determined by A_n and let Z_p and $(Z\circ Q^{-1})_p$ denote the singular integral and tame singular integrals determined by A. For f in $L^p(H)$,

$$\|Z_p(f)-(Z\circ Q^{-1})_p(f)\|_p$$
$$\leqslant \|Z_p(f)-Z_p^n(f)\|_p+\|Z_p^n(f)-(Z\circ Q^{-1})_p^n(f)\|_p+\|(Z\circ Q^{-1})_p^n(f)-(Z\circ Q^{-1})_p(f)\|_p.$$

As has been shown in [1], the first and third terms on the right are each dominated by a constant multiple of $\|A-A_n\|_1$. So for $\varepsilon>0$ there is an integer N such that for $n\geqslant N$, the first and third terms on the right of this inequality are each $<\varepsilon/3$. Fix $n\geqslant N$. By the above argument we know that the second term on the right converges to zero as Q tends strongly to the identity through $\mathscr{F}$. Thus Z_p is the strong limit of the net $\{(Z\circ Q^{-1})_p \,|\, Q\in\mathscr{F}\}$ when $A(y)$ is an absolutely integrable odd function. A similar argument completes the proof for even r-power integrable $(r\geqslant 1)$ tame functions $A(y)$ with $E(A)=0$.

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THE JOHN HOPKINS UNIVERSITY

Reçu par la Rédaction le 2. 2. 1968

STUDIA MATHEMATICA, T. XXXII. (1969)

Some remarks on the multiple Weierstrass transform and Abel summability of multiple Fourier-Hermite series

by

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INTRODUCTION

The purpose of this paper is to extend to the m-dimensional case some theorems given in [2], [3] and [4] concerning the inversion formula of the Weierstrass Transform and the Abel summability of Fourier-Hermite series. The theorems of the present paper are referred to the measure

$$e^{-|x|^2}dx = e^{-\sum_1^m x_j^2}dx_1\ldots dx_m,$$

case which is not included in [2], [3], [4] and [6]; on the other hand, we also give maximal theorems with respect to Abel Summability of multiple Fourier-Hermite series and to the inversion formula for the multiple Weierstrass Transform.

The first part of the paper is devoted to the study of theorems of general character concerning differentiation of multiple integrals which have to be used in the second part, the specific problem.

I would like to thank Prof. A. González Dominguez who proposed the problem to me and to Prof. A. P. Calderón for many helpful suggestions.

NOTATION

1. By x we denote a point $(x_1, \ldots, x_m)$ of the Euclidean m-dimensional space:

$$|x| = \Big(\sum_{j=1}^m x_j^2\Big)^{1/2}.$$

2. If μ is an elementary measure defined on $\boldsymbol{R}^m$, it is an additive function of the subsets of $\boldsymbol{R}^m$ which are finite union of m-dimensional intervals. The variation W of μ on a cube $Q\subset \boldsymbol{R}^m$ is defined in the following way:

$$W(Q) = \sup_{S\subset Q}\sum_{j=1}^l |\mu(S_j)|,\qquad S=\bigcup_{j=1}^l S_j,\qquad S_i\cap S_j=\emptyset\quad \text{if } i\neq j,$$

where the sup is taken over all the possible elementary sets contained in Q. If the measure μ is completely additive we shall call it σ-additive.

3. If $\mu \geqslant 0$ is σ-additive on the Borel subsets of $\boldsymbol{R}^m$, we shall use the following classes of μ-measurable functions:

a) $L_\mu^p(\boldsymbol{R}^m)$ is the set of functions such that

$$\Big(\int_{\boldsymbol{R}^m} |f|^p\, d\mu\Big)^{1/p} = \|f\|_{p,\mu} < \infty, \qquad p \geqslant 1.$$

b) $L_\mu(\log^+ L_\mu)^s$ is the set of functions such that

$$\int_{\boldsymbol{R}^m} |f|(\log^+|f|)^s\, d\mu < \infty, \qquad s > 0.$$

If $d\mu = e^{-|x|^2}dx$, we shall use the notation $L_G^p(\boldsymbol{R}^m)$, $L_G(\log^+ L_G)^s$ for a) and b) respectively.

Finally, we shall use an auxiliary class of functions $I_\gamma(\boldsymbol{R}^m)$, $0 < \gamma < 1$, the class of measurable functions such that

$$\int_{\boldsymbol{R}^m} f\, e^{-\gamma|x|^2}\, dx < \infty$$

and $I_{-1/2}(\boldsymbol{R}^m)$ will denote the class of functions or measures such that

$$\int_{\boldsymbol{R}^m} e^{|x|^2/2} f(x)\, dx < \infty, \qquad \int_{\boldsymbol{R}^m} e^{|x|^2/2}\, dw < \infty,$$

where dw denotes the variation of μ.

4. $E(\lambda, f)$, $f \geqslant 0$, will denote the set of points where $f > \lambda$. $\mu\{E(\lambda, f)\}$, $G\{E(\lambda, f)\}$ and $|E(\lambda, f)|$ denote the μ-measure of $E(\lambda, f)$, $\int_{E(\lambda,f)} e^{-|x|^2}\, dx$ and the Lebesgue measure of $E(\lambda, f)$, respectively.

1. PREVIOUS LEMMAS

1.1. We shall denote by $I(t, h, f)(x)$ an operator of restrictive derivation defined on $\boldsymbol{R}^m$, i.e.,

$$I(t, h, f)(x) = \frac{1}{\prod_{j=1}^{m} 2h_j(t)} \int_{Q(h,t,x)} f(y)\, dy, \tag{1.1.1}$$

where the $h_j(t)$, $j = 1, \ldots, m$, are continuous functions of $t \geqslant 0$ strictly increasing to $+\infty$ and 0 at $t = 0$; $Q(h, t, x)$ is an m-dimensional rectangle, centered at the point x, with edges of length $2h_j(t)$ parallel to the coordinate axes. (The edge of length $2h_j(t)$ being parallel to the axis x_j).

By f we denote a measurable function, locally integrable, defined on $\boldsymbol{R}^m$. We can also define the operator over measures defined on $\boldsymbol{R}^m$.

1.2. It is well known that the following inequality holds:

$$|E\{\mu^*, \lambda\}| < \frac{C}{\lambda} \int_{\boldsymbol{R}^m} dw, \tag{1.2.1}$$

where

$$\mu^*(x) = \sup_{0<t<\infty} |I(t, h, \mu)(x)|$$

and μ is an additive measure defined on the elementary subsets of $\boldsymbol{R}^m$ (finite union of intervals), having bounded variation there, and denoting by dw its variation.

The constant C of (1.2.1) depends neither on the measure μ nor on the functions $h_j(t)$ of the operator (see [8], Vol II, p. 309 and 310). An important consequence of inequality (1.2.1) is that the limit

$$\lim_{t\to 0} I(t, h, \mu)(x) \tag{1.2.2}$$

exists a.e. and it is a.e. equal to the density function associated to μ.

1.3. LEMMA. *Let $K_t(\mu)(x)$ be a family of sublinear operators defined on the space of the elementary measures (in the sense of 1.2) defined on $\boldsymbol{R}^m$ such that*

(a) $$|K_t(\mu)(x)| \leqslant \sum_0^\infty b_k |I^{(k)}(t, \mu)(x)|, \qquad b_k \geqslant 0,\ t > 0,$$

(b) $$\sum b_k^{1/2} < \infty,$$

where for each $k > 0$, $I^{(k)}(t, \mu)(x)$ is an operator of restrictive derivation.

Under the preceding assumptions we have

(i) $$|E\{K^*(\mu), \lambda\}| < \frac{C_0}{\lambda} \int_{\boldsymbol{R}^m} dw,$$

where $K^*(\mu)$ denotes $\sup_{0<t<\infty} |K_t(\mu)(x)|$, and dw denotes the variation of the measure μ. C_0 does not depend on μ.

(ii) *If μ is singular we have*

$$\lim_{t\to 0} K_t(\mu)(x) = 0 \qquad \textit{a.e. in } \boldsymbol{R}^m.$$

Proof. Without loss of generality we may assume that $\mu \geqslant 0$, and

$$\sum_0^\infty b_k^{1/2} = 1.$$

Let X_λ be the set of points where

$$\sup_{0<t<\infty} \sum_0^\infty b_k^{1/2} b_k^{1/2} I^{(k)}(t, \mu)(x) > \lambda$$

and $X_\lambda^{(k)}$ the set of points where

$$\sup_{0<t<\infty} b_k^{1/2} I^{(k)}(t, \mu)(x) > \lambda.$$

Thus we have

$$X_\lambda \subset \bigcup_{k=0}^\infty X_\lambda^{(k)} \tag{1.3.1}$$

and so, according to (1.2.1) we obtain

$$|X_\lambda| \leqslant \sum_{k=0}^\infty |X_\lambda^{(k)}| \leqslant \frac{C}{\lambda} \sum_{k=0}^\infty b_k^{1/2} \int_{R^m} dw,$$

which proves part (i).

Now let $\mu \geqslant 0$ be singular and $\varepsilon > 0$, then there exists an integer number $N > 0$, such that $\sum_{k=N}^\infty b_k^{1/2} < \varepsilon$. If we denote by X_ε^N the set of points where

$$\sup_{0<t<\infty} \sum_{k=N}^\infty b_k^{1/2} b_k^{1/2} I^{(k)}(t, \mu)(x) > \varepsilon; \tag{1.3.2}$$

it is clear that

$$X_\varepsilon^N \subset \bigcup_{k=N}^\infty X^{(k)}_{\varepsilon/\sum_N^\infty b_s^{1/2}} \tag{1.3.3}$$

and so from (1.2.1) again, we have

$$|X_\varepsilon^N| \leqslant \sum_{k=N}^\infty |X^{(k)}_{\varepsilon/\sum_N^\infty b_s^{1/2}}| \leqslant \frac{C}{\varepsilon}\Big[\sum_N^\infty b_s^{1/2}\Big] \cdot \sum_{k=N}^\infty b_k^{1/2} \int_{R^m} dw \tag{1.3.4}$$
$$\leqslant \varepsilon C \int_{R^m} dw.$$

So in $R^m - X_\varepsilon^N$ we have

$$\overline{\lim_{t\to 0}}\, |K_t(\mu)(x)| \leqslant \overline{\lim_{t\to 0}} \sum_0^N b_k I^{(k)}(t, \mu)(x) + \sup_{0<t<\infty} \sum_N^\infty b_k I^{(k)}(t, \mu)(x) \leqslant \varepsilon.$$

Since $|X_\varepsilon^N| \leqslant \varepsilon C \int_{R^m} dw$, part (ii) follows.

1.4. Definition. Let $F_j(x), j = 1, 2, \ldots, m$, be real functions of the single variable x defined on the real line and belonging to $L^1(R)$. Then we can form with them the kernel

$$\prod_{j=1}^m n_j(t) F_j[n_j(t) x_j] = K(t, x), \tag{1.4.1}$$

where the functions $1/n_j(t)$ play the same role as $h_j(t)$ of 1.1, that is, the $1/n_j(t)$ are continuous functions of $t \geqslant 0$, strictly increasing to $+\infty$ and 0 at $t = 0$.

We shall impose to the $F_j(x)$ the following conditions:

a) the $F_j(x)$, $j = 1, 2, \ldots, m$, are symmetric and non-increasing on $x > 0$;

b) there exists $\varepsilon > 0$ and $1 > \theta > 0$, such that

$$\int_{|x|<\varepsilon} |F_j(x)|^{1+\theta} dx < \infty, \quad j = 1, 2, \ldots, m. \tag{i}$$

$$\int_{|x|>\varepsilon} |F_j(x)|^{1-\theta} dx < \infty, \quad j = 1, 2, \ldots, m. \tag{ii}$$

1.5. Lemma. *If the functions of the kernel* (1.4.1) *have the properties* a) *and* b), *and if μ is an additive function of the elementary subsets of R^m (finite union of intervals), and with bounded variation there, then the operator*

$$\sup_{0<t<\infty} \Big|\int_{R^m} K(t, x-y)\, d\mu(y)\Big| = \sup_{0<t<\infty} |K_t * \mu| = \bar{\mu} \tag{1.5.1}$$

has the following property:

1) $|E(\bar{\mu}, \lambda)| < \frac{C}{\lambda} \int_{R^m} dw(\mu)$, *where C does not depend on μ, and $W(\mu)$ denotes the variation of the measure μ on R^m, and, furthermore, the operators $K_t * \mu$ have the property*:

2) $\lim_{t\to 0} K_t * \mu = 0$ *a.e. if μ is singular.*

Proof. From properties a) and b) of (1.4) we have for $|y| > \varepsilon$

$$\sup_y |y| F_j(y)^{1-\theta} < \int_{-\infty}^\infty F_j(y)^{1-\theta} dy \tag{1.5.2}$$

so, if $|y| \geqslant \varepsilon$,

$$F_j(y) < \frac{A}{|y|^{1/(1-\theta)}},$$

where A depends on the bound of $\int_{-\infty}^\infty F_j^{1-\theta} dy$ and on $\varepsilon > 0$. On the other hand, for $|y| < \varepsilon$ we have

$$(1.5.3)\qquad \sup_y |y| F_j(y)^{1+\theta} < \int_{-\varepsilon}^{\varepsilon} F_j(y)^{1+\theta}\, dy,$$

$$F_j(y) < \frac{B}{|y|^{1/(1+\theta)}} \quad \text{if } |y| < \varepsilon,$$

where B depends on $\varepsilon > 0$ and on the value of $\int_{-\varepsilon}^{\varepsilon} F_j^{1+\theta} dy$. From the form of the F_j it is easy to see that we can take $\varepsilon = 1$. Now calling $\varphi_{2^k}(y)$ to the characteristic function of the interval $[2^k, 2^{k+1}]$, and $\varphi_{2^{-k}}(y)$ to the characteristic function of the interval $[2^{-(k+1)}, 2^{-k}]$, then from (1.5.2) and (1.5.3) we have the following estimate for $y \geqslant 0$:

$$(1.5.4)\qquad F_j(y) \leqslant C_0\Big(\sum_{k=0}^{\infty} 2^{-k/(1-\theta)}\varphi_{2^k}(y) + \sum_{k=0}^{\infty} 2^{(1+k)/(1+\theta)}\varphi_{2^{-k}}(y)\Big).$$

Finally, if we call Ψ_k to the characteristic function of the interval $[-2^{k+1}, 2^{k+1}]$ and Ψ_{-k} to the characteristic function of the interval $[-2^{-k}, 2^{-k}]$ we have again

$$F_j(y) \leqslant C_0\Big(\sum_{k=0}^{\infty} 2^{-k/(1-\theta)}\Psi_k(y) + \sum_{k=0}^{\infty} 2^{(1+k)/(1+\theta)}\,\Psi_{-k}(y)\Big)$$

and

$$(1.5.5)\qquad n_j(t)F_j[n_j(t)y_j] \leqslant C_0\Big\{2^{-2}\sum_{k=0}^{\infty} 2^{-k(1/(1-\theta)-1)}\frac{n_j(t)}{2^{k+2}}\Psi_k[n_j(t)y_j] + {}$$
$$+2^2\sum_{k=0}^{\infty} 2^{-k(1-1/(1+\theta))} n_j(t)\,2^{k+1}\Psi_{-k}[n_j(t)y_j]\Big\}.$$

Setting $a_k = 2^{-k(1/(1-\theta)-1)}$ and $a_{-k} = 2^{-k(1-1/(1+\theta))}$ and according to (1.5.5) and (1.4.1) we have

$$(1.5.6)\qquad K(t, y) \leqslant 8^m C_0^m \sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} a_{k_1}\ldots a_{k_m} D_{k_1\ldots k_m}(t, y),$$

where $D_{k_1\ldots k_m}(t, x)$ are functions which generate each one of them an operator of restrictive derivation; and since

$$\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} a_{k_1}^{1/2}\ldots a_{k_m}^{1/2} < \infty,$$

from the lemma (1.3) the parts 1) and 2) of the thesis follow.

1.6. Corollary. *In the preceding lemma it is easy to see that if in the place of the $n_j(t)$ we take functions of the form $a_j n$ $(j = 1, \ldots, m)$, where the a_j are fixed constants and the n's run over the non-negative integers, then the same conclusions hold.*

1.7. Definition. Let $\mu_j \geqslant 0$, $j = 1, 2, \ldots, m$, be non-negative and σ-additive measures defined on the Borel subsets of $\boldsymbol{R}^{n_j}$. We say that the integral $\int_J f d\mu$, where $d\mu$ is the product measure $d\mu_1\ldots d\mu_m$ and f is a μ-measurable and μ-locally integrable function, is *strongly differentiable* at the point $x_0 = (\mathring{x}_1, \ldots, \mathring{x}_m)$, $\mathring{x}_j \in R^{n_j}$, if the limit

$$\lim_{d[Q^1(\mathring{x}_1)\times\ldots\times Q^m(\mathring{x}_m)]\to 0} \frac{1}{\prod_{j=1}^{m}\mu_j\{Q^j(\mathring{x}_j)\}}\int_{Q^1(\mathring{x}_1)\times\ldots\times Q^m(\mathring{x}_m)} f d\mu_1\ldots d\mu_m$$

exists and is finite.

$d[Q^1(\mathring{x}_1)\times\ldots\times Q^m(\mathring{x}_m)]$ denotes the diameter of $Q^1(\mathring{x}_1)\times\ldots\times Q^m(\mathring{x}_m)$, and each $Q^j(\mathring{x}_j)$ denotes an n^j-dimensional cube with sides parallel to the coordinate axes and centered at the point $\mathring{x}_j$.

1.8. Theorem. *Under the conditions of (1.7) if f is μ-measurable and if $|f|(\log^+|f|)^{m-1}$ is locally integrable, then the integral $\int_J f d\mu$ is strongly differentiable a. e. with respect to the measure $\mu = \mu_1\ldots\mu_m$.*

Furthermore, setting

$$\bar f(x) = \sup_{Q^1(x_1)\times\ldots\times Q^m(x_m)}\left|\frac{1}{\prod_{j=1}^{m}\mu_j(Q^j(x_j))}\int_{Q^1(x_1)\times\ldots\times Q^m(x_m)} f\,d\mu\right|$$

we have

$$\text{(i)}\quad \Big(\int_{\boldsymbol{R}^{n_1+\ldots+n_m}} (\bar f)^p d\mu\Big)^{1/p} < C(p)\Big(\int_{\boldsymbol{R}^{n_1+\ldots+n_m}} |f|^p d\mu\Big)^{1/p}\quad (p > 1)$$

where the constant $C(p)$ depends only on p and m.

(ii) *If A is an elementary subset of $\boldsymbol{R}^{n_1+\ldots+n_m}$ having bounded measure we have*

$$\text{(a)}\quad \int_A \bar f\, d\mu \leqslant B_1\{\mu(A), m\} + \Big[\int_A |f|(1+\log^+|f|)^m d\mu\Big] B_2\{\mu(A), m\},$$

$$\text{(b)}\quad \int_A (\bar f)^a d\mu \leqslant C_1\{\mu(A), m, a\} + C_2\{\mu(A), m, a\}\int_A |f|(1+\log^+|f|)^{m-1} d\mu,$$

where $0 < a < 1$.

The proof follows very closely that of the correlative result of Jessen-Marcinkiewicz-Zygmund and therefore we are going to sketch it only. First we shall prove some auxiliary lemmas of very well known type.

1.9. *If for each point x of a bounded set $S \subset \boldsymbol{R}^m$ there is given a cube $Q(x)$ (with sides parallel to the coordinate axes), it is possible to select a sequence $\{Q(x_n)\}$ of these cubes such that:*

(i) $S \subset \bigcup_{j=1}^{\infty} Q(x_j)$;

(ii) *each point of $\boldsymbol{R}^m$ belongs at most to 4^m different cubes of the family $\{Q(x_j)\}$.*

For the proof see [1], p. 125-128.

Remark. The same conclusion holds if we consider an open subset of R^m as space.

1.10. Lemma. *If $\nu \geqslant 0$ is an additive elementary measure defined on $\boldsymbol{R}^m$ such that $\nu(\boldsymbol{R}^m) < \infty$, and if $\mu \geqslant 0$ is a σ-additive measure defined on the Borel subsets of $\boldsymbol{R}^m$, then the operation*

$$\overset{*}{\nu}(x) = \sup_{Q(x)\supset\{x\}} \frac{\nu[Q(x)]}{\mu[Q(x)]} \tag{1.10.1}$$

(*where the $Q(x)$ are cubes centered at x*) *has the property*

(i) $\mu\{E(\overset{*}{\nu}, \lambda)\} < \dfrac{4^m}{\lambda} \displaystyle\int_{\boldsymbol{R}^m} d\nu$.

(ii) *If $f \in L_\mu^p(\boldsymbol{R}^m)$, $p > 1$, then*

$$\Big(\int_{\boldsymbol{R}^m} \overset{*}{f}{}^p d\mu\Big)^{1/p} < O_p\Big(\int_{\boldsymbol{R}^m} |f|^p d\mu\Big)^{1/p}.$$

(iii) *If A is an elementary subset of bounded μ-measure, we have*

(a) $$\int_A \overset{*}{f} d\mu \leqslant \mu(A) + O' \int_A |f|(1+\log^+|f|)\, d\mu,$$

(b) $$\int_A (\overset{*}{f})^a d\mu \leqslant O_a'' \mu(A) + O_a''' \int_A |f|\, d\mu, \qquad 0 < a < 1.$$

(iv) *If f is locally μ-integrable, then*

$$\lim_{Q(x)\to\{x\}} \frac{1}{\mu\{Q(x)\}} \int_{Q(x)} f\, d\mu = f(x)$$

a. e. with respect to μ.

Proof. Let S be a bounded elementary set on $\boldsymbol{R}^m$ and consider $E\{\overset{*}{\nu}, \lambda\} \frown S$. From (1.10.1) for each point x of $E\{\overset{*}{\nu}, \lambda\} \frown S$ there exists a cube $Q(x)$ such that

$$\lambda\mu[Q(x)] < \nu[Q(x)]. \tag{1.10.2}$$

From 1.9 we can select a sequence of such cubes verifying (i) and (ii) of 1.9.

Let us take a finite number of such cubes, $Q(x_1), \ldots, Q(x_k)$, and denote by $\varphi_j(x)$, $j = 1, \ldots, k$, the corresponding characteristic functions; then

$$\mu\{\bigcup_1^k Q(x_j)\} \leqslant \int \sum_1^k \varphi_j(x)\, d\mu \leqslant \sum_1^k \int_{Q(x_j)} d\mu \tag{1.10.3}$$
$$\leqslant \sum_1^k \frac{1}{\lambda} \int_{Q(x_j)} \varphi_j(x)\, d\nu = \frac{1}{\lambda} \int \sum_1^k \varphi_j(x)\, d\nu \leqslant \frac{4^m}{\lambda}\nu(\boldsymbol{R}^m).$$

The last inequality holds since $\sum_1^k \varphi_j(x) \leqslant 4^m$ everywhere. Then

$$\mu\{\bigcup_1^k Q(x_j)\} \leqslant \frac{4^m}{\lambda}\nu(\boldsymbol{R}^m) \tag{1.10.4}$$

and since k is arbitrary, we have

$$\mu\{E(\overset{*}{\nu}, \lambda) \frown S\} \leqslant \mu\{\bigcup_1^\infty Q(x_j)\} < \frac{4^m}{\lambda}\nu(\boldsymbol{R}^m).$$

Finally, letting S tend to the whole $\boldsymbol{R}^m$ we have part (i) of the thesis.

Part (ii) immediately follows combining inequality (i) with $\|\overset{*}{f}\|_\infty \leqslant \|f\|_\infty$ and using the well known theorem of Marcinkiewicz on interpolation of sublinear operators.

Now let us consider $f \geqslant 0$, μ-measurable and such that $f\log^+ f$ is μ-locally integrable and let us take an open neighborhood A as space, provided that $\mu(A) < \infty$. Putting $f = f^{\lambda/2} + f_{\lambda/2}$, where

$$f^{\lambda/2} = f \quad \text{if} \quad f(x) \leqslant \lambda/2 \quad \text{and 0 otherwise}, \tag{1.10.5}$$

$$f_{\lambda/2} = f \quad \text{if} \quad f(x) > \lambda/2 \quad \text{and 0 otherwise}, \tag{1.10.5}$$

we obtain

$$\mu\{E(\lambda, f^*)\} \leqslant \mu\{E[\lambda/2, (f^{\lambda/2})^*]\} + \mu\{E[\lambda/2, (f_{\lambda/2})^*]\} \tag{1.10.6}$$

but $\mu\{E[\lambda/2, (f^{\lambda/2})^*]\} = 0$ since $\|\overset{*}{g}\|_\infty < \|g\|_\infty$.

Using the estimate (i) of the thesis and (1.10.6), we have

$$\int_A f^* d\mu = \int_0^\infty \mu\{E(x\lambda, f^*)\}\, d\lambda \leqslant \mu(A) + \int_1^\infty \frac{2\cdot 4^m}{\lambda}\, d\lambda \int_A f_{\lambda/2}\, d\mu \tag{1.10.7}$$
$$= \mu(A) + 2\cdot 4^m \int_{A\frown\{f>1/2\}} f\, d\mu \int_1^{2f} \frac{d\lambda}{\lambda}$$
$$\leqslant \mu(A) + 2\cdot 4^m \int_A f(1+\log^+ f)\, d\mu$$

and the part (iii), (a), holds.

Now we can suppose $f \geqslant 0$ is locally integrable:

$$(1.10.8)\qquad \int_A (f^*)^a d\mu = a\int_0^\infty \mu\{E(\lambda, f^*)\}\lambda^{a-1}d\lambda$$

$$\leqslant a\mu(A) + a\cdot 2\cdot 4^m \int_1^\infty \lambda^{a-2}d\lambda \int_A f_{\lambda/2}d\mu$$

$$\leqslant a\mu(A) + a\cdot 2\cdot 4^m \int_{A\cap\{f>1/2\}} f d\mu \int_1^{2f} \lambda^{a-2}d\lambda$$

$$\leqslant a\mu(A) + O(a)\int_A f d\mu$$

which proves (iii), (b).

Finally, (iv) is verified if f is a step function.

Let us take a general f, locally integrable; then we can find a step function f' for each $\varepsilon > 0$ such that

$$(1.10.9)\qquad \int_A |f-f'|\,d\mu < \varepsilon,$$

where A is a bounded open set such that $\mu(A) < \infty$.

Now, by inequality (i) of the thesis we have

$$\mu\{E(\varepsilon^{1/2}, |f-f'|^*)\} < \frac{2\cdot 4^m}{\varepsilon^{1/2}}\int_A |f-f'|\,d\mu < 2\cdot 4^m\varepsilon^{1/2}$$

and then

$$\left|\overline{\lim}\,\frac{1}{\mu[Q(x)]}\int_{Q(x)} f d\mu - \underline{\lim}\,\frac{1}{\mu[Q(x)]}\int_{Q(x)} f d\mu\right| > 2\varepsilon^{1/2}$$

only in a set of μ-measure smaller than $2\cdot 4^m\varepsilon^{1/2}$.

1.11. Lemma. *If $\mu \geqslant 0$ and $\nu \geqslant 0$ are σ-additive bounded measures defined on the Borel subsets of $\boldsymbol{R}^m$, then:*

(i) $\lim\limits_{Q(x)\to\{x\}} \mu[Q(x)]/\nu[Q(x)]$ *exists and is finite a.e. with respect to the measure ν.*

(ii) $\lim\limits_{Q(x)\to\{x\}} \{\mu[Q(x)]/\nu[Q(x)]\}^{-1}$ *exists and is finite a.e. with respect to the measure μ.*

Proof. We may consider only (ii), since (i) is symmetric. The well known theorem on decomposition gives

$$(1.11.1)\qquad \nu(A) = \int_A g\,d\mu + \nu(N\cap A),$$

where $\nu(N\cap A)$ is the singular part and $\mu(N) = 0$, and A runs over all the Borel subsets of $\boldsymbol{R}^m$.

Setting $\nu(N\cap A) = \nu_1(A)$, it is sufficient to prove the differentiability of ν_1 since the differentiability of $\int_A g\,d\mu$ holds from the preceding lemma.

For each $\varepsilon > 0$ we can find an open set G such that $\nu_1(G) < \varepsilon$ and $\mu(G^*) < \varepsilon$, where G^* denotes the complementary set. Setting now $\nu_1 = \nu_1' + \nu_1''$, where

$$\nu_1'(A) = \nu_1(A\cap G),\qquad \nu_2''(A) = \nu_1(A\cap G^*),$$

for each point $x\in G$ there is a cube $Q(x)$ such that $Q(x)\cap G^* = \varphi$, then in G

$$\overline{\lim_{Q(x)\to\{x\}}}\frac{\nu_1\{Q(x)\}}{\mu\{Q(x)\}} = \overline{\lim_{Q(x)\to\{x\}}}\frac{\nu_1'\{Q(x)\}}{\mu\{Q(x)\}}.$$

Now using inequality (i) of lemma 1.10, we have

$$\mu\{E[\varepsilon^{1/2}, \overset{*}{\nu}{}_1'(x)]\} < \frac{4^m}{\varepsilon^{1/2}}\int_{\boldsymbol{R}^m} d\nu_1' < 4^m\varepsilon^{1/2};$$

then

$$\overline{\lim_{Q(x)\to\{x\}}}\frac{\nu_1\{Q(x)\}}{\mu\{Q(x)\}} < \varepsilon^{1/2}$$

except in a set of μ-measure at most equal to $4^m\varepsilon^{1/2}+\varepsilon$. Thus the lemma is proved.

Remark. The extension of the preceding lemma to the case $\mu(\boldsymbol{R}^m) = \infty$ or $\nu(\boldsymbol{R}^m) = \infty$ or $\mu(\boldsymbol{R}^m) = \infty$ and $\nu(\boldsymbol{R}^m) = \infty$ is not difficult, therefore we are not going to state it here explicitly.

1.12. Let μ_j be measures in the conditions of [1.7], $j = 1, \ldots, m$, and let us consider the operators

$$(1.12.1)\qquad M_j(f)(\mathring{x}) = \sup_{Q_j(\mathring{x}_j)\supset\{\mathring{x}_j\}}\left|\frac{1}{\mu_j[Q_j(\mathring{x}_j)]}\int_{Q_j(\mathring{x}_j)} f(\mathring{x}_1, \ldots, x_j, \mathring{x}_{j+1}, \ldots, \mathring{x}_m)\,d\mu_j\right|,$$

where f is locally $\mu_1\ldots\mu_m$ integrable; then the operator is well defined a.e. in the measure $\mu = \mu_1\ldots\mu_m$.

If $f \geqslant 0$ is a μ-measurable, locally integrable function and $\varphi(u)$ is a convex non-decreasing function, we have the following easy to prove inequalities:

$$(1.12.2)\qquad \begin{aligned} &\text{(a)}\quad \bar{f} \leqslant M_m M_{m-1}\ldots M_1(f)\\ &\text{(b)}\quad \varphi(\bar{f}) \leqslant \overline{\varphi(f)},\qquad \varphi\geqslant 0,\\ &\text{(c)}\quad \varphi\big(M_j(f)\big) \leqslant M_j\big(\varphi(f)\big),\qquad j = 1, \ldots, m;\end{aligned}$$

$(\ )^-$ denotes the maximal operator associated to the strong differentation in the hypotesis of Theorem 1.8. Let us observe that the operator M_j acting on the variable x_j is the operator $*$ of lemma 1.10.

1.13. LEMMA. *If $s \geqslant 1$ and $M(f) = f^*$ is the operator defined in lemma 1.10, then*

(i) $$\int_A M(f)[1+\log^+ M(f)]^s d\mu \leqslant \mu(A) + s\cdot 4^{m+1} \int_A f(1+\log^+ f)^{s+1} d\mu,$$

where $f \geqslant 0$, and A is an open set of bounded μ-measure.

Proof. Observing that
$$u(1+\log^+ u)^s$$
is positive increasing and convex on $u > 0$ with $s \geqslant 1$, we have

(1.13.1) $$\int_A M(f)\big(1+\log^+ M(f)\big)^s d\mu \leqslant \int_A M\{f(1+\log^+ f)^s\} d\mu,$$

(1.13.2) $$\int_A M\{f(1+\log^+ f)^s\} d\mu$$
$$\leqslant \mu(A) + 2\cdot 4^{m+1} \int_A f(1+\log^+ f)^s [1+\log^+\{f(1+\log^+ f)^s\}] d\mu.$$

On the other hand, we have

(1.13.3) $$\log^+[f(1+\log^+ f)^s] \leqslant \log^+ f + s\log^+(1+\log^+ f)$$
$$\leqslant s\{\log^+ f + \log^+(1+\log^+ f)\} \leqslant 2s\log^+ f.$$

Then
$$1+\log^+[f(1+\log^+ f)^s] \leqslant 2s(1+\log^+ f).$$

Combining this last inequality with (1.13.2) we have the desired result.

1.14. Now to finish the proof of theorem 1.8 we may take $m = 2$, a case entirely typical.

Let $f \geqslant 0$ belong to $L^p_{\mu_1\cdot\mu_2}(\boldsymbol{R}^{n_1+n_2})$, $p > 1$. According to inequality (ii) of lemma 1.10, we have

(1.14.1) $$\int_{\boldsymbol{R}^{n_1}} \big(M_1 M_2(f)\big)^p d\mu_1 \leqslant C(p) \int_{\boldsymbol{R}^{n_1}} \big(M_2(f)\big)^p d\mu_1.$$

The same argument gives
$$\int_{\boldsymbol{R}^{n_2}} d\mu_2 \int_{\boldsymbol{R}^{n_1}} [M_1 M_2(f)]^p d\mu_1 \leqslant C(p) \int_{\boldsymbol{R}^{n_2}} d\mu_2 \int_{\boldsymbol{R}^{n_1}} [M_2(f)]^p d\mu_1$$
$$\leqslant C(p)^2 \iint_{\boldsymbol{R}^{n_1+n_2}} f^p d\mu_1 d\mu_2.$$

Now, using (1.12.2), (a), we obtain

(1.14.2)
$$\iint_{\boldsymbol{R}^{n_1+n_2}} (\bar{f})^p d\mu_1 d\mu_2 \leqslant \iint_{\boldsymbol{R}^{n_1+n_2}} [M_1 M_2(f)]^p d\mu_1 d\mu_2 \leqslant C(p) \iint_{\boldsymbol{R}^{n_1+n_2}} f^p d\mu_1 d\mu_2$$

which gives (i) of 1.8.

Let A be a set of the form $Q_1 \times Q_2$, where the $Q_i \epsilon \boldsymbol{R}^{n_i}$ $(i = 1, 2)$ are cubes, and let $f \geqslant 0$ be a μ-measurable function, such that
$$\int_A f\log^+ f d\mu < \infty.$$

Using inequality (iii), (b) of lemma 1.10, we have

(1.14.3) $$\int_{Q_1} (M_1 M_2(f))^a d\mu_1 < C_a'' \mu_1(Q_1) + C_a''' \int_{Q_1} M_2(f) d\mu_1.$$

Integrating this inequality with respect to $d\mu_2$ and using (iii), (a) of lemma 1.10 we obtain

(1.14.4) $$\iint_{Q_1\times Q_2} (M_1 M_2(f))^a d\mu_1 d\mu_2$$
$$\leqslant C_a'' \mu(A) + C_a''' \mu_2(Q_2) + C' C_a''' \iint_{Q_1\times Q_2} f(1+\log^+ f) d\mu_1 d\mu_2$$

which gives (ii), (b) of theorem 1.8.

Now if we impose $\int_A f(1+\log^+ f)^2 d\mu < \infty$, as in the preceding case, we have from (iii), (a) of lemma 1.10

(1.14.5) $$\int_{Q_1} M_1 M_2(f) d\mu_1 \leqslant \mu_1(Q_1) + C' \int_{Q_1} M_2(f)[1+\log^+ M_2(f)] d\mu_1.$$

Integrating with respect to $d\mu_2$ and using lemma 1.13 we obtain

(1.14.6)
$$\iint_{Q_1\times Q_2} M_1 M_2(f) d\mu_1 d\mu_2 \leqslant \mu(A) + C' \mu_2(Q_2) + 4^{m+1} C' \int_A f(1+\log^+ f)^2 d\mu.$$

which gives (ii), (a) of theorem 1.8.

Remark. If $\mu_j(R^{n_j}) < \infty$ $(j = 1, \ldots, m)$, we can take $A = \boldsymbol{R}^{n_1+\ldots n_m}$ and the same conclusions hold.

Finally, since we have pointwise convergence for a dense set (step functions), the corresponding pointwise convergence result follows from the maximal inequality proved in (1.14.4).

1.15. LEMMA. *Let $k_{a_j}(x_j, y_j) \geqslant 0$ $(j = 1, \ldots, m)$, $a_j \epsilon \Delta_j$ be a family of real-valued functions with the properties*

(i) $$\int k^j_{a_j}(x_j, y_j) d\mu_j(y_j) < A, \quad j = 1, 2, \ldots, m,$$

where $d\mu_j$ is a σ-additive and non-negative measure defined on the real line; the bound A does not depend on the parameters j, x_j, a_j (x_j runs over the real line).

(ii) *For each (j, x_j, a_j), $k^j_{a_j}(x_j, y_j)$ is a non-increasing function of y_j for $y_j > x_j$ and non-decreasing for $y_j < x_j$.*

Under the two preceding conditions the operator

$$\tilde{f}(x) = \sup_{a\in\Delta} \Big| \int_{R^m} \Big(\prod_{j=1}^{m} k^j_{a_j}(x_j, y_j)\Big) f(y)\, d\mu(y) \Big| \tag{1.15.1}$$

verifies all the inequalities proved for $\bar{f}$ (with other constants);

$$a = (a_1, \ldots, a_m), \quad \Delta = \Delta_1 \times \ldots \times \Delta_m, \quad d\mu = d\mu_1 \ldots d\mu_m.$$

Proof. We shall only show that $\tilde{f}(x) \leqslant C\bar{f}(x)$ for $f \geqslant 0$, where $\bar{f}(x)$ denotes the maximal function of the strong differentiation associated to f.

Given $\varepsilon > 0$ and fixed a and $x = (x_1, \ldots, x_m)$ we can find an auxiliary function $k'(y_1, \ldots, y_m) \geqslant 0$ with the following properties:

(i) $$k'(y_1, \ldots, y_m) = \sum_{n_1,\ldots,n_m} C_{n_1,\ldots,n_m} \varphi_{n_1}(y_1) \ldots \varphi_{n_m}(y_m),$$

where $\varphi_{n_j}(y_j)$ are characteristic functions of 1-dimensional intervals I_{n_j} centered at the point x_j;

(ii) $$k'(y_1, \ldots, y_m) \geqslant \prod_{j=1}^{m} k^j_{a_j}(x_j, y_j);$$

(iii) $$\int_{R^m} k'(y_1, \ldots, y_m)\, d\mu \leqslant \varepsilon + \int_{R^m} \Big(\prod_{j=1}^{m} k^j_{a_j}(x_j, y_j)\Big) d\mu(y) \leqslant A^m + \varepsilon.$$

An easy examination of the form of the $k^j_{a_j}(x_j, y_j)$ shows the existence of the function $k'(y_1, \ldots, y_m)$. Then, we have

$$\int_{R^m} \Big(\prod_{j=1}^{m} k^j_{a_j}(x_j, y_j)\Big) f(y)\, d\mu(y) \leqslant \int_{R^m} k'(y) f(y)\, d\mu(y) \tag{1.15.3}$$

$$= \sum_{n_1,\ldots,n_m} C_{n_1,\ldots,n_m} \int_{I_{n_1}\times\ldots\times I_{n_m}} f(y)\, d\mu$$

$$= \sum_{n_1,\ldots,n_m} C_{n_1,\ldots,n_m} \mu_1(I_{n_1}) \ldots \mu_m(I_{n_m}) \frac{1}{\mu_1(I_{n_1}) \ldots \mu_m(I_{n_m})} \int_{I_{n_1}\times\ldots\times I_{n_m}} f\, d\mu$$

$$\leqslant \Big(\int_{R^m} k'(y)\, d\mu(y) \Big) \bar{f}(x) \leqslant A^m + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary we have:

$$\int_{R^m} \Big(\prod_{j=1}^{m} k^j_{a_j}(x_j, y_j)\Big) f(y)\, d\mu(y) \leqslant A^m \bar{f}(x). \tag{1.15.4}$$

This last bound does not depend on a, therefore we get

$$\overset{*}{f}(x) \leqslant A^m \bar{f}(x) \tag{1.15.5}$$

which is the desired result.

2. MULTIPLE WEIERSTRASS TRANSFORMATION AND ABEL SUMMABILITY OF THE MULTIPLE FOURIER-HERMITE SERIES

2.1. Let μ be an elementary measure defined on R^m, and w its variation, which is well defined on the elementary subsets of R^m; then if

$$\int_{R^m} \Big| \exp\Big\{-\sum_{j=1}^{m} (z_j - t_j)^2\Big\} \Big|\, dw < \infty$$

for all $z = (z_1, \ldots, z_m)$ belonging to C^m, we shall define the multiple Weierstrass transform $I_\mu(z)$ as follows:

$$I_\mu(z) = \frac{1}{\pi^{m/2}} \int_{R^m} \exp\Big\{-\sum_{j=1}^{m} (z_j - t_j)^2\Big\}\, d\mu.$$

2.2. We shall denote by $H^*_n(x)$, where $n = (n_1, \ldots, n_m)$, the set of orthonormal functions in $L^2_G(R^m)$ defined in the following way:

$$H^*_n(x) = \frac{1}{\pi^{m/4}} \prod_{j=1}^{m} \frac{1}{2^{n_j/2}(n_j!)^{1/2}} H_{n_j}(x_j),$$

where

$$H_{n_j}(x_j) = e^{x_j^2} \frac{d^{n_j}}{dx_j^{n_j}} \{e^{-x_j^2}\}$$

is the n_j-th Hermite polynomial on the single variable x_j.

2.3. From a formula due to Mehler (see [3], p. 439) we have

$$\sum_n r^n H^*_n(x) H^*_n(y) = \sum_{n_1,\ldots,n_m} r_1^{n_1} \ldots r_m^{n_m} \frac{1}{\pi^{m/2} 2^{n_1} n_1! \ldots 2^{n_m} n_m!} \times$$

$$\times H_{n_1}(x_1) \ldots H_{n_m}(x_m) \cdot H_{n_1}(y_1) \ldots H_{n_m}(y_m)$$

$$= \frac{1}{\pi^{m/2} \prod_{j=1}^{m} (1 - r_j^2)^{1/2}} \exp\Big\{-\sum_{j=1}^{m} \frac{r_j^2(x_j^2 + y_j^2) - 2r_j x_j y_j}{1 - r_j^2}\Big\}, \quad 0 < r_j < 1,$$

We shall call the last member of the equality a *multiple singular kernel of Abel-Hermite* and we shall use for it the notation

$$K^*(r, x, y) = \prod_{j=1}^{m} K_j(r_j, x_j, y_j),$$

where the K_j are the single kernels.

2.4. If $f \in L_G^p(R^m)$, $p > 1$, its H.S. is well defined; in fact

$$C_n = \int_{R^m} f H_n^*(x) e^{-|x|^2} dx \tag{2.4.1}$$

and

$$|C_n| = \Big| \int_{R^m} f H_n^*(x) e^{-|x|^2} dx \Big| \leqslant \|f\|_{p,G} \cdot \|H_n^*\|_{p',G} < \infty,$$

where $1/p + 1/p' = 1$.

2.5. If $f \sim \sum C_n H_n^*(x)$ and if $\sum r^n C_n H_n^*(x)$ is absolutely convergent for $0 < r_j < 1$, $j = 1, \ldots, m$, then we call Abel sum of $\sum C_n H_n^*(x)$ to the limit

$$\lim_{(r_1, \ldots, r_m) \to (1, \ldots, 1)} \sum r^n C_n H_n^*(x) \quad (r_j \neq 1; j = 1, 2, \ldots, m).$$

2.6. The set $H_n^*(x)$ is not a set of uniformly bounded functions as in the case of trigonometrical functions; nevertheless, they are uniformely bounded over each compact subset of $\boldsymbol{R}^m$ (see [3], p. 436) and we have the estimates

$$|H_{n_j}(x_j)| \leqslant k \cdot 2^{n_j/2} (n_j!)^{1/2} e^{x_j^2/2}, \tag{2.6.1}$$

where the constant $K > 0$ does not depend on the pair (n_j, x_j); consequently, we have

$$|H_n^*(x)| \leqslant K^m \pi^{-m/4} e^{|x|^2/2}. \tag{2.6.2}$$

2.7. Lemma. *If $f \in L_G^p(R^m)$, $p > 1$, then I_f is an analytic function of $(z_1, \ldots, z_m)$ on the whole complex m-plane C^m. If μ belongs to the class J_γ, $\gamma \neq 1$, we have the same conclusion.*

Proof. If $f \in L_G^p, p > 1$, we shall show that $f \in J_\gamma$ for some $\gamma \neq 1$. In fact, Hölder's inequality gives

$$\int_{R^m} |f| e^{-(1-\varepsilon)|t|^2} dt \leqslant \|f\|_{p,G} \Big(\int_{R^m} e^{\varepsilon p'|t|^2} e^{-|t|^2} dt \Big)^{1/p'}, \tag{2.7.1}$$

where $1/p + 1/p' = 1$ and $\varepsilon > 0$ is submitted to the condition $\varepsilon p' < 1$. Now let $\mu \geqslant 0$ be an elementary measure and observe that

$$\Big| \int_{R^m} \exp\Big\{-\sum_1^m (z_j - t_j)^2\Big\} d\mu \Big| \tag{2.7.2}$$

$$\leqslant \Big|\exp\Big\{-\sum_1^m z_j^2\Big\}\Big| \int_{R^m} \exp\Big\{2\sum_1^m |z_j|\,|t_j|\Big\} \exp\{-|t|^2\} d\mu$$

$$= \Big|\exp\Big\{-\sum_1^m z_j^2\Big\}\Big| \int_{R^m} \exp\Big\{2\sum_1^m |z_j t_j|\Big\} \exp\{-(1-\gamma)|t|^2\} \exp\{-\gamma|t|^2\} d\mu.$$

Now observing that

$$\sup_{t \in R^m} \Big[\exp\Big\{2\sum_1^m |z_j|\,|t_j|\Big\} \exp\{-(1-\gamma)|t|^2\}\Big] < A(\gamma, z_1, \ldots, z_m),$$

we have for every finite value of $z = (z_1, \ldots, z_m)$

$$\int_{R^m} \exp\Big\{2\sum_1^m |z_j|\,|t_j|\Big\} \exp\{-|t|^2\} d\mu \tag{2.7.3}$$

$$= \int_{R^m} \exp\Big\{2\sum_1^m |z_j|\,|t_j|\Big\} \exp\{-(1-\gamma)|t|^2\} \exp\{-\gamma|t|^2\} d\mu$$

$$\leqslant A(\gamma, z_1, \ldots, z_m) \int_{R^m} \exp\{-\gamma|t|^2\} d\mu < \infty.$$

Finally, from Beppo-Levi's theorem we obtain

$$\int_{R^m} \exp\Big\{2\sum_1^m |z_j|\,|t_j|\Big\} \exp\{-|t|^2\} d\mu \tag{2.7.4}$$

$$= \sum_{n_1, \ldots, n_m} \frac{2^{n_1 + \ldots + n_m}}{n_1! \ldots n_m!} |z_1|^{n_1} \ldots |z_m|^{n_m} \int_{R^m} |t_1|^{n_1} \ldots |t_m|^{n_m} e^{-|t|^2} d\mu.$$

The analyticity of

$$\int_{R^m} \exp\Big\{2\sum_1^m z_j t_j\Big\} \exp\{-|t|^2\} d\mu$$

in the whole m-plane C^m follows since its formal McLaurin series is majorized by the series of the second member of (2.7.4); this completes the proof.

If we have a signed measure μ, making the classical decomposition $\mu = \mu_1 - \mu_2$, where $\mu_j \geqslant 0$, $i = 1, 2$, the same conclusion holds.

Remark. If we make a restriction on the variables z_j, for instance taking $z_j = it_j$, where the t_j are real, we can define Weierstrass Transform of functions of $L^1_G(R^m)$ and also for measures belonging to $J_\gamma(R^m)$ with $\gamma = 1$, as functions of the variables t_j.

2.8. Let y_j, w_j be real variables, $j = 1, \ldots, m$, and the s_j real parameters such that $0 < s_j < 1$; $I_\mu(z_1, \ldots, z_m)$ denotes the Weierstrass transform of a measure μ. The integral

$$(2.8.1)\qquad \frac{1}{\pi^{m/2}} \int_{R^m} \exp\Big\{-\sum_1^m (iw_j+y_j)^2\Big\} I_\mu(is_1y_1, \ldots, is_my_m)\,dy$$

will give an inversion formula for the Weierstrass transform when the $s_j \to 1$. Knowing the result for the 1-dimensional case (see [3], p. 453, and [4]), the validity of the inversion formula (2.8.1) holds for a dense subset of $L^p_G(R^m)$, $p \geq 1$, for instance the set formed by functions of the form

$$\sum_{n_1,\ldots,n_m} C_{n_1,\ldots,n_m}\varphi_{n_1}(x_1)\ldots\varphi_{n_m}(x_m),$$

where the sum has a finite number of terms, the $C_{n_1,\ldots,n_m}$ are constants and the $\varphi_{n_j}(x_j)$ are indefinitely differentiable and compact supported functions defined on the real line.

An easy calculation as in the 1-dimensional case also gives (see [3], p. 453)

$$(2.8.2)\qquad \frac{1}{\pi^{m/2}} \int_{R^m} \exp\Big\{-\sum_1^m (iw_j+y_j)^2\Big\} I_\mu(is_1y_1, \ldots, is_my_m)\,dy = \int_{R^m} K^*(s, w, y)e^{-|y|^2}d\mu,$$

where $K^*(s, w, y)$ denotes the multiple Abel-Hermite kernel.

2.9. Lemma. *Let $k(r, x, y)$ be the 1-dimensional Abel-Hermite kernel; then, there exists a kernel $h(r, x, y)$ which has the following properties:*

(i) $h(r, x, y) \geq k(r, x, y)$, $0 < r < 1$, $-\infty < x < +\infty$, $-\infty < y < +\infty$;

(ii) $\int_{-\infty}^{\infty} h(r, x, y)e^{-y^2}dy < A$, *where A does not depend on the pair (r, x);*

(iii) *for each pair (r, x), $h(r, x, y)$ is non-increasing on $y > x$ and non-decreasing on $y < x$.*

Proof. Let us fix the pair (x, r) in $k(r, x, y)$. From the symmetry of $k(r, x, y)$ we may suppose $x > 0$.

Differentiating partially with respect to y we have

$$(2.9.1)\qquad \frac{\partial k}{\partial y} = -\frac{1}{1-r^2}\{2r^2y-2rx\}k(r, x, y),$$

consequently,

$$\operatorname{sign}\frac{\partial k}{\partial y} = \operatorname{sign}\{x/r-y\}.$$

Then $k(r, x, y)$ is strictly decreasing for $y > x/r$ and strictly increasing for $y < x/r$.

We define $h(r, x, y)$ in the following way:

(2.9.2)

$$h(r, x, y) = k(r, x, y) = \frac{1}{\pi^{1/2}}\frac{1}{(1-r^2)^{1/2}}\exp\Big\{-\frac{r^2(x^2+y^2)-2rxy}{1-r^2}\Big\}$$

if $y \notin (x, x/r)$;

$$h(r, x, y) = k\Big(r, x, \frac{x}{r}\Big) = \frac{1}{\pi^{1/2}}\frac{1}{(1-r^2)^{1/2}}e^{x^2}$$

if $y \in (x, x/r)$.

Conditions (i) and (iii) of the thesis are satisfied by the kernel $h(r, x, y)$; it remains to show that condition (ii) is also satisfied.

Since

$$\int_{-\infty}^{\infty} k(r, x, y)e^{-y^2}dy = 1, \qquad -\infty < x < \infty, 0 < r < 1,$$

we must only prove the uniformly boundedness of

$$\frac{e^{x^2}}{\pi^{1/2}(1-r^2)^{1/2}}\int_x^{x/r} e^{-y^2}dy.$$

Fixed $\delta > 0$, we have three cases for $x > 0$:

1° $0 < r \leq \delta$.

$$\int_x^{x/r} e^{-y^2}dy < \int_x^{+\infty} e^{-y^2}dy < \sum_{n=0}^{\infty} e^{-(x+n)^2} < e^{-x^2}\sum_0^{\infty} e^{-n^2}.$$

Consequently, we have

$$(2.9.3)\qquad \frac{e^{x^2}}{\pi^{1/2}(1-r^2)^{1/2}}\int_x^{x/r} e^{-y^2}dy \leq \frac{1}{\pi^{1/2}(1-\delta^2)^{1/2}}\sum_0^{\infty} e^{-n^2}.$$

2° $\delta < r < 1$ and $x\sqrt{1-r} \leqslant 1$.

$$\text{(2.9.4)}\qquad \frac{e^{x^2}}{\pi^{1/2}(1-r^2)^{1/2}}\int\limits_x^{x/r} e^{-y^2}dy \leqslant \frac{1}{\{\pi(1+r)\}^{1/2}}\cdot\frac{1}{(1-r)^{1/2}}\int\limits_x^{x/r} dy$$
$$= \frac{1}{\{\pi(1+r)\}^{1/2}}\,\frac{1}{(1-r)^{1/2}}\,\frac{x}{r}(1-r) \leqslant \frac{1}{\delta\pi^{1/2}}.$$

3° $\delta < r < 1$ and $x\sqrt{1-r} > 1$.

In this case $x(1-r) > \sqrt{1-r}$ and also $x(1-r)/r > \sqrt{1-r}$ since $0 < r < 1$. Now

$$\text{(2.9.5)}\qquad \frac{e^{x^2}}{\{\pi(1+r)\}^{1/2}(1-r)^{1/2}}\int\limits_x^{x/r} e^{-y^2}dy$$
$$= \frac{1}{\{\pi(1+r)\}^{1/2}}\left[\frac{e^{x^2}}{(1-r)^{1/2}}\int\limits_x^{x+\sqrt{1-r}} e^{-y^2}dy + \frac{e^{x^2}}{(1-r)^{1/2}}\int\limits_{x+\sqrt{1-r}}^{x/r} e^{-y^2}dy\right].$$

We have

$$\text{(2.9.6)}\qquad \frac{e^{x^2}}{(1-r)^{1/2}}\int\limits_x^{x+\sqrt{1-r}} e^{-y^2}dy < 1.$$

On the other hand, we get

$$\text{(2.9.7)}\qquad \frac{e^{x^2}}{(1-r)^{1/2}}\int\limits_{x+\sqrt{1-r}}^{x/r} e^{-y^2}dy$$
$$\leqslant \frac{e^{x^2}}{(1-r)^{1/2}}\exp\{-[x+(1-r)^{1/2}]^2\}\left[\frac{x}{r}-\big(x+(1-r)^{1/2}\big)\right]$$
$$\leqslant \frac{\exp\{-(1-r)\}}{(1-r)^{1/2}}\exp\{-2x(1-r)^{1/2}\}\left[\frac{x}{r}-\big(x+(1-r)^{1/2}\big)\right]$$
$$\leqslant \frac{1}{(1-r)^{1/2}}\exp\{-2x(1-r)^{1/2}\}\,\frac{x}{r}(1-r)$$
$$\leqslant \frac{1}{2\delta}\sup_u e^{-|u|}|u|.$$

Consequently, we get from (2.9.5), (2.9.6) and (2.9.7):

$$\text{(2.9.8)}\qquad \frac{e^{x^2}}{\{\pi(1-r^2)\}^{1/2}}\int\limits_x^{x/r} e^{-y^2}dy \leqslant \frac{1}{\pi^{1/2}}\left[1+\frac{1}{2\delta}\sup_u e^{-|u|}|u|\right].$$

For $x = 0$, we have $h(r, 0, y) = k(r, 0, y)$, but $k(r, 0, y)$ has the required form; then this case does not offer difficulty. Finally, collecting the estimates (2.9.8), (2.9.4) and (2.9.3) we have part (ii) of the thesis.

2.10. If we call

$$f^*(x) = \sup_{r_1,\ldots,r_m}\left|\int\limits_{\boldsymbol{R}^m} K^*(r, x, y)f(y)e^{-|y|^2}dy\right|,$$

the operator $(\ )^*$ has the same properties as that of the strong differentiation; more precisely:

2.11. Theorem. *If $I_\mu(z)$ is a Weierstrass Transform, possibly defined only on $z_j = iy_j$, $j = 1, \ldots, m$, where the y_j are real, and calling $f(s, x)$ to*

$$\int\limits_{\boldsymbol{R}^m} K^*(s, x, y)f(y)e^{-|y|^2}dy,$$

we have the following estimates for the operator $f^*(x)$:

(2.11.1)

$$f^*(x) = \sup_{s_1,\ldots,s_m}\left|\frac{1}{\pi^{m/2}}\int\limits_{\boldsymbol{R}^m}\exp\Big\{-\sum_{j=1}^m (ix_j+y_j)^2\Big\}I_f(is_1y_1, \ldots, is_my_m)\,dy\right|$$
$$= \sup_{s_1,\ldots,s_m}\left|\int\limits_{\boldsymbol{R}^m} K^*(s, x, y)f(y)e^{-|y|^2}dy\right|.$$

(i) *If $p > 1$ and $f \in L^p_G(\boldsymbol{R}^m)$ we have*

(a) $$\|f^*(x)\|_{p,G} \leqslant C(p)\,\|f(x)\|_{p,G},$$

(b) $$\|f(s, x)-f(x)\|_{p,G} \to 0 \quad \textit{with } (s_1, \ldots, s_m) \to (1, \ldots, 1).$$

(ii) *If $|f|(\log^+|f|)^m \in L^1_G(\boldsymbol{R}^m)$ we have*

(a) $$\int\limits_{\boldsymbol{R}^m} f^*(x)e^{-|x|^2}dx \leqslant A_m + B_m\int\limits_{\boldsymbol{R}^m}|f|(1+\log^+|f|)^m e^{-|x|^2}dx,$$

(b) $$\|f(s, x)-f(x)\|_{1,G} \to 0 \quad \textit{with } (s_1, \ldots, s_m) \to (1, \ldots, 1).$$

(iii) *If $|f|(\log^+|f|)^{m-1} \in L^1_G(\boldsymbol{R}^m)$ we have for $0 < \alpha < 1$:*

(a) $$\int\limits_{\boldsymbol{R}^m}(f^*(x))^\alpha e^{-|x|^2}dx \leqslant C_{m,\alpha} + D_{m,\alpha}\int\limits_{\boldsymbol{R}^m}|f|(1+\log^+|f|)^{m-1}e^{-|x|^2}dx,$$

(b) $$\int\limits_{\boldsymbol{R}^m}|f(s, x)-f(x)|^\alpha e^{-|x|^2}dx \to 0 \quad \textit{with } (s_1, \ldots, s_m) \to (1, \ldots, 1).$$

(iv) *In the cases* (i), (ii) *and* (iii), *$f(s, x)$ also converges pointwise a.e.; if condition* (iii) *is verified only locally we have pointwise convergence a.e. in this neighborhood and it is the best possible result.*

(v) *If $m = 1$ we have for a non-negative σ-additive measure $\mu \geqslant 0$:*

(a) $$G\{E(\overset{*}{\mu}, \lambda)\} < \frac{C}{\lambda} \int_{-\infty}^{\infty} e^{-t^2} dt,$$

(b) $\mu(s, x)$ *converges a.e. with* $s \to 1$.

(c) $$\int_{-\infty}^{\infty} \{\overset{*}{\mu}(x)\}^a d\mu \leqslant O_a + O'_a \int_{-\infty}^{\infty} e^{-|t|^2} d\mu.$$

Proof. From lemma 2.9 we have

(2.11.2) $$\overset{*}{K}(s, x, y) \leqslant \prod_{j=1}^{m} h(s_j, x_j, y_j) = h^*(s, x, y).$$

On the other hand, $h^*(s, x, y)$ is a particular case of lemma 1.15, taking $d\mu_j = e^{-x_j^2} dx_j$; then, parts (a) of (i), (ii) and (iii) follow.

Since we have pointwise convergence for a dense subset of $L_G^p(\boldsymbol{R}^m)$, $p \geqslant 1$, using the maximal inequality (iii), (a), we have pointwise convergence in the class $L_G(\log^+ L_G)^{m-1}$ and also for the classes L_G^p, $p > 1$, and $L_G(\log^+ L_G)^s$ with $s \geqslant m-1$.

The parts (b) of (i), (ii) and (iii) follow since we have pointwise convergence which is also dominated (from the corresponding (a) maximal inequalities).

If f belongs locally to $L_G(\log^+ L_G)^{m-1}$, its strong differentiation maximal operator with respect to the measure $e^{-|x|^2}dx$ behaves in the same way as that of the classical operator of Jessen-Marcinkiewicz-Zygmund, which shows that (iv) is the best possible result.

Finally, if we consider in lemma 1.15, $m = 1$, we see that the estimates of this case are of the same type as those of lemma 1.10; then (v) follows.

2.12. THEOREM. *The following two conditions are equivalent:*

(i) *The function $I(z_1, \ldots, z_m)$, defined in the whole complex m-plane C^m, is a Weierstrass transform of a function of the class $L_G^p(\boldsymbol{R}^m)$, $p > 1$.*

(ii) *The function $I(z_1, \ldots, z_m)$, analytic in the whole complex m-plane C^m, has the properties:*

(a) $$\int_{\boldsymbol{R}^m} |I(is_1y_1, \ldots, is_my_m)| e^{-|y|^2} dy < \infty$$

if $0 < s_j < 1$ $(j = 1, \ldots, m)$;

(b) $$\int_{\boldsymbol{R}^m} \Big| \int_{\boldsymbol{R}^m} \exp\Big\{-\sum_{j=1}^{m} (iw_j + y_j)^2\Big\} I(is_1y_1, \ldots, is_my_m)\, dy \Big|^p e^{-|w|^2} dw < A,$$

where the s_j, w_j, y_j are real and A does not depend on the s_j.

Proof. From lemma 2.7 and part (i), (a), of Theorem 2.11 it follows that (i) implies (ii).

Now, let $I(z_1, \ldots, z_m)$ be in the conditions of (ii) and for the sake of simplicity, let us put

(2.12.1) $$f(s, w) = \frac{1}{\pi^{m/2}} \int_{\boldsymbol{R}^m} \exp\Big\{-\sum_{j=1}^{m} (iw_j + y_j)^2\Big\} I(is_1y_1, \ldots, is_my_m)\, dy.$$

From condition (d) we have

(2.12.2) $$\int_{\boldsymbol{R}^m} |f(s, w)|^p e^{-|w|^2} dw < A.$$

Now, using the weak compacity of the spheres in $L_G^p(\boldsymbol{R}^m)$, we can select a sequence $F(s_n, w)$, $s_n = (s_1^{(n)}, \ldots, s_m^{(n)})$ satisfying the following conditions:

(2.12.3)

(1) $$\int_{\boldsymbol{R}^m} f(s_n, w)\overline{g(w)} e^{-|w|^2} dw \to \int_{\boldsymbol{R}^m} f(w)\overline{g(w)} e^{-|w|^2} dw$$

for $s_n = (s_1^{(n)}, \ldots, s_m^{(n)}) \to (1, \ldots, 1)$, $0 < s_j^{(n)} < 1$, and $j = 1, \ldots, m$; and for all function g belonging to $L_G^{p'}(\boldsymbol{R}^m)$, $1/p + 1/p' = 1$;

(2) $$\|f(w)\|_{p,G} \leqslant A^{1/p}.$$

Developing formula (2.12.1) for s_n, we obtain

(2.12.4) $$f(s_n, w) = \frac{1}{\pi^{m/2}} \exp\Big\{\sum_{1}^{m} w_j^2\Big\} \int_{\boldsymbol{R}^m} e^{-2i<w,y>} e^{-|y|^2} I(is_n y)\, dy.$$

On the other hand, we have

(2.12.5) $$\frac{1}{\pi^{m/2}} \int_{\boldsymbol{R}^m} \exp\Big\{-\sum_{j=1}^{m} (iy_j - w_j)^2\Big\} f(s_n, w)\, dw = \frac{\exp\{\sum_1^m y_j^2\}}{\pi^{m/2}} \int_{\boldsymbol{R}^m} e^{2i<y,w>} f(s_n, w) e^{-|w|^2} dw.$$

Consequently, from (2.12.4), (2.12.5) and the unicity of the Fourier transform we get

(2.12.6) $$I(is_n y) = \frac{1}{\pi^{m/2}} \int_{\boldsymbol{R}^m} \exp\Big\{-\sum_{j=1}^{m} (iy_j - w_j)^2\Big\} f(s_n, w)\, dw.$$

Now letting s_n tend to $(1, \ldots, 1)$ in (2.12.6) and using the fact that $e^{-2i\langle y,w\rangle}$ belongs to $L^p_G(\boldsymbol{R}^m)$, one can obtain the following equality:

$$(2.12.7)\quad I(iy_1, \ldots, iy_m) = \frac{1}{\pi^{m/2}} \int_{\boldsymbol{R}^m} \exp\Big\{-\sum_{j=1}^{m} (iy_j - w_j)^2\Big\} f(w)\,dw.$$

According to lemma 2.7 the second member of (2.12.7) is an analytic function in the whole m-plane C^m; since I is also analytic and both are identical on the lines iy_j, $j = 1, \ldots, m$, they must be identical in the whole C^m.

2.13. Let us set

$$f(r, x) = \sum_n r^n C_n H^*_n(x), \qquad 0 < r_j < 1;\ j = 1, 2, \ldots, m,$$

where the C_n are the Fourier-Hermite coefficients of f.

We also call $f(r, x)$ an Abel approximating associated to the Fourier-Hermite series of f.

A Fourier-Hermite series of a function belonging to a class $L^p_G(\boldsymbol{R}^m)$ with $p > 1$ is well defined; nevertheless, if $1 < p < 2$ we shall need further conditions to ensure the absolute convergence of its Abel approximation (some of them are given in [3], p. 450, in the 1-dimensional case).

2.14. Theorem. *The operators*

$$f(r, x) = \sum_{n_1, \ldots, n_m} r^n C_n H^*_n(x)$$

and

$$\overset{*}{f}(x) = \sup_{r_1, \ldots, r_m} |f(r, x)|, \qquad 0 < r_j < 1;\ j = 1, 2, \ldots, m,$$

verify the following conditions:

(i) *If* $f \in L^p_G(\boldsymbol{R}^m)$, $p \geqslant 2$, *then*

(a) $$\|\overset{*}{f}\|_{p,G} \leqslant C(p)\|f\|_{p,G},$$

(b) $$\|f(r, x) - f(x)\|_{p,G} \to 0 \quad \text{as } (r_1, \ldots, r_m) \to (1, \ldots, 1),$$

(c) $$f(r, x) \to f(x) \ \text{a.e.}$$

(ii) *If* $f \in L^p_G(\boldsymbol{R}^m) \cap J_{1/2}(\boldsymbol{R}^m)$, $p > 1$, *then we have the same conclusion as in* (i).

(iii) *If* $\int_{\boldsymbol{R}^m} |f|(1 + \log^+ |f|)^m e^{-|x|^2} dx < \infty$ *and also* $f \in J_{1/2}(\boldsymbol{R}^m)$, *then*

(a) $$\int_{\boldsymbol{R}^m} \overset{*}{f} e^{-|x|^2} dx < A_m + B_m \int_{\boldsymbol{R}^m} |f|(1 + \log^+ |f|)^m e^{-|x|^2} dx,$$

(b) $$\|f(r, x) - f(x)\|_{1,G} \to 0 \quad \text{as } (r_1, \ldots, r_m) \to (1, \ldots, 1),$$

(c) $$f(r, x) \to f(x) \ \text{a.e.}$$

(iv) *If* $\int_{\boldsymbol{R}^m} |f|(1 + \log^+ |f|)^{m-1} e^{-|x|^2} dx < \infty$ *and also* $f \in J_{1/2}(\boldsymbol{R}^m)$, *then for* $0 < a < 1$ *we have*

(a) $$\int_{\boldsymbol{R}^m} (\overset{*}{f})^a e^{-|x|^2} dx < A_{m,a} + B_{m,a} \int_{\boldsymbol{R}^m} |f|(1 + \log^+ |f|)^{m-1} e^{-|x|^2} dx,$$

(b) $$\int_{\boldsymbol{R}^m} |f(r, x) - f(x)^a| e^{-|x|^2} dx \to 0 \quad \text{as } (r_1, \ldots, r_m) \to (1, \ldots, 1).$$

(c) $$f(r, x) \to f(x) \quad \text{a.e.}$$

(v) *If* f *verifies locally the condition*

$$\int_A |f|(1 + \log^+ |f|)^{m-1} e^{-|x|^2} dx < \infty$$

and also $f \in J_{1/2}(\boldsymbol{R}^m)$, *we have in this neighborhood convergence a.e. of* $f(r, x)$ *to* $f(x)$.

Proof. It will only be necessary to prove the following equality for the different cases:

$$(2.14.1)\qquad \sum_n r^n C_n H^*_n(x) = \int_{\boldsymbol{R}^m} f(y) K^*(r, x, y) e^{-|y|^2} dy.$$

We shall begin with case (i).

If we choose a function f of the form $\sum C_n H_n(x)$, where the sum has a finite number of terms, we have

$$(2.14.2)\quad \sum_{\{n_j < N_j\}} r^n C_n H^*_n(x) = \sum_{\{n_j < N_j\}} r^n \Big(\int_{\boldsymbol{R}^m} f H^*_n(y) e^{-|y|^2} dy\Big) H^*_n(x)$$
$$= \int_{\boldsymbol{R}^m} f(y) \Big\{\sum_{\{n_j < N_j\}} r^n H^*_n(y) H^*_n(x)\Big\} e^{-|y|^2} dy.$$

Now if we denote by M the set of multi-indices for which $C_n \neq 0$ and by M' the set of multi-indices for which $C_n = 0$, we get

$$(2.14.3)\quad \int_{\boldsymbol{R}^m} f(y) \Big\{\sum_n r^n H^*_n(y) H^*_n(x)\Big\} e^{-|y|^2} dy$$
$$= \int_{\boldsymbol{R}^m} f(y) \Big\{\sum_{n \in M} r^n H^*_n(y) H^*_n(x)\Big\} e^{-|y|^2} dy + \int_{\boldsymbol{R}^m} f(y) \Big\{\sum_{n \in M'} r^n H^*_n(y) H^*_n(x)\Big\} e^{-|y|^2} dy$$
$$= \int_{\boldsymbol{R}^m} f(y) \Big\{\sum_{n \in M} r^n H^*_n(x) H^*_n(y)\Big\} e^{-|y|^2} dy$$

according to the fact that for $0 < r_j < 1$, $j = 1, \ldots, m$, and fixed $(x_1, \ldots, x_m)$:

(*) $\sum_{n \in M'} r^n H^*_n(x) H^*_n(y)$ belongs to $L^2_G(R^m)$ as function of $(y_1, \ldots, y_m)$.

Then, from (2.14.3) and Mehler's formula (2.3) we have for every function f of the form $\sum_{\{n_j<N_j\}} C_n H_n^*(x)$ the identity

$$\sum_n r^n C_n H_n^*(x) = \int_{R^m} K^*(r, x, y) f(y) e^{-|y|^2} dy. \tag{2.14.4}$$

Now, for fixed r, $0<r_j<1$, $j=1,\ldots,m$, and $x=(x_1,\ldots,x_m)$, the absolutely convergent series

$$\sum r^n C_n H_n^*(x) = \langle T, f\rangle \tag{**}$$

constitutes a linear, continuous functional on $L_G^2(R^m)$, which has a representation of the form (2.14.4) for a dense subset. Then, since $K^*(r,x,y)$ belongs to $L_G^2(R^m)$ (for a fixed $x=(x_1,\ldots,x_m)$, $0<r_j<1$ and $j=1,\ldots,m$) as function of $y=(y_1,\ldots,y_m)$, we have the same representation for all $L_G^2(R^m)$ and consequently for all $L_G^p(\boldsymbol{R}^m)$, $p\geqslant 2$.

Finally, if we prove (2.14.1) for functions of the class $J_{1/2}(\boldsymbol{R}^m)$ the parts (ii), (iii), (iv) and (v) will be established.

Using the fact that

$$|H_n^*(w)| \leqslant K e^{|w|^2/2}$$

the following estimates will give the desired result:

(2.14.5)

$$\int_{R^m}\Big\{\sum_n |r^n H_n^*(x) H_n^*(y)|\Big\} |f(y)| e^{-|y|^2} dy \leqslant \int_{R^m} \frac{K^2}{\prod_{j=1}^m (1-r_j)} e^{|x|^2/2} e^{|y|^2/2} |f(y)| e^{-|y|^2} dy$$
$$= \frac{K^2 e^{|x|^2/2}}{\prod_{j=1}^m (1-r_j)} \int_{R^m} |f(y)| e^{-|y|^2/2} dy < \infty.$$

(*) and (**) follow from the estimate $|H^*(w)| < K e^{|w|^2/2}$.

2.15. Now, we are going to study the inversion formula for the Weierstrass Transform in the case of measures.

If μ is an elementary measure defined on R^m and if w denotes its variation, then the condition:

$$\Big|\int_{R^m} e^{-|x|^2} d\mu\Big| \leqslant \int_{R^m} e^{-|x|^2} dw < \infty \tag{2.15.1}$$

ensures the existence of its W.T. on the lines iy_j ($j=1,\ldots,m$). According to (2.8.2) we need to study the integral

$$\frac{1}{\pi^{m/2}} \int_{R^m} e^{-\sum_{j=1}^m (iw_j+y_j)^2} I_\mu(is_1y_1, \ldots, is_my_m) dy \tag{2.15.2}$$
$$= \int_{R^m} \overset{*}{K}(s, w, y) e^{-|y|^2} d\mu = \mu(s, w)$$

where $0<s_j<1$, $j=1,\ldots,m$ and the variables y_j, w_j are real.

2.16. We shall say that $\mu(s,w)$ *converges restrictively when* $(s_1,\ldots,s_m)\to(1,\ldots,1)$ if the limit

$$\lim_{(s_1,\ldots,s_m)\to(1,\ldots,1)} \mu(s,w)$$

exists submitted to the condition $\theta^{-1} < (1-s_i)/(1-s_j) < \theta$ for some θ such that $0<\theta<\infty$ and for all pair (i,j).

2.17. Lemma. *If* $|x_j|\leqslant b$, $j=1,\ldots,m$, $b>1$, $M>4b^2$ *and* $1<r_t<\frac{1}{2}$, *then*

(i)
$$\overset{*}{K}(r,x,t) e^{-|t|^2} \leqslant \frac{e^{-|t|^2}}{\pi^{m/2}} \prod_{j=1}^m \Big\{\frac{A(M,b)}{(1-r_j^2)^{1/2}} \exp\Big[-\Big(\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big)^2\Big] + \frac{1}{(1-r_j^2)^{1/2}} \exp\Big[-a(M,b)\Big(\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big)^2\Big]\Big\}.$$

Proof. It will only be necessary to show that

$$K(r_j, x_j, t_j) e^{-t_j^2} \leqslant \frac{A(M,b)}{\pi^{1/2}(1-r_j^2)^{1/2}} e^{-t_j^2} \exp\Big[-\Big\{\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big\}^2\Big] + \frac{e^{-t_j^2}}{\pi^{1/2}(1-r_j^2)^{1/2}} \exp\Big[-a(M,b)\Big\{\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big\}^2\Big]. \tag{2.17.1}$$

If $|t_j|\leqslant M$, we have:

$$K(r_j, x_j, t_j) e^{-t_j^2} = \frac{1}{[\pi(1-r_j^2)]^{1/2}} \exp\Big[-\Big(\frac{t_j-x_jr_j}{\sqrt{1-r_j^2}}\Big)^2\Big] \tag{2.17.2}$$
$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{1-r_j^2}} \exp\Big[-\Big\{\frac{t_j-x_j+x_j(1-r_j)}{\sqrt{1-r_j^2}}\Big\}^2\Big]$$
$$\leqslant \frac{1}{[\pi(1-r_j^2)]^{1/2}} \exp\Big[-\Big\{\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big\}^2\Big] \exp\Big[\frac{2}{1+r_j}|t_j-x_j||x_j|\Big] \exp\Big[-\frac{(1-r_j)^2}{1-r_j^2}x_j^2\Big]$$
$$\leqslant \frac{1}{[\pi(1-r_j^2)]^{1/2}} e^{4Mb} \exp\Big[-\Big(\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big)^2\Big] e^{t_j^2} e^{-t_j^2}$$
$$\leqslant \frac{e^{M(M+4b)}}{[\pi(1-r_j^2)]^{1/2}} \exp\Big[-\Big(\frac{t_j-x_j}{\sqrt{1-r_j^2}}\Big)^2\Big] e^{-t_j^2}.$$

If $|t_j| > M$, we have

$$(2.17.3)\quad \frac{1}{[\pi(1-r_j^2)]^{1/2}}\exp\left[-\left\{\frac{t_j-r_jx_j}{\sqrt{1-r_j^2}}\right\}^2\right]$$
$$= \frac{e^{-t_j^2}}{[\pi(1-r_j^2)]^{1/2}}\exp\left\{\frac{-r_j^2x_j^2-r_j^2t_j^2+2r_jx_jt_j}{1-r_j^2}\right\}$$
$$\leqslant \frac{e^{-t_j^2}}{[\pi(1-r_j^2)]^{1/2}}\exp\left\{-\frac{r_j^2}{1-r_j^2}\right\}\exp\left\{\frac{2r_j}{1-r_j^2}|x_j||t_j|\right\}.$$

Since $|t_j| > M$, $|t_j| > M > 4b^2 > 4b|x_j|$; therefore, the last term of (2.17.3) is dominated by

$$(2.17.4)\quad \frac{\exp\{-t_j^2\}}{[\pi(1-r_j^2)]^{1/2}}\exp\left\{-\frac{r_j}{1-r_j^2}\left[r_jt_j^2-\frac{2}{4b}t_j^2\right]\right\}$$
$$\leqslant \frac{\exp\{-t_j^2\}}{[\pi(1-r_j^2)]^{1/2}}\exp\left\{-\frac{1}{4}\left(\frac{b-1}{b}\right)\frac{t_j^2}{1-r_j^2}\right\}$$
$$= \frac{\exp\{-t_j^2\}}{[\pi(1-r_j^2)]^{1/2}}\exp\left\{-\frac{1}{4}\left(\frac{b-1}{b}\right)\left(\frac{t_j}{\sqrt{1-r_j^2}}\right)^2\right\}.$$

Now, since $|t_j| > M$ and $|x_j| < b < M$, we have $|t_j-x_j| < a_0|t_j|$ for some $a_0 > 0$; consequently,

$$(2.17.5)\quad \frac{e^{-t_j^2}}{[\pi(1-r_j^2)]^{1/2}}\exp\left[-\frac{1}{4}\left(\frac{b-1}{b}\right)\left(\frac{t_j}{\sqrt{1-r_j^2}}\right)^2\right]$$
$$\leqslant \frac{e^{-t_j^2}}{[\pi(1-r_j^2)]^{1/2}}\exp\left[-\frac{a_0^{-1}}{4}\left(\frac{b-1}{b}\right)\left(\frac{t_j-x_j}{\sqrt{1-r_j^2}}\right)^2\right],$$

which ends the proof.

2.18. THEOREM. *If μ is an elementary measure such that*

$$\int_{R^m} e^{-|t|^2}dw < \infty,$$

where dw denotes the variation of μ, then

$$\text{(i)}\quad \mu(s, x) = \frac{1}{\pi^{m/2}}\int_{R^m}\exp\left\{-\sum_{j=1}^{m}(ix_j+y_j)^2\right\}I_\mu(is_1y_1, \ldots, is_my_m)\,dy$$

converges restrictively a.e. to the density function of μ, and furthermore, if

$$\overset{*}{\mu}(x) = \sup_{s_1,\ldots,s_m}|\mu(s, x)|$$

with $0 < s_j < 1$, $\theta^{-1} < (1-s_i)/(1-s_j) < \theta$ $(i, j = 1, \ldots, m)$ *we have*

$$\text{(ii)}\quad |E(\mu^*, \lambda)\cap Q_b| < \frac{A_{b,\theta}}{\lambda}\int_{R^m} e^{-|t|^2}dw,$$

where Q_b denotes a cube centered at the origin and with edges of length equal to $b > 1$.

(iii) *If μ is an elementary measure belonging to $J_{-1/2}(\boldsymbol{R}^m)$, then its multiple Fourier-Hermite series converges restrictively a.e. to the density function associated to μ and, furthermore, the maximal operator associated to its Abel approximating has the same type of* (ii) *without the exponential factor.*

Proof. If $|x_j| < b$, $|\mu(s, x)|$ is dominated by

$$(2.18.1)\quad \int_{R^m}\left[\prod_{j=1}^{m}\left\{\frac{A(M, b)}{(1-r_j^2)^{1/2}}\exp\left[-\left(\frac{x_j-y_j}{\sqrt{1-r_j^2}}\right)^2\right]+\right.\right.$$
$$\left.\left.+\frac{1}{(1-r_j^2)^{1/2}}\exp\left[-a(M, b)\left(\frac{x_j-y_j}{\sqrt{1-r_j^2}}\right)^2\right]\right\}\right]e^{-|y|^2}dw$$

which follows according to lemma 2.17.

Now, from lemma 1.5, we have (ii), since the role of $n_j(t)$ is played by $1/(1-r_j^2)^{1/2}$.

The pointwise convergence follows from the estimate (ii) and from the fact that we have pointwise convergence in a dense subset of $L_G^1(\boldsymbol{R}^m)$.

Part (iii) follows from the fact that $\mu \in J_{-1/2}(\boldsymbol{R}^m)$ and, consequently, its Abel approximation can be expressed by the Abel-Hermite singular kernel; in fact, if $\mu \in J_{-1/2}(\boldsymbol{R}^m)$ its F-H coefficients are well defined,

$$C_n = \int_{R^m} H_n^*(x)\,d\mu \quad\text{and}\quad |C_n| \leqslant K\int_{R^m} e^{|x|^2/2}dw < \infty$$

and, as we did in (2.14.5), one can also show that the Abel Approximation associated to μ can be expressed by the integral

$$\int_{R^m} K^*(r, x, y)\,d\mu(y)$$

which is in modulus dominated by

$$\int_{R^m}\left(\prod_{j=1}^{m}\left\{\frac{A(M, b)}{(1-r_j^2)^{1/2}}\exp\left[-\left(\frac{x_j-y_j}{\sqrt{1-r_j^2}}\right)^2\right]+\right.\right.$$
$$\left.\left.+\frac{1}{(1-r_j^2)^{1/2}}\exp\left[-a(M, b)\left(\frac{x_j-y_j}{\sqrt{1-r_j^2}}\right)^2\right]\right\}\right)dw.$$

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Reçu par la Rédaction le 20. 2. 1968

STUDIA MATHEMATICA, T. XXXII. (1969)

On the control of linear periodic time lag systems

by

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Let us consider a linear time lag system

$$\sum_{k=0}^{p}\sum_{j=0}^{q} F_{kj}(t)x^{(k)}(t-h_j) = \sum_{j=0}^{q} G_j(t)u(t-h_j), \tag{1}$$

where $x(t)$ is an n-dimensional vector-function called *state*, $u(t)$ is an m-dimensional vector-function called *control*, $F_{kj}(t)$ are $n\times n$ matrix functions, $G_i(t)$ are $n\times m$ matrix functions, all of the real variable t.

We assume that all functions $u(t)$, $F_{kj}(t)$ and $G_j(t)$ are measurable and locally integrable on the real line. An n-dimensional vector-function is called a *solution* of (1) if there exists a $(p-1)$-th derivative of $x(t)$ which is absolutely continuous and if $x(t)$ satisfies (1) almost everywhere.

Let $h_0 = 0$ and let h_j be commensurable. Then we have an $r \neq 0$ and integers n_j such that $h_j = n_j r$ for $j = 1, 2, \ldots, q$. Let us assume that the functions $F_{kj}(t)$ and $G_j(t)$ are r-periodic ([1]). Let N be a common multiple of numbers $n_1, \ldots, n_q$ (not necessarily the smallest one) and let $\omega = Nr$. We shall consider system (1) in the class of ω-periodic functions.

Suppose we are given the following performance functional:

$$\mathscr{K}(u,x) = \int_0^{\omega} K\big(t, [x(t)-x^0(t), u(t)-u^0(t)]\big)\,dt, \tag{2}$$

where $[x, u]$ is an $(n+m)$-dimensional vector $(x_1, \ldots, x_n, u_1, \ldots, u_m)$ and $K(t, [x, u])$ for each fixed t is a non-negative quadratic form defined. on an $(n+m)$-dimensional space and $x^0(t), u^0(t)$ are given functions. We assume that $K(t, [x, u])$ is an r-periodic square integrable function with respect to t.

The aim of this note is to minimalize the performance functional (2) under the assumption that $x(t), u(t)$ satisfies equation (1).

The manner in which the proposed question will be solved is based on the method of involution (see [3], also [5]).

([1]) A periodic function with period s will be called briefly an *s-periodic function*.