On equations with several involutions of different orders
and its applications to partial differential-difference equations

by

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In [8] we have shown that a differential-difference equation

\[ \sum_{k=1}^{n} \sum_{j=1}^{m} a_{kj} x^{(j)}(t - c_{k}) = y(t) \] (1)

(y is a given periodic function, \(a_{kj}\), \(c_{k}\) are scalars, \(c_{0} = 0\) and \(x^{(j)}\) denotes the \(k\)-th derivative of \(x\)) is equivalent in the class of periodic functions to a finite system of ordinary differential equations with constant coefficients. This permits us to find all the periodic solutions of (1).

The method used appears to be more general. This paper contains general theorems on equations with several involutions and a generalization of the result described above to partial differential-difference equations with periodic coefficients.

1. Involutions of order \(N\). We shall enumerate here without proofs those properties on involutions of order \(N\) which will be needed later. The reader can find the respective proofs in papers [7], [8] and in book [9].

Let \(X\) be a linear space (over complex scalars). A linear operator \(S\) transforming \(X\) onto \(X\) is called an involution of order \(N\) if \(N\) is the smallest positive integer \((N > 2)\) such that \(S^{N} = I\), where \(I\) denotes the identity operator.

Let \(\varepsilon = e^{i\pi/N}\),

\[ P_{v} = \frac{1}{N} (I + \varepsilon^{-v} S + ... + \varepsilon^{-(N-1)v} S^{N-1}) \], \(v = 1, 2, ..., N\).

If \(S\) is an involution of order \(N\), we have the following important properties of operators \(P_{v}\):

\[ \sum_{v=1}^{N} P_{v} = I \], \(P_{v} P_{v} = P_{v} P_{v} = \delta_{v,v} P_{v} \), \(P_{v} S = S P_{v}\)

\((\mu, v = 1, 2, ..., N)\)

(1.1)
where $\delta_{\mu}$ is the Kronecker symbol;

\begin{equation}
SP_{\nu} = e^\nu P_{\nu}, \quad (\nu = 1, 2, \ldots, N).
\end{equation}

This implies that $X$ is a direct sum,

\begin{equation}
X = \bigoplus_{\nu=1}^{N} X_{|\nu|},
\end{equation}

of spaces $X_{|\nu|}$ such that

$s_{\nu|\nu|} = \delta_{\nu|\nu|}$ for $\nu|\nu| \in X_{|\nu|}$ \quad $(\nu = 1, 2, \ldots, N)$.

Every element $x \in X$ can be written in a unique manner as a sum:

\begin{equation}
x = \sum_{\nu=1}^{N} x_{|\nu|}, \quad \text{where} \quad x_{|\nu|} = P_{\nu} s_{\nu} x_{|\nu|} \quad (\nu = 1, 2, \ldots, N).
\end{equation}

If a linear operator $A$ acting in $X$ is commutative with an involutive $S$ of order $N$, then

\begin{equation}
A(D_{\nu} \cap X_{|\nu|}) \subset X_{|\nu|} \quad \text{for} \quad \nu = 1, 2, \ldots, N,
\end{equation}

where $D_{\nu} = X$ denotes the domain of $A$. In fact, suppose that we have an arbitrary $x \in D_{\nu}$. Then $x = P_{\nu} s_{\nu} x_{|\nu|}$ for $\nu = 1, 2, \ldots, N$ and

\begin{align*}
Ax_{|\nu|} &= A P_{\nu} x_{|\nu|} = A \left( \sum_{\nu=0}^{N-1} e^{-i\nu S_{\nu}} x_{|\nu|} \right) = \left( \sum_{\nu=0}^{N-1} e^{-i\nu S_{\nu}} \right) A x_{|\nu|} \\
&= P_{\nu} (Ax_{|\nu|}) = s_{\nu} x_{|\nu|}.
\end{align*}

For any polynomial $a(t) = \sum_{k=0}^{n} a_{k} t^{k}$ ($a_{k}$ being scalars) we have

\begin{equation}
a(S) = \sum_{k=0}^{n} a_{k} S^{k},
\end{equation}

because

\begin{align*}
a(S) = \sum_{k=0}^{n} a_{k} S^{k} &= \sum_{k=0}^{N} a_{k} S^{k} \left( \sum_{\nu=1}^{n} P_{\nu} \right) = \sum_{k=0}^{N} \sum_{\nu=1}^{n} a_{k} S^{k} P_{\nu} \\
&= \sum_{k=0}^{N} \sum_{\nu=1}^{n} a_{k} e^{\nu x P_{\nu}} = \sum_{\nu=1}^{n} \left( a_{k} (e^{x P_{\nu}}) P_{\nu} \right) = \sum_{\nu=1}^{n} a_{k} e^{\nu x P_{\nu}}.
\end{align*}

Then any equation with the involutive of order $N$,

\begin{equation}
a(S) x = y, \quad y = y_{1} \times X,
\end{equation}

is equivalent to $N$ equations

\begin{equation}
a(\nu) a(\nu) = y(\nu), \quad \nu = 1, 2, \ldots, N,
\end{equation}

each of them being considered in the space $X_{|\nu|}$. The respective theorems on the solvability of (1.8) are given in [8].

2. Multi-involutions. Let us suppose that we have $q$ given involutions $S_{1}, \ldots, S_{q}$ of orders $N_{1}, \ldots, N_{q}$ respectively acting in $X$. Let us write

\begin{equation}
\epsilon_{p} = e^{\pi i N_{p}}, \quad P_{p,r} = \frac{1}{N_{p}} \sum_{k=0}^{N_{p}-1} \epsilon_{p}^{-kr} S_{k}^{p} \quad (p = 1, 2, \ldots, q).
\end{equation}

From the preceding considerations we obtain

\begin{align}
\sum_{p=1}^{N} P_{p,r} &= I, \quad P_{p,r}, P_{p,r} = \delta_{p,r} P_{p,r}, \quad S_{p} P_{p,r} = \epsilon_{p} P_{p,r} \\
&(p, r, s = 1, 2, \ldots, q).
\end{align}

To simplify the theorems to be given later on, we shall now introduce multi-involutions.

Let us consider $q$-dimensional multi-indices $k = (k_{1}, \ldots, k_{q})$ and $m = (m_{1}, \ldots, m_{q})$, where $k_{p}$ and $m_{p}$ are non-negative integers. As usual, we assume

\begin{align*}
|k| &= k_{1} + \cdots + k_{q}, \quad k + m = (k_{1} + m_{1}, \ldots, k_{q} + m_{q}) \\
lk &= (\lambda k_{1}, \ldots, \lambda k_{q}) \quad \text{for any non-negative integer} \lambda,
\end{align*}

and

\begin{align*}
km &= (k_{1} m_{1}, \ldots, k_{q} m_{q}).
\end{align*}

We shall also write $(a_{n}) = (a_{n_{1}}, \ldots, a_{n_{q}})$ for $n = 0, 1, 2, \ldots$ We write

\begin{align*}
k &\leq m \text{ if and only if } k_{p} \leq m_{p} \text{ for } p = 1, 2, \ldots, q \\
&= m \text{ if and only if } k_{p} = m_{p} \text{ for } p = 1, 2, \ldots, q.
\end{align*}

Let $N = (N_{1}, \ldots, N_{q})$ and $\epsilon = (\epsilon_{1}, \ldots, \epsilon_{q})$, where $\epsilon_{p} = e^{\pi i N_{p}}$ ($p = 1, 2, \ldots, q$). We write

\begin{align*}
\epsilon^{k} &= (\epsilon_{1}^{k_{1}}, \ldots, \epsilon_{q}^{k_{q}}),
\end{align*}

where $k = (k_{1}, \ldots, k_{q})$ is a multi-index. By definition, $e^{-ik} = i^{N - \epsilon k}$, where $N, \epsilon$ and $k$ are the respective multi-indices.

A superposition $S = S_{1}, \ldots, S_{q}$ of operators $S_{1}, \ldots, S_{q}$ acting in a linear space $X$ is called multi-involution of order $N = (N_{1}, \ldots, N_{q})$ if $S_{p}$ is an involutive of order $N_{p}$ and $S_{p}$ are commutative with $S_{r}$ for $p, r = 1, 2, \ldots, q$.

Let us write

\begin{align*}
S^{k} &= S_{1}^{k_{1}} \ldots S_{q}^{k_{q}}, \quad \text{where} \quad k = (k_{1}, \ldots, k_{q}) \\
S^{N} &= S_{1}^{N_{1}} \ldots S_{q}^{N_{q}} = I.
\end{align*}
If we write
\[ P_r = P_r, \quad v = (v_1, \ldots, v_q), \]
\[ a_{v_0} = P_r, \quad X_{v_0} = P_r, \]
we find that the following formulae (which are analogous to (1.1) and
(1.2)) are true:
\[ \sum_{v \in S^r} P_r = I, \]
\[ P_r P_r = \delta_{r, r} P_r, \]
\[ \mu \leq v, \quad (1, 2, \ldots, q), \quad \delta^{v, v} = \delta_0^{v, v} P_r. \]

To prove (2.2) by induction we use the first of the formulae (2.1).
We have further
\[ P_r P_r = \left( \prod_{1 \leq p \leq q} P_{r, p} \right) \left( \prod_{1 \leq p \leq q} P_{r, p} \right) = \left( \prod_{1 \leq p \leq q} P_{r, p} P_{r, p} \right) = \prod_{1 \leq p \leq q} \delta^{v, v} P_{r, p}, \]
\[ = \begin{cases} 1, & \text{if } v = \mu (p = 1, 2, \ldots, q), \\ 0, & \text{otherwise.} \end{cases} \]

Finally,
\[ S^{r, r} = \prod_{1 \leq p \leq q} S^{r, r} P_{r, p}, \quad S^{r, r} = \left( \prod_{1 \leq p \leq q} S^{r, r} P_{r, p} \right) = \left( \prod_{1 \leq p \leq q} S^{r, r} P_{r, p} \right) = \left( \prod_{1 \leq p \leq q} S^{r, r} P_{r, p} \right) = \prod_{1 \leq p \leq q} \delta^{v, v} P_{r, p}. \]

Let us remark that the space \( X \) is a direct sum:
\[ X = \bigoplus_{0 \leq r < N} X_r, \quad \text{where } X_r = P_r X. \]

If a linear operator \( A \) acting in \( X \) commutes with involutions \( S_1, \ldots, S_q \) of orders \( N_1, \ldots, N_q \) respectively, then \( A \) commutes also with a multi-involution \( S = S_1 \ldots S_q \) of order \( N = (N_1, \ldots, N_q) \) and
\[ A(D_0 \times X_0) = X_0 \quad \text{for } (1, 2, \ldots, q), \]
where \( D_0 \subset X \) denotes the domain of \( A \). In fact, let \( v \in D_0 \). Then \( x_{v_0} = P_r x_{v_0} \) for \( (1, 2, \ldots, q) \)
\[ A(x_{v_0}) = P_r (Ax) \times X, \]
since \( P_r \) commute with \( S \). Hence \( A(D_0 \times X_0) = X_0 \).

3. Equations with several involutions. By \( A(t) \) we denote an arbitrary polynomial of variables \( t = (t_1, \ldots, t_q) \), which we shall write further in one of the following manners:
\[ A(t) = \sum_{0 \leq v < N} a_v t_1^{v_1} \ldots t_q^{v_q} = \sum_{0 \leq v < N} a_v t^v. \]

Let \( S = S_1 \ldots S_q \) be a multi-involution of order \( N = (N_1, \ldots, N_q) \).

Let
\[ A(S) = \sum_{0 \leq v < N} \delta^v S^v. \]

Then
\[ A(S) = \sum_{0 \leq v < N} A(P_r) P_r. \]

In fact,
\[ A(S) = \sum_{0 \leq v < N} A(S) P_r = \sum_{0 \leq v < N} \sum_{0 \leq v < N} a_v S^v P_r = \sum_{0 \leq v < N} \sum_{0 \leq v < N} A(P_r) P_r. \]

This implies that any equation
\[ A(S_1, \ldots, S_q) = y \]
is equivalent to a system of equations
\[ A(P_r) x_{v_0} = y_{v_0} \quad \text{for } (1, 2, \ldots, q), \]
where \( X_r = \bigoplus_{0 \leq r < N} X_0 \).

THEOREM 3.1. Let \( X \) be the linear space. Let
\[ A(S) = \sum_{0 \leq v < N} A(S) P_r, \]
where
\[ 1^v S = S_1 \ldots S_q \]
and \( S_1, \ldots, S_q, A_1, \ldots, A_m \) are linear operators acting on \( X_1 \); \( 2^v S_r S_r S_r S_r = 0, S_r A_r A_r S_r = 0 \)
\[ \mu = 1, 2, \ldots, q, \quad \text{for } \mu = 1, 2, \ldots, q, \quad \text{and} \]
\[ j = 0, 1, \ldots, m; \]
and \( n_{v, r} = (n_{v, r}, \ldots, n_{v, r}) \), \( n_{v, r} \) are non-negative integers and
\[ \sum_{v \in S^v} n_{v, r} > 0 \quad (j = 0, 1, \ldots, m). \]

Let \( X_r \) be a common multiple of numbers \( n_{v, r} \) \( (j = 0, 1, \ldots, m) \) and let us suppose that there is a subspace \( X \subset X(X) \) such that \( S \) is a multi-involution of order \( N = (N_1, \ldots, N_q) \) on \( X \).

Let
\[ A(S) = \sum_{0 \leq v < N} A(P_r) P_r \quad \text{on } X, \]
where \( \delta^v S^v, \quad \delta^v S^v, \quad \delta^v S^v \)
\[ P_r P_r = \frac{1}{N_r} \sum_{n_{v, r} \geq 0} e^{\delta n_{v, r} S_r} P_r. \]
Proof. Since \( S_j \) and \( A_j \) are commutative, we find
\[
A(S) = A(S) \sum_{0 \leq j \leq N} P_j = \sum_{0 \leq j \leq N} A_j S_j P_j = \sum_{0 \leq j \leq N} A_j e^{\alpha j} P_j = \sum_{0 \leq j \leq N} A_j e^{\alpha j} \sum_{0 \leq j \leq N} f_j P_j = \sum_{0 \leq j \leq N} \left( \sum_{0 \leq j \leq N} e^{\alpha j} A_j \right) P_j = \sum_{0 \leq j \leq N} A(e^\alpha) P_j,
\]
which was to be proved.

**Theorem 3.2.** Under the assumptions of Theorem 3.1, the equation
\[
A(S)x = y, \quad y \in \mathbb{X},
\]
is equivalent to \( N_x = N_1 \ldots N_q \) independent equations
\[
A(e^\alpha)x = y_0, \quad (1) \leq r \leq N,
\]
where \( y_0 = P_r y \), and if each of the equations (3.6) has a solution \( x_r \), then a solution of (3.5) is given by the formula
\[
x = \sum_{0 \leq j \leq N} P_j x_r.
\]

Proof. Since the operator \( S_j \) as a superposition of operators \( S_1, \ldots, S_q \), commutes with \( A_j \) (\( j = 0, 1, \ldots, m \)), every space \( X_{0j} \) is preserved by operator \( A(e^\alpha) \). Then equation (3.6) for \( y \in \mathbb{X} \) can be written in the following manner:
\[
0 = A(S)x - y = \sum_{0 \leq j \leq N} A(e^\alpha) P_j x - \sum_{0 \leq j \leq N} P_j y
\]
\[
= \sum_{0 \leq j \leq N} P_j [A(e^\alpha)x - P_j y]
\]
\[
= \sum_{0 \leq j \leq N} P_j [A(e^\alpha)x - y_0]
\]
Since the space \( X \) is a direct sum of spaces \( X_{0j} \), we infer that equation (3.5) is equivalent to \( N_x = N_1 \ldots N_q \) independent equations
\[
A(e^\alpha)x = y_0, \quad (1) \leq r \leq N,
\]
and if \( x \) is a solution of equation (3.5) in \( \mathbb{X} \), then each of these equations has \( x \) as a solution.

Conversely, let us suppose that each of the equations \( A(e^\alpha)x = y_0 \) has a solution. Let us denote by \( x_r \), the solution of the \( r \)-th equation. Then, if we write
\[
x = \sum_{0 \leq j \leq N} P_j x_r,
\]
we obtain
\[
A(S)x = A(S) \sum_{0 \leq j \leq N} P_j x_r = \sum_{0 \leq j \leq N} A(S) P_j x_r
\]
\[
= \sum_{0 \leq j \leq N} \left( \sum_{0 \leq j \leq N} A(e^\alpha) P_j \right) A(S) P_j x_r = \sum_{0 \leq j \leq N} A(e^\alpha) P_j [A(e^\alpha) x_r] = \sum_{0 \leq j \leq N} A(e^\alpha) P_j x_r = \sum_{0 \leq j \leq N} P_j y = y,
\]
which proves that \( x \) is a solution of equation (3.5).

Let us remark that the second part of Theorem 3.2 can be formulated more strongly:

**Corollary 3.3.** Under the assumptions of Theorem 3.1, if each of the equations
\[
A(e^\alpha)x = y, \quad y \in \mathbb{X}, \quad (1) \leq r \leq N,
\]
has a solution \( x_r \), then
\[
x = \sum_{0 \leq j \leq N} P_j x_r
\]
is a solution of the equation \( A(S)x = y \) on \( \mathbb{X} \).

Proof. In the same manner as in the proof of Theorem 3.2, we obtain
\[
A(S)x = A(S) \sum_{0 \leq j \leq N} P_j x_r = \sum_{0 \leq j \leq N} P_j [A(e^\alpha)x_r].
\]

But \( A(e^\alpha)x = y \), whence
\[
A(S)x = \sum_{0 \leq j \leq N} P_j y = y,
\]
and \( x \) is a solution of \( A(S)x = y \).
4. Application to partial differential-difference equations with periodic coefficients. Let \( R^q \) be a \( q \)-dimensional real euclidean space. Let \( t = (t_1, \ldots, t_q) \in R^q \), \( q \geq 1 \). As usual, we write

\[
D^k x(t) = \frac{\partial^{k_1} \cdots \partial^{k_q} x(t)}{\partial t_1^{k_1} \cdots \partial t_q^{k_q}} , \quad k = (k_1, \ldots, k_q).
\]

Let \( \omega_j = (\omega_{1,j}, \ldots, \omega_{q,j}) \), \( j = 0, 1, \ldots, m \). Let us consider the partial differential-difference equation

\[
\sum_{(\omega_j) \in \mathbb{C}^q \times \mathbb{C}^q} A_{\omega_j} D^k x(t - \omega_j) = y(t), \tag{4.1}
\]

Without loss of generality we can assume that

\[
0 = \omega_{0,1} < \omega_{0,2} < \cdots < \omega_{0,m} \quad (p = 1, 2, \ldots, q).
\]

We assume also that all numbers \( \omega_{p,j} \) are commensurable. This implies that there is a number \( r \neq 0 \) and there are positive integers \( n_{p,j} \) such that

\[
(\omega_{p,j}) = n_{p,j} r \quad \text{for} \quad p = 1, 2, \ldots, q; \quad j = 1, 2, \ldots, m; \quad n_{p,q} = 0.
\]

By \( n_i \) we denote the multi-index \( n_i = (n_{i,j}) \). We say that a function \( x(t) \) is \( \omega \)-periodic if \( x \) is \( \omega_{p,j} \)-periodic with respect to the \( p \)-th variable \( t_p \) \( (p = 1, 2, \ldots, q) \) and \( \omega \) = \((\omega_{1,j}, \ldots, \omega_{q,j})\). The vector \( \omega \) will be called the period of function \( x \). Obviously, if \( x \) is \( \omega \)-periodic, then for any multi-index \( n \) \( =(n_1, \ldots, n_q)\)

\[
x(t - n) = x(t_1 - n_1, t_2, \ldots, t_q) = x(t_1, \ldots, t_q).
\]

**Theorem 4.1.** Let a real function \( y(t) \) determined for \( t \in R^q \) be \( \omega_{m,q} \)-periodic with period \( \omega_{m,q} = (\omega_{1,m}, \ldots, \omega_{q,m}) \), where \( \omega_{p,m} \) are commensurably real commensurable numbers \( \omega_{p,j} \) \( (p = 1, 2, \ldots, q; \quad j = 1, 2, \ldots, m) \).

Let \( r \) be a common divisor of numbers \( \omega_{p,j} \) (not necessarily the greatest one) and let \( \hat{r} = (\hat{r}_j) = (r, \ldots, r) \). Let the real functions \( A_{\omega_j} \) determined for \( t \in R^q \) be \( \hat{r} \)-periodic, \( (\hat{r}_0) = \hat{r}, \quad 0 \leq j \leq m \). Then equation (4.1) has \( \hat{r} \)-periodic solutions belonging to the class \( C^\infty \) if and only if all partial differential equations

\[
A_{\omega_j} x(t) = \sum_{\omega_j \in \mathbb{C}^q} b_{\omega_j}(t) D^k x(t) = y(t), \quad (1)_1 \leq r \leq N,
\]

have \( \hat{r} \)-periodic solutions belonging to the class \( C^\infty \), where

\[
b_{\omega_j}(t) = \sum_{t \in \mathbb{C}^q} e^{-\omega_j} A_{\omega_j} x(t),
\]

\[
y(t) = \sum_{t \in \mathbb{C}^q} e^{-\omega_j} y(t - k\hat{r}),
\]

\[
e = (e_1, \ldots, e_q), \quad e_p = e^{2\pi i e_p}, \quad p = 1, 2, \ldots, q,
\]

\[
n_j = (n_{1,j}, \ldots, n_{q,j}), \quad n_j = n_j \hat{r} \quad \text{for} \quad j = 0, 1, \ldots, m - 1,
\]

\( N \) is a common multiple (not necessarily the smallest one) of numbers \( n_{p,1}, \ldots, n_{p,m-1} \) and \( \hat{r} = (\hat{r}_1, \ldots, \hat{r}_q) \), where \( \hat{r}_p = N \hat{r} \).

The number of the equations (4.3) is \( N \). The solutions are of the form

\[
x(t) = \sum_{t \in \mathbb{C}^q} e^{-\omega_j(t - k\hat{r})},
\]

where \( x \) is an \( \hat{r} \)-periodic solution of the \( r \)-th equation (4.3).

**Proof.** Let us consider the space \( X \) of all \( \hat{r} \)-periodic real functions \( x(t) \) determined for \( t \in R^q \) with period \( \hat{r} \) described above. Let

\[
S_p x = x(t_1, \ldots, t_{p-1}, t_p - r, t_{p+1}, \ldots, t_q) \quad \text{for} \quad x \in X, \quad p = 1, 2, \ldots, q.
\]

Every \( S_p \) is a linear operator transforming \( X \) onto \( X \) and, moreover, \( S_p \) is an involution of order \( N_p \). In fact,

\[
S_{N_p} x = x(t_1, \ldots, t_{p-1}, t_p - N_p r, t_{p+1}, \ldots, t_q) = x(t_1, \ldots, t_{p-1}, t_p - \hat{r}_p, t_{p+1}, \ldots, t_q) = x(t_1, \ldots, t_{p-1}, t_p, \ldots, t_q) = x(t).
\]

and \( N_p \) is the smallest number satisfying (4.4). Let

\[
S x = x(t - \hat{r})
\]

Then \( S \) is a multi-involution of order \( N = (N_1, \ldots, N_q) \) because

\[
S x = S_{N_1} \cdots S_{N_q} x = x(t_1 - N_1 r, \ldots, t_q - N_q r) = x(t - \hat{r}_1, \ldots, t_q - \hat{r}_q) = x(t - \hat{r}) = x(t).
\]

Let \( a(t) \) be an arbitrary real \( \hat{r} \)-periodic function determined on \( R^q \). Then the operator \( a \) of multiplication by the function \( a(t) \) acting in \( X \) is commutative with \( S \):

\[
S a x = a S x = 0.
\]
Indeed,
\[ (\delta a - a D) x = a(t - \tau) x(t - \tau) - a(t) x(t - \tau) = [a(t - \tau) - a(t)] x(t - \tau) = 0. \]

Let \( \hat{X} \) be the subspace of all \( k \)-times differentiable functions belonging to \( X \). The operator \( S \), as a shift-operator, is commutative with the differential operator \( D^k \), \( k \leq n \). Hence the superposition \( a D^k \) of the operator of multiplication by an \( \tau \)-periodic function \( a(t) \) with the differential operator \( D^k \) is also commutative with operator \( S \) on \( \hat{X} \):
\[ S(a D^k) - a D^k S = 0 \text{ on } \hat{X}, \quad k \leq n. \]

According to (2.5) we can decompose the space \( X \) into a direct sum of spaces \( X_{\eta j} \), where \( X_{\eta j} = P_{\eta j} X \), \( P_{\eta j} = P_{\eta 0} \ldots P_{\eta \eta j} \) and
\[ a_{\eta 0 \eta j} = P_{\eta 0 \eta j} x = \frac{1}{N_{p \eta j}} \sum_{\eta \eta j = 1}^{N_{p \eta j}} e^{-t\eta \tau} x(t - \tau k). \]

Let us write
\[ A(S) x = \sum_{\eta \eta j \in \pi C N} A_{\eta j}(t) D^k x(t - \eta j) = \sum_{\eta \eta j \in \pi C N} \sum_{\eta \eta j \in \pi C N} A_{\eta j}(t) D^k S^\eta j x = \sum_{\eta \eta j \in \pi C N} A_{\eta j} S^\eta j x, \quad \text{where } A_{\eta j} = \sum_{\eta \eta j \in \pi C N} A_{\eta j} D^k. \]

Hence equation (4.1) can be written as follows:
\[ A(S) x = y. \]

Basing ourselves on theorem 3.1, we obtain
\[ A(S) = \sum_{\eta \eta j \in \pi C N} A(\eta) P_{\eta j}. \]

Let us write
\[ A_{\eta j} = A(\eta) = \sum_{\eta \eta j \in \pi C N} b_{\eta j} D^k, \quad b_{\eta j} = \sum_{\eta \eta j \in \pi C N} e^{-\eta \tau} A_{\eta j}(t). \]

Theorem 3.2 implies that equation (4.6) (i.e. (4.1)) is equivalent to \( N_{p} = N_{1} \ldots N_{q} \) partial differential equations,
\[ A_{\eta j} x = y_{\eta j}, \quad (1)_{\eta j} \leq \eta \leq N_{\eta j}, \]
on \( X \). If this system has an \( \omega \)-periodic solution \( x_{\eta j} \), \( (1)_{\eta j} \leq \tau \leq N_{\eta j} \), then the given equation (4.6) has an \( \omega \)-periodic solution
\[ x(t) = \sum_{\eta \eta j \in \pi C N} P_{\eta j} x_{\eta j}(t) = \frac{1}{N_{1} \ldots N_{q}} \sum_{\eta \eta j \in \pi C N} e^{-t\eta \tau} x(t - \tau k), \]

which was to be proved.

The assumption that \( 0 < c_{p 1} < \ldots < c_{p q} \) (\( p = 1, \ldots, g \)) is not essential. Indeed, if \( c_{p j} = 0 \), then \( c_{p j} = -\gamma_{p j} \tau \), where \( \gamma_{p j} \) is a positive integer and
\[ x(t_{1}, \ldots, t_{p-1}, t_{p} = c_{p j}, t_{p+1}, \ldots, t_{q}) = x(t_{1}, \ldots, t_{p-1}, t_{p} + c_{p j}, t_{p+1}, \ldots, t_{q}) \]

But \( S^{-\gamma_{p j} \tau} = S^{\gamma_{p j} \tau} \), which follows from the fact that \( S \) is an involution of order \( N_{p} \).

The assumption that all numbers \( c_{p j} \) are commensurable is not essential either. It is enough to assume that for any fixed \( p \) all numbers \( c_{p j} \) are commensurable, and in place of vector \( \tau = (\tau_{1}, \ldots, \tau_{g}) \) to consider a vector \( \tau = (\tau_{1}, \ldots, \tau_{q}) \), where
\[ c_{p j} = \gamma_{p j} \tau_{p} \quad (p = 1, 2, \ldots, g). \]

In the same manner we can consider the case where \( x \) and \( y \) are vector-functions and \( A_{\eta j}(t) \) are square matrices of respective orders. This is also true without any essential changes for functions with values in a Banach space, even in a linear metric space.

References

Tame singular integrals

by

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Introduction. Let $H$ be a real separable Hilbert space, $B$ be a one-one Hilbert-Schmidt operator on $H$, and $y \to T_y$ be the regular representation of the additive group of $H$ acting in $L^p(H), 1 < p < \infty$.

In [1] we studied singular integral operators

$$Z_p(f) = \lim_{t \to \pm \infty} \int \left[ \int_{B^{-1}(y)} f(y,t) \, d\sigma(y) \right] dt,$$

acting on $L^p(H)$, where $\int_B a(y) \, d\sigma B^{-1}(y) = 0$ and $a(y)$ satisfies an integrability condition with respect to the Gaussian measure $\mu B^{-1}$. In this note we shall restrict $a(y)$ to be either an absolutely integrable odd function or an $r$-power integrable even tame function for some $r > 1$. Under these conditions $Z_p$ is a bounded operator on $L^p(H)$ as was shown in [1].

Extension of the results of the present note to the more general functions $a(y)$ used in [1] is a simple matter.

Singular integral operators $Z_p$ generally map tame functions $f$ in $L^p(H)$ to non-tame functions $Z_p(f)$. In this note we shall consider the tame singular integrals (introduced in [1]) which map tame functions to tame functions. Corresponding to each singular integral $Z_p$ there is a net $(Z_p Q^{-1}_t \phi)$ of tame singular integrals determined by the finite-dimensional orthogonal projections $Q_t \phi$ on $H$ and this net converges strongly to $Z_p$ as $Q$ tends strongly to the identity through the directed set $\mathcal{F}$. We shall prove this result in this note.


Definition (Segal). A weak distribution on a real Hilbert space $H$ is an equivalence class $\mathcal{F}$ of linear maps from the conjugate space $H^*$.