References

[1] M. Cotlar, *Condiciones de continuidad de operadores potenciales y de Hilbert*, Cursos y Seminarios de Matemática 2 (1959), Universidad Nacional de Buenos Aires.

[2] A. González Dominguez, *Contribución a la teoria de funciones de Hille*, Ciencia y Técnica 475 (1941), p. 475-521.

[3] E. Hille, *A class of reciprocal functions*, Ann. Math., 2nd series, 27 (1926), p. 426-464.

[4] — *Bemerkung zu einer Arbeit des Herrn Müntz*, Math. Zeitschrift 32 (1930), p. 421-425.

[5] I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton 1955.

[6] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Coll. Publ. 23 (1959) (revised edition).

[7] D. V. Widder, *Necessary and sufficient conditions for the representation of functions by a Weierstrass transform*, Trans. Amer. Math. Soc. 71 (1951), p. 430-439.

[8] A. Zygmund, *Trigonometrical series*, Vols I and II, 1959.

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On the control of linear periodic time lag systems

by

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Let us consider a linear time lag system

$$\sum_{k=0}^{p}\sum_{j=0}^{q} F_{kj}(t)x^{(k)}(t-h_j) = \sum_{j=0}^{q} G_j(t)u(t-h_j), \tag{1}$$

where $x(t)$ is an n-dimensional vector-function called *state*, $u(t)$ is an m-dimensional vector-function called *control*, $F_{kj}(t)$ are $n\times n$ matrix functions, $G_i(t)$ are $n\times m$ matrix functions, all of the real variable t.

We assume that all functions $u(t)$, $F_{kj}(t)$ and $G_j(t)$ are measurable and locally integrable on the real line. An n-dimensional vector-function is called a *solution* of (1) if there exists a $(p-1)$-th derivative of $x(t)$ which is absolutely continuous and if $x(t)$ satisfies (1) almost everywhere.

Let $h_0 = 0$ and let h_j be commensurable. Then we have an $r\neq 0$ and integers n_j such that $h_j = n_j r$ for $j = 1, 2, \ldots, q$. Let us assume that the functions $F_{kj}(t)$ and $G_j(t)$ are r-periodic ([1]). Let N be a common multiple of numbers $n_1, \ldots, n_q$ (not necessarily the smallest one) and let $\omega = Nr$. We shall consider system (1) in the class of ω-periodic functions.

Suppose we are given the following performance functional:

$$\mathscr{K}(u, x) = \int_0^{\omega} K\big(t, [x(t)-x^0(t), u(t)-u^0(t)]\big)\,dt, \tag{2}$$

where $[x, u]$ is an $(n+m)$-dimensional vector $(x_1, \ldots, x_n, u_1, \ldots, u_m)$ and $K(t, [x, u])$ for each fixed t is a non-negative quadratic form defined on an $(n+m)$-dimensional space and $x^0(t), u^0(t)$ are given functions. We assume that $K(t, [x, u])$ is an r-periodic square integrable function with respect to t.

The aim of this note is to minimalize the performance functional (2) under the assumption that $x(t), u(t)$ satisfies equation (1).

The manner in which the proposed question will be solved is based on the method of involution (see [3], also [5]).

([1]) A periodic function with period s will be called briefly an *s-periodic function*.

Let X be a space of n-dimensional measurable ω-periodic square integrable vector-functions $x(t) = (x_1(t), \ldots, x_n(t))$ of a real variable. Let U be a space of m-dimensional measurable ω-periodic square integrable vector-functions $u(t) = (u_1(t), \ldots, u_m(t))$. Let us consider the space $Y = X \times U$. The performance functional (2) induces in the space $Y = X \times U$ an inner product. Namely

$$(3) \qquad ([x, u], [\xi, \eta]) = \int_0^\omega \tilde{K}(t, [x(t), u(t)], [\overline{\xi(t)}, \overline{\eta(t)}])\, dt,$$

where $\tilde{K}(t, [x, u], [\xi, \eta])$ is the bilinear form induced by the quadratic form $K(t, [x, u])$.

Now we shall define an operator S acting in X (respectively, in U and in Y)

$$(Sx)(t) = x(t-r), \qquad (Su)(t) = u(t-r),$$
$$S[x(t), u(t)] = [x(t-r), u(t-r)].$$

Since X and U are spaces of ω-periodic functions, the operator S is an involution of order N in X and in U and also in Y, i.e. $S^N = I$, where I denotes the identity operator.

Using the same method as in paper [3] we can decompose the spaces X and U into direct sums

$$X = X_{(1)} \oplus \ldots \oplus X_{(N)}, \qquad U = U_{(1)} \oplus \ldots \oplus U_{(N)}$$

such that

$$(4) \qquad \begin{aligned} (Sx)(t) &= e^{\frac{2\pi i}{N}j} x(t) \quad \text{for } x(t) \in X_{(j)}, \\ (Su)(t) &= e^{\frac{2\pi i}{N}j} u(t) \quad \text{for } u(t) \in U_{(j)}. \end{aligned} \qquad (j = 1, 2, \ldots, N)$$

Formulae (4) imply that for $[x(t), u(t)] \in Y_{(j)} = X_{(j)} \times U_{(j)}$ equation (1) is transformed into a differential equation without time lag (obtained in the same manner as in [4]):

$$(5) \qquad \sum_{k=0}^{p} \tilde{F}_{kj}(t) x_{(j)}^{(k)}(t) = \tilde{G}_j(t) u_{(j)}(t), \qquad x_{(j)} \in X_{(j)},\ u_{(j)} \in U_{(j)},$$

where

$$\tilde{F}_{kj}(t) = \sum_{\nu=0}^{q} e^{\frac{-2\pi i}{N}\nu n_j} F_{k\nu}(t), \qquad \tilde{G}_j(t) = \sum_{\nu=0}^{q} e^{\frac{-2\pi i}{N}\nu n_j} G_\nu(t).$$

Formulae (4) imply also that the spaces $Y_{(j)}$ are orthogonal one to another with respect to the inner product (3). Indeed, since $K(t, [x, u])$

is r-periodic with respect to t, S is a unitary transformation. Therefore, if $y_{(j)} \in Y_{(j)}$, $y_{(k)} \in Y_{(k)}$, then

$$(y_{(j)}, y_{(k)}) = (Sy_{(j)}, Sy_{(k)}) = e^{\frac{2i}{N}j - \frac{2i}{N}k}(y_{(j)}, y_{(k)}).$$

Hence $(y_{(j)}, y_{(k)}) = 0$ if $j \neq k$.

Then the method of solving the problem is the following. We decompose the space Y into an orthogonal direct sum $Y = Y_{(1)} \oplus \ldots \oplus Y_{(N)}$. This decomposition induces also a decomposition of the element $y^0(t) = [x^0(t), u^0(t)]$:

$$\begin{aligned} x^0(t) &= x^0_{(1)}(t) + \ldots + x^0_{(N)}, \quad \text{where } x^0_{(j)} \in X_{(j)}, \\ u^0(t) &= u^0_{(1)}(t) + \ldots + u^0_{(N)}(t), \quad \text{where } u^0_{(j)} \in U_{(j)}. \end{aligned} \qquad (j = 1, 2, \ldots, N)$$

We consider the differential equation (5) in $Y_{(j)}$ and minimalize the functional

$$(6) \qquad \int_0^\omega K(t, [x_{(j)}(t) - x^0_{(j)}(t) u_{(j)}(t) - u^0_{(j)}(t)])\, dt.$$

It is easy to check that for any $[x_{(j)}, u_{(j)}] \in Y_{(j)}$

$$(7) \qquad \int_0^\omega K(t, [x_{(j)}(t) - x^0_{(j)}(t), u_{(j)}(t) - u^0_{(j)}(t)])\, dt = N \int_0^r K(t, [x_{(j)}(t) - x^0_{(j)}, u_{(j)}(t) - u^0_{(j)}(t)])\, dt.$$

The solution of the initial problem is of the form

$$x(t) = x_{(1)}(t) + \ldots + x_{(N)}(t), \qquad u(t) = u_{(1)}(t) + \ldots + u_{(N)}(t),$$

where $[x_{(j)}, u_{(j)}] \in Y_{(j)}$ is a pair minimalizing (6) under condition (5).

The method described above can also be used in cases where we have some constraints. We give here two examples.

Let us consider system (1) with the additional assumption that $N = 2$. Let us minimalize functional (2). Let us assume that $u(t) = (u_1(t), \ldots, u_m(t))$ is constrained on each coordinate, i.e. that $|u_j(t)| \leq M_j$.

It is not difficult to verify that $|u_i(t)| \leq M_i$ if and only if $|u_{(1)i}(t)| + |u_{(2)i}(t)| \leq M_i$, where $u_{(1)}(t) = \frac{1}{2}[u(t) - u(t-r)]$, $u_{(2)}(t) = \frac{1}{2}[u(t) + u(t-r)]$. Then we can consider $u(t)$ as a $2m$-dimensional vector $[u_{(1)1}(t), u_{(2)1}(t), \ldots, u_{(1)m}(t), u_{(2)m}(t)]$. Similarly, $x(t)$ can be considered as a $2n$-dimensional vector $[x_{(1)1}(t), x_{(2)1}(t), \ldots, x_{(1)n}(t), x_{(2)n}(t)]$. Thus system (5) can be considered as one $2n$-dimensional system.

Formula (7) implies that we minimalize the performance functional

$$2\Big[\int\limits_0^r K\big(t,[x_{(1)}(t)-x_{(1)}^0(t),\,u_{(1)}(t)-u_{(1)}^0(t)]\big)\,dt+ + \int\limits_0^r K\big(t,[x_{(2)}(t)-x_{(2)}^0(t),\,u_{(2)}(t)-u_{(2)}^0(t)]\big)\,dt\Big]$$

with constraints

$$|u_{(1)j}(t)|+|u_{(2)j}(t)|\leqslant M_j \quad (j=1,2,\ldots,m).$$

The second example is the following. We consider a non-negative quadratic form $C(t,[x,u])$ defined for each fixed t on an $(m+n)$-dimensional space. Let us assume that $C(t,[x,u])$ is r-periodic and square integrable with respect to t. Let the constraint be the following:

$$\mathscr{C}(x,u)=\int\limits_0^\omega C\big(t,[x(t),u(t)]\big)\,dt\leqslant M^2. \tag{8}$$

Obviously the spaces $Y_{(j)}$ are also orthogonal with respect to the inner product induced by $\mathscr{C}(x,u)$. Therefore the method of minimalizing (2) under condition (1) with constraint (8) is the following. We assume that

$$|y_{(j)}(t)|\leqslant M_j \quad (j=1,2,\ldots,N), \tag{9}$$

and minimalize (6) under condition (5) with constraint (9). This minimum depends on M . We obtain in this way the minimum of (2) under condition (1) with constraints (9). This minimum depends on $(M_1,\ldots,M_N)$. Then we minimalize this minimum with respect to $(M_1,\ldots,M_N)$ under the condition

$$M_1^2+\ldots+M_N^2\leqslant M^2.$$

We obtain a minimum of functional (2) under condition (1) with the constraint (8).

References

[1] A. Manitius, *Optymalne sterowanie procesów z opóźnieniami zmiennych stanu*, Ph. d. thesis, Politechnika Warszawska 1968.

[2] N. Namik Oğuztöreli, *Time lag control systems*, New York—London 1966.

[3] D. Przeworska-Rolewicz, *Sur les involutions d'ordre n*, Bull. Acad. Polon. Sci. 8 (1960), p. 735-739.

[4] — *On periodic solutions of linear differential-difference equations with constant coefficients*, Studia Math. 31 (1968), p. 69-73.

[5] — and S. Rolewicz, *Equations in linear spaces*, Warszawa 1968.

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О задаче Коши для нелинейных параболических операторных уравнений*

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ВВЕДЕНИЕ

Теория краевых задач для квазилинейных эллиптических и параболических уравнений или систем уравнений высшего порядка, создание которой началось совсем недавно, успешно развивается в настоящее время. Первые результаты в этой области принадлежат Вишику [7, 8]. Уравнения, исследованные Вишиком, характеризуются положительной определенностью первой вариации оператора, соответствующего главной части уравнения. Этот выделенный класс нелинейных уравнений, имеющих дивергентные формы, им назван *сильно эллиптическим*. Эти исследования были продолжены Браудером [3, 4] в цикле его работ. Основываясь на им же доказанных общих теоремах существования для нелинейных функциональных уравнений, обладающих более слабым свойством непрерывности, а также свойством монотонности, он показал разрешимость краевых задач для квазилинейных эллиптических и параболических уравнений, обладающих теми же свойствами. Эти работы Браудера по идее явились развитием работы Минти [18] по монотонным операторам. Отметим, что впервые монотонные функциональные уравнения были рассмотрены Вайнбергом и Качуровским [6].

В 1965 году Дубинский [10-11] дал простые доказательства основных теорем Вишика и Браудера. Одновременно он получил ряд новых результатов, относящихся к краевым задачам для квазилинейных эллиптических и параболических уравнений недивергентной формы, а также для вырождающихся уравнений. Примененный им метод основывается на слабой сходимости в лебеговских пространствах L_p.

* Основные результаты работы были доложены в октябре-ноябре 1967 года на заседаниях Математического общества в Институте математики ПАН и в его Краковском отделении.