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On symmetric Schauder bases in a Fréchet space

by

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1. Let a Fréchet space E be given. A sequence $\{x_i\}$ of elements of E is called a *Schauder basis* for E if there exists a sequence $\{f_i\}$ of elements of the dual E^* of E such that $f_i(x_j) = \delta^{ij}$, the Kronecker delta and such that every x of E could be written as

$$x = \sum t_i x_i, \quad t_i = f_i(x),$$

in a unique way. Our summations are always from 1 to ∞ unless other limits of summation are expressly indicated. The Schauder basis $\{x_i\}$ is called *symmetric* if for each permutation π of the positive integers, the sequence $\{x_{\pi(i)}\}$ is also a Schauder basis and there exists a topological isomorphism T_π of E onto itself such that $T_\pi(x_i) = x_{\pi(i)}$ for each i. Hence the bases $\{x_i\}$ and $\{x_{\pi(i)}\}$ are similar in the sense of [1].

If E is a Banach space, then it is proved in [6] that $\{x_i\}$ is symmetric if and only if

$$\sup_{\pi\in\mathscr{P}} \sup_{|a_i|\leqslant 1, 1\leqslant n} \Big\| \sum_{i=1}^{i=1} a_i f_i(x) x_{\pi(i)} \Big\| < \infty, \quad x \in E,$$

where $\mathscr{P}$ is the collection of all permutations of the positive integers.

Recently Ruckle [5] extends this result to Fréchet spaces. He shows that if the topology of the Fréchet space E is determined by the sequence of seminorms $\{p_1, p_2, \ldots\}$, then $\{x_i\}$ is symmetric if and only if for each p_k

$$(1) \qquad \sup_{\pi\in\mathscr{P}} \sup_{|a_i|\leqslant 1, 1\leqslant n} p_k\Big(\sum_{i=1}^{i=n} a_i f_i(x) x_{\pi(i)}\Big) < \infty, \quad x \in E.$$

It seems to us that this is not true for all Fréchet spaces. In fact, consider the topological product $\prod C$ of countably many 1-dimensional spaces, each of which equipped with the natural topology. This is a Fréchet space and its topology can be determined by the sequence of seminorms $\{p_k\}$ defined by

$$p_k(t) = \sup_{1\leqslant i\leqslant k} |t_i|, \quad t = (t_i) \in \prod C.$$

It is not difficult to see that $\mathscr{E} = \{e_1, e_2, \ldots\}$, where e_i has 1 in its i-th coordinate and 0 elsewhere, is a Schauder basis and, in fact, a symmetric one for $\prod C$ (see Proposition 1 further down). However $\sup_{\pi\in\mathscr{P}} p_1((t_{\pi(i)}))$, where $(t_i) = (i)$ is not finite and therefore (1) does not hold for this space.

In this short note we prove among other things that if the Fréchet space E is not isomorphic to $\prod C$, then a basis $\{x_i\}$ of it is symmetric if and only if (1) holds. It seems to us that our approach is different from that of [5].

2. The collection of all sequences $(f_i(x))$, $i = 1, 2, \ldots$, $x \in E$, equipped with the topology transferred from E is the FK-space S associated with E and its basis $\{x_i\}$ ([7], p. 208). S is isomorphic to E. It is not difficult to see that the dual S^* of S consists of all sequences (s_i) such that $\sum s_i t_i$ converges for each $(t_i) \in S$. If the basis $\{x_i\}$ is symmetric, then the coordinate spaces associated with $\{x_i\}$ and $\{x_{\pi(i)}\}$ are the same for each permutation π of the positive integers [1] and if $(t_i) \in S$, then $(t_{\pi(i)}) \in S$. This shows that for each $f \in E^*$ and each π the series $\sum f_i(x) f(x_{\pi(i)})$ converges unconditionally and hence absolutely, $\sum |f_i(x) f(x_{\pi(i)})| < \infty$.

LEMMA 1. *If E is a Fréchet space with a symmetric basis $\{x_i\}$, then for each $f \in E^*$ the sequence $(f(x_i))$ is bounded.*

Proof. Suppose that $(f(x_i))$ is not bounded. Since $\sum |f_i(x) f(x_{\pi(i)})| < \infty$ for each permutation π, it is not difficult to see that $f_i(x) = 0$ except for a finite number of i for each x. The dual of S is then the space of all sequences. If we denote by $\sum \oplus C$ the topological direct sum of countably many 1-dimensional spaces, each of which equipped with the usual topology ([2], p. 214), then the dual of $\sum \oplus C$ is the space of all sequences. $S = \sum \oplus C$ (algebraically) and they have the same dual. The topologies of S and of $\sum \oplus C$ are their strong topologies, therefore they should coincide and the topology of $\sum \oplus C$ were metrizable which is not the case.

LEMMA 2. *If E is a Fréchet space with a symmetric basis $\{x_i\}$ and if for an $x \in E$ the sequence $(f_i(x))$ is not bounded, then the FK-space S associated with E and $\{x_i\}$ is equal (algebraically and topologically) to $\prod C$, the topological product of a sequence of 1-dimensional spaces.*

Proof. This Lemma is the "dual" of the last one.

Since $\sum |f_i(x) f(x_{\pi(i)})| < \infty$, $f \in E^*$, it is not difficult to see that $f(x_i) = 0$ except for a finite number of i. The dual of S is then the space of all finite sequences, i.e. the duals of S and of $\prod C$ are the same. S is a subset of $\prod C$ and since S is total on the dual of $\prod C$, S is dense in $\prod C$. Moreover, the topology induced on S by $\prod C$ is metrizable, hence it is the Mackey topology ([2], p. 264) and therefore coincides with the own topology of S. It is then not difficult to see that $S = \prod C$.

COROLLARY. *If E is a Banach space with a symmetric Schauder basis $\{x_i\}$, then for each $x \in E$ the sequence $(f_i(x))$ is bounded and for each $f \in E^*$ the sequence $f((x_i))$ is also bounded.*

LEMMA 3. *If the sequence (t_i) is neither finite nor unbounded and if the sequence (s_i) is such that for each permutation π of the positive integers $\sum |t_i s_{\pi(i)}| < \infty$, then*

$$\sup_{\pi\in\mathscr{P}} \sum |t_i s_{\pi(i)}| < \infty.$$

Proof. See [4].

PROPOSITION 1. *A Schauder basis $\{x_i\}$ of a Fréchet space E is symmetric if for each p_k of the countable family $\{p_1, p_2, \ldots\}$ of seminorms determining the topology of E we have*

$$\sup_{n\geqslant 1} p_k\Big(\sum_{i=1}^{i=n} f_i(x) x_{\pi(i)}\Big) < \infty, \qquad x \in E,$$

for each permutation π of the positive integers.

Proof. $\{x_{\pi(i)}\}$ is a basis for E because the sequence $\{x_{\pi(i)}\}$ is fundamental and basic ([7], p. 209). Consider the mappings $T_{\pi,n}$ of E into itself defined by

$$T_{\pi,n}(x) = \sum_{i=1}^{i=n} f_i(x) x_{\pi(i)}, \qquad n = 1, 2, \ldots$$

For each x the set $\{T_{\pi,n}(x)\}$ is bounded and for x belonging to the linear hull of $\{x_1, x_2, \ldots\}$ the sequence $\{T_{\pi,n}(x)\}$ converges. Therefore, by the Banach-Steinhaus theorem ([2], p. 173), the sequence $\{T_{\pi,n}(x)\}$ converges for all x and the mapping T_π defined by $T_\pi(x) = \sum f_i(x) x_{\pi(i)}$ is continuous. Moreover, $T_\pi(x_i) = x_{\pi(i)}$ for each i and it is not difficult to see that T_π^{-1} is also continuous. Thus the bases $\{x_i\}$ and $\{x_{\pi(i)}\}$ are similar.

PROPOSITION 2. *If the Fréchet space E is not isomorphic to $\prod C$, then a Schauder basis $\{x_i\}$ of E is symmetric if and only if for each p_k of the countable family $\{p_1, p_2, \ldots\}$ of seminorms determining the topology of E we have*

$$\sup_{\pi\in\mathscr{P}} \sup_{|a_i|\leqslant 1, 1\leqslant n} p_k\Big(\sum_{i=1}^{i=n} a_i f_i(x) x_{\pi(i)}\Big) < \infty, \qquad x \in E.$$

Proof. Sufficiency has been proved in Proposition 1. We only need prove necessity. In the discussion preceding Lemma 1, we have seen that if $\{x_i\}$ is a symmetric Schauder basis, then $\sum |f_i(x) f(x_{\pi(i)})| < \infty$ for each $x \in E$, each $f \in E^*$ and each $\pi \in \mathscr{P}$. Therefore by Lemmas 1, 2 and 3

$$\sup_{\pi\in\mathscr{P}} \sup_{|a_i|\leqslant 1, 1\leqslant n} \Big|\sum_{i=1}^{i=n} a_i f_i(x) f(x_{\pi(i)})\Big| < \infty.$$

But a weakly bounded subset of a locally convex space is also bounded ([2], p. 255), hence

$$\sup_{\pi\in\mathcal{P}} \sup_{|a_i|\leqslant 1, 1\leqslant n} p_k\Big(\sum_{i=1}^{i=n} a_i f_i(x) f(x_{\pi(i)})\Big) < \infty$$

for each seminorm p_k.

Remarks. (1) By the corollary to Lemmas 1 and 2, we reobtain the result of Singer [6] from Proposition 2.

(2) Suppose now that E is a sequentially complete barrelled space. Because the Banach-Steinhaus theorem is valid for such a space, Proposition 1 holds. Since two Schauder bases $\{x_i\}$ and $\{y_i\}$ of a barrelled space are similar if and only if the convergence of $\sum t_i x_i$ implies and is implied by the convergence of $\sum t_i y_i$, $(t_i)\in\prod C$ (see [3]), a slightly weaker version of Proposition 2 holds.

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Corrigenda to the paper "From triangular matrices to separated inductive limits"

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476_{10}	f	q
477^{8}	h	h^j
477_{16}	complicated	complete
477_{14}	omit the whole line	