \beta(\delta_1 + \delta_2) \leq \beta(\delta_1) + \beta(\delta_2) \), so \( \beta \) is subadditive.

Consider the sequence \( (2^n \beta(1/2^n))_{n=0}^\infty \). Since \( \beta(2t) \leq 2\beta(t) \) for \( t < \delta \), we have \( 2^n \beta(1/2^n) \leq 2\beta(1/2^{n-1}) \) ultimately. Also for \( 1/2^{n+1} < \delta \), we have

\[
2^n \beta(1/2^{n+1}) \leq \beta(t) \leq 2^{n+1} \beta(1/2^n).
\]

Hence, if \( \beta(t) \) is not bounded, then \( \lim \beta(t) = \infty \).

4.2. Theorem. Let \( (X, d) \) be a uniformly locally starlike metric space. Let \( f \) be an \( F \)-valued function on \( X \). If for any modulus of continuity \( \beta \) such that \( \lim \beta(t) = \infty \), there exists \( K > 0 \) such that

\[
|f(x) - f(y)| \leq K \beta\circ d(x, y) \quad \text{for all } (x, y) \in X \times X,
\]

then \( f \in \text{Lip}(X, d) \).

Proof. Set \( a(t) = \sup \{ |f(x) - f(y)| : \|x - y\| < t \} \). Suppose \( f \) does not satisfy a Lipschitz condition. Since \( f \) must be bounded, we may assume that \( f \) does not satisfy a local Lipschitz condition. Thus \( a(t) \leq Kt \) for \( t < \delta \). Then (4.2) implies that \( a(t) \) is not bounded in any interval \((0, \delta)\). By (4.1), \( \lim a(t)/t = \infty \).

Let \( \gamma(t) = [a(t)/t]^{1/2} \). Then \( \gamma(t) \) is a modulus of continuity, and \( \lim \gamma(t) = \infty \). Thus there exists \( K \) such that \( |f(x) - f(y)| \leq K \gamma d(x, y) \).

Since \( a(t)/t \to \infty \) as \( t \to 0 \), there exists \( \varepsilon > 0 \) such that \( a(t)/t > 2K \) for \( t < \varepsilon \). Then

\[
|f(x) - f(y)| \leq K \gamma d(x, y) = K \gamma d(x, y) \circ d(x, y) \leq 2K \gamma d(x, y) \circ d(x, y) \leq 2K \gamma d(x, y).
\]

But \( a(t) = \sup \{ |f(x) - f(y)| : \|x - y\| < t \} \). The contradiction arose from the assumption that \( f \) did not satisfy a local Lipschitz condition.

4.3. Corollary. \( \cap \{ \text{Lip}(X, \beta) : \text{lim sup} \beta(t)/t = \infty \} \) is \( \text{Lip}(X, \delta) \) whenever \( (X, d) \) is a uniformly locally starlike metric space.

It should be noted that the theorem is false if one considers only a countable number of moduli of continuity \( \beta_n \). By a result in [2], p. 12, we can construct a function \( \beta \) which has an infinite derivative at zero, but satisfies \( \lim \beta_n(t)/\beta(t) = \infty \) for each \( \beta_n \). The constructed function can even be chosen subadditive and piecewise linear.

The proofs given above depend strongly on the fact that \( (X, d) \) is uniformly locally starlike. This seems to be an unnatural restriction, but we are unable to remove it.

References


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