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Formulae of Fredholm type for solutions of linear equations with generalized Fredholm operator

by

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1. Introduction. Sikorski [3] gave formulae of Fredholm type for solutions of a Fredholm linear equation

$$(I+T)x = x_0$$

in a Banach space X, and for the adjoint equation

$$\omega(I+T) = \omega_0$$

in a conjugate space Ω in case T is a quasi-nuclear operator. Later I proved the same formulae in a more general case by an algebraic argument [2].

The purpose of this paper is to give a further generalization of Sikorski's formulae to a larger class of linear equations

$$(U+T)x = x_0$$

in a linear space X, and for the adjoint equation

$$\omega(U+T) = \omega_0$$

in a conjugate space Ω, where U is a generalized identity and T is any operator such that $U+T$ is a generalized Fredholm operator of finite defect [1]. The formulae obtained are abstract analogues of the original Fredholm formulae for solutions of inhomogeneous integral equations with a continuous kernel in the space $C_{[a,b]}$.

2. Terminology and notation. Ω and X denote two fixed linear spaces over the real or complex field F. The letters ω, ξ, η will always denote elements of Ω and the letters x, y, z — elements of X. Every mapping into F will be called a *functional.*

We assume that Ω and X are conjugate in the sense that there is a bilinear functional ωx defined on $\Omega\times X$ such that

(a) if $\omega x = 0$ for every $\omega\in\Omega$, then $x = 0$;

(a′) if $\omega x = 0$ for every $x\in X$, then $\omega = 0$.

Let A be a bilinear functional defined on $\Omega\times X$. The value of A at a point (ω, x) will be denoted by $\omega A x$.

In the following $\mathfrak{A}$ will denote the class of all bilinear functionals on $\Omega \times X$ such that

(b) for every fixed $x \in X$, there exists a $y \in X$ such that $\omega A x = \omega y$ for every $\omega \in \Omega$ (this unique element y will be denoted by Ax);

(b′) for every fixed $\omega \in \Omega$ there exists an $\eta \in \Omega$ such that $\omega A x = \eta x$ for every $x \in X$ (this unique element η will be denoted by ωA).

Thus every bilinear functional $A \in \mathfrak{A}$ can simultaneously be interpreted as an endomorphism $y = Ax$ in X or as an endomorphism $\eta = \omega A$ in Ω. The three possible interpretations of A will systematically be used throughout the paper and the elements $A \in \mathfrak{A}$ will simply be called *operators*.

If ω_0 and x_0 are fixed non-zero elements, then the bilinear function K defined by the formula

$$\omega K x = \omega x_0 \cdot \omega_0 x$$

will be called a *one-dimensional operator* and will be denoted by $x_0 \cdot \omega_0$.

Every finite sum of one-dimensional operators will be called a *finite dimensional operator*.

Let $U \in \mathfrak{A}$ be a fixed generalized Fredholm operator [1] of order $r(U) = 0$ and defect $d(U) = -d$ where $d > 0$. There exists a quasi-inverse $S \in \mathfrak{A}$ of U such that $r(S) = 0$ and $d(S) = d$, i.e.

$$SUS = S, \qquad USU = U.$$

Clearly

$$SU = I \quad \text{and} \quad US = I - \sum_{i=1}^{d} s_i \cdot \varepsilon_i, \tag{1}$$

where $\varepsilon_1, \ldots, \varepsilon_d$ and $s_1, \ldots, s_d$ are complete systems of solutions of the equations $\omega U = 0$ and $Sx = 0$, respectively such that $\varepsilon_i s_j = \delta_{i,j}$ for $i, j = 1, \ldots, d$.

Suppose that $T \in \mathfrak{A}$ is any fixed operator such that $U+T$ is a generalized Fredholm operator of order r and defect $-d$. The operator $U+T$ has a determinant system $D_0, D_1, \ldots$ also of order r and defect $-d$, D_n being a multilinear functional on $\Omega^n \times X^{n+d}$ whose value at a point $(\omega_1, \ldots, \omega_n, x_1, \ldots, x_{n+d})$ is

$$D_n \begin{pmatrix} \omega_1, \ldots, \omega_n \\ x_1, \ldots, x_{n+d} \end{pmatrix}.$$

Since r is the order of the determinant system, $D_r \neq 0$ but all D_i with $i < r$ vanish identically.

Let $\eta_1, \ldots, \eta_r$ and $y_1, \ldots, y_{r+d}$ be points such that

$$\delta_r = D_r \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_{r+d} \end{pmatrix} \neq 0.$$

Then the complete system $\xi_1, \ldots, \xi_{r+d}$ of solutions of $\omega(U+T) = 0$ is given by the formulae

$$\xi_i x = \frac{1}{\delta_r} D_r \begin{pmatrix} \eta_1, \ldots\ldots\ldots\ldots\ldots, \eta_r \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \quad \ldots, \quad y_{r+d} \end{pmatrix} \quad \text{for every } x \in X \tag{2}$$

and the complete system $z_1, \ldots, z_r$ of solutions of $(U+T)z = 0$ is given by

$$\omega z_j = \frac{1}{\delta_r} D_r \begin{pmatrix} \eta_1, \ldots, \eta_{j-1}, \quad \omega, \quad \eta_{j+1}, \ldots, \eta_r \\ y_1, \ldots\ldots\ldots\ldots\ldots, y_{r+d} \end{pmatrix} \quad \text{for every } \omega \in \Omega, \tag{3}$$

where $\xi_i y_j = \delta_{ij}$ $(i, j = 1, \ldots, r+d)$ and $\omega_i z_j = \delta_{ij}$ $(i, j = 1, \ldots, r)$. The operator B defined by the formula

$$\omega B x = \frac{1}{\delta_r} D_{r+1} \begin{pmatrix} \omega, \eta_1, \ldots, \eta_r \\ x, y_1, \ldots, y_{r+d} \end{pmatrix} \tag{4}$$

is a quasi-inverse of $U+T$.

Moreover, using properties of the determinant system for $U+T$, it can be shown that

$$(U+T)B = I - \sum_{i=1}^{r+d} y_i \cdot \xi_i, \qquad B(U+T) = I - \sum_{i=1}^{r} z_i \cdot \eta_i. \tag{5}$$

Having (5) we easily obtain the formula

$$STB - \sum_{i=1}^{r} z_i \cdot \eta_i S = B(US + TS - I) - \sum_{i=1}^{r+d} S y_i \cdot \xi_i. \tag{6}$$

Since the determinant system $D_0, D_1, \ldots$ for $U+T$ is determined by T up to a scalar factor $\neq 0$, we may assume [1] that this system is of the form

$$D_n = 0 \quad \text{for } n = 0, \ldots, r-1, \tag{7}$$

$$D_r \begin{pmatrix} \omega_1, \ldots, \omega_r \\ x_1, \ldots, x_{r+d} \end{pmatrix} = \begin{vmatrix} \omega_1 z_1, \ldots, \omega_1 z_r \\ \ldots\ldots \\ \omega_r z_1, \ldots, \omega_r z_r \end{vmatrix} \cdot \begin{vmatrix} \xi_1 x_1, \ldots, \xi_1 x_{r+d} \\ \ldots\ldots \\ \xi_{r+d} x_{r+d}, \ldots, \xi_{r+d} x_{r+d} \end{vmatrix}, \tag{8}$$

and for $k = 1, 2, \ldots$

$$D_{r+k} \begin{pmatrix} \omega_1, \ldots, \omega_{r+k} \\ x_1, \ldots, x_{r+d+k} \end{pmatrix} = \sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q \begin{vmatrix} \omega_{p_1} B x_{q_1}, \ldots, \omega_{p_1} B x_{q_k} \\ \ldots\ldots \\ \omega_{p_k} B x_{q_1}, \ldots, \omega_{p_k} B x_{q_k} \end{vmatrix} \times \tag{9}$$

$$\times D_r \begin{pmatrix} \omega_{p_{k+1}}, \ldots, \omega_{p_{k+r}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r+d}} \end{pmatrix},$$

where $\sum_{p,q}$ is extended over all permutations $p = (p_1, \ldots, p_{k+r})$ and $q = (q_1, \ldots, q_{r+d+k})$ of the integers $1, \ldots, r+k$ and $1, \ldots, r+d+k$, respectively such that

$$
(10) \qquad \begin{array}{ll} p_1 < p_2 < \ldots < p_k, & p_{k+1} < p_{k+2} < \ldots < p_{k+r}, \\ q_1 < q_2 < \ldots < q_k, & q_{k+1} < q_{k+2} < \ldots < q_{k+r+d}. \end{array}
$$

2. Formulae of Fredholm type. We precede the proof of these formulae by the proof of the following theorem:

THEOREM 1. *If $D_0, D_1, \ldots$ is a determinant system for $U+T$ of order r and defect $-d < 0$, then*

$$
(11) \quad D_n\begin{pmatrix} \omega_1 ST, \ldots, \omega_n ST \\ x_1, \ldots, x_{n+d} \end{pmatrix} = (-1)^d D_n\begin{pmatrix} \omega_1, \ldots\ldots\ldots\ldots, \omega_n \\ (US+TS-I)x_1, \ldots, (US+TS-I)x_{n+d} \end{pmatrix}
$$

for $n = 0, 1, \ldots$

Moreover

$$
(12) \qquad D_r\begin{pmatrix} \omega_1 ST, \ldots, \omega_r ST \\ x_1, \ldots, x_{r+d} \end{pmatrix} = (-1)^r D_r\begin{pmatrix} \omega_1, \ldots, \omega_r \\ x_1, \ldots, x_{r+d} \end{pmatrix}.
$$

Since $Uz_i = -Tz_i$ for $i = 1, \ldots, r$ and $-\xi_j = \xi_j(US+TS-I)$ for $j = 1, \ldots, r+d$, formulae (12) and (11) for $n = r$ follow from (8). The proof of (11) is based on the well-known formula

$$
(13) \quad \begin{vmatrix} a_{1,1} \ldots a_{1,k+r} \\ \ldots\ldots\ldots \\ a_{k+r,1} \ldots a_{k+r,k+r} \end{vmatrix} = \sum_p \operatorname{sgn} p \begin{vmatrix} a_{p_1,1} \ldots a_{p_1,k} \\ \ldots\ldots\ldots \\ a_{p_k,1} \ldots a_{p_k,k} \end{vmatrix} \cdot \begin{vmatrix} a_{p_{k+1},1} \ldots a_{p_{k+1},k+r} \\ \ldots\ldots\ldots \\ a_{p_{k+r},1} \ldots a_{p_{k+r},k+r} \end{vmatrix},
$$

where the permutation p is the same as in (10). Therefore by (8), (9), (6), (13), (12) and well-known properties of classical determinants, we obtain

$$
D_{r+k}\begin{pmatrix} \omega_1 ST, \ldots, \omega_{r+k} ST \\ x_1, \ldots, x_{r+d+k} \end{pmatrix} = (-1)^r \sum_{p,q} \operatorname{sgn} p \cdot \operatorname{sgn} q \begin{vmatrix} \omega_{p_1} STBx_{q_1} \ldots \omega_{p_1} STBx_{q_k} \\ \ldots\ldots\ldots \\ \omega_{p_k} STBx_{q_k} \ldots \omega_{p_k} STBx_{q_k} \end{vmatrix} \times D_r\begin{pmatrix} \omega_{p_{k+1}}, \ldots, \omega_{p_{k+r}} \\ x_{q_{k+1}}, \ldots, x_{q_{k+r+d}} \end{pmatrix}
$$

$$
= (-1)^r \sum_q \operatorname{sgn} q \times
$$

$$
\times \begin{vmatrix} \omega_1\big(STB - \sum_{i=1}^{r} z_i \cdot \eta_i S\big)x_{q_1} \ldots \omega_1\big(STB - \sum_{i=1}^{r} z_i \cdot \eta_i S\big)x_{q_k} \, \omega_1 z_1 \ldots \omega_1 z_r \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_{k+r}\big(STB - \sum_{i=1}^{r} z_i \cdot \eta_i S\big)x_{q_1} \ldots \omega_{k+r}\big(STB - \sum_{i=1}^{r} z_i \cdot \eta_i S\big)x_{q_k} \, \omega_{k+r} z_1 \ldots \omega_{k+r} z_r \end{vmatrix} \times
$$

$$
\times \begin{vmatrix} \xi_1 x_{q_{k+1}} \ldots \xi_1 x_{q_{k+r+d}} \\ \ldots\ldots\ldots \\ \xi_{r+d} x_{q_{k+1}} \ldots \xi_{r+d} x_{q_{k+r+d}} \end{vmatrix}
$$

$$
= (-1)^r \sum_{p,q} \operatorname{sgn} p \operatorname{sgn} q \times
$$

$$
\times \begin{vmatrix} \omega_{p_1}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_1} \ldots \omega_{p_1}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_k} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_{p_k}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_1} \ldots \omega_{p_k}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_k} \end{vmatrix} \times
$$

$$
\times D_r\begin{pmatrix} \omega_{p_{k+1}}, \ldots, \omega_{p_{k+r}} \\ \ldots\ldots\ldots \\ x_{q_{k+1}}, \ldots, x_{q_{k+r+d}} \end{pmatrix}
$$

$$
= (-1)^r \sum_p \operatorname{sgn} p \times
$$

$$
\times \begin{vmatrix} \omega_{p_1}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_1 \ldots \omega_{p_1}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_{k+r+d}} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \omega_{p_k}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_1 \ldots \omega_{p_k}\big[B(US+TS-I) - \sum_{i=1}^{r+d} Sy_i \cdot \xi_i\big]x_{q_{k+r+d}} \\ \xi_1 x_1 \ldots \xi_1 x_{q_{k+r+d}} \\ \ldots\ldots\ldots \\ \xi_{r+d} x_1 \ldots \xi_{r+d} x_{q_{k+r+d}} \end{vmatrix} \times
$$

$$
\times \begin{vmatrix} \omega_{p_{k+1}} z_1 \ldots \omega_{p_{k+1}} z_r \\ \ldots\ldots\ldots \\ \omega_{p_{k+r}} z_1 \ldots \omega_{p_{k+r}} z_r \end{vmatrix}
$$

$$= (-1)^d \sum_{p,q} \operatorname{sgn} p \cdot \operatorname{sgn} q \times$$

$$\times \begin{vmatrix} \omega_{p_1} B(US+TS-I)x_{q_1} & \ldots & \omega_{p_1} B(US+TS-I)x_{q_k} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \omega_{p_k} B(US+TS-I)x_{q_1} & \ldots & \omega_{p_k} B(US+TS-I)x_{q_k} \end{vmatrix} \times$$

$$\times D_r \begin{pmatrix} \omega_{p_{k+1}}, \ldots, \omega_{p_{k+r}} \\ (US+TS-I)x_{q_1}, \ldots, (US+TS-I)x_{q_{k+r+d}} \end{pmatrix}$$

$$= (-1)^d D_{r+k} \begin{pmatrix} \omega_1, \ldots, \omega_{k+r} \\ (US+TS-I)x_1, \ldots, (US+TS-I)x_{q_{k+r+d}} \end{pmatrix}.$$

This completes the proof.

Now we are in a position to prove the formulae of Fredholm type.

THEOREM 2 (cf. [1], p. 152-153). *For* $n = 0, 1, \ldots,$ *let*

$$D_n^* \begin{pmatrix} \omega_1, \ldots, \omega_n \\ x_1, \ldots, x_{n+d} \end{pmatrix} = D_n \begin{pmatrix} \omega_1 ST, \ldots, \omega_n ST \\ x_1, \ldots, x_{n+d} \end{pmatrix},$$

and let $\eta_1, \ldots, \eta_r, y_1, \ldots, y_{r+d}$ *be fixed points such that*

$$\delta^* = D_r^* \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_{r+d} \end{pmatrix} \neq 0.$$

Let ξ_i, z_j $(i = 1, \ldots, r+d, j = 1, \ldots, r)$ *be defined as follows:*

$$\xi_i x = \frac{1}{\delta^*} D_r^* \begin{pmatrix} \eta_1, \ldots\ldots\ldots\ldots, \eta_r \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_{r+d} \end{pmatrix} \quad \textit{for every } x \in X,$$

$$\omega z_j = \frac{1}{\delta^*} D_r^* \begin{pmatrix} \eta_1, \ldots, \eta_{j-1}, \omega, \eta_{j+1}, \ldots, \eta_r \\ y_1, \ldots\ldots\ldots\ldots, y_{r+d} \end{pmatrix} \quad \textit{for every } \omega \in \Omega$$

and define an operator B^* *by*

$$\omega B^* x = \frac{1}{\delta^*} D_{r+1}^* \begin{pmatrix} \omega, \eta_1, \ldots, \eta_r \\ x, y_1, \ldots, y_{r+d} \end{pmatrix}.$$

Then the equation

$$(*) \qquad (U+T)x = x_0$$

has a solution x *iff* $\xi_i x_0 = 0$ *for* $i = 1, \ldots, r+d$, *and the equation*

$$(**) \qquad \omega(U+T) = \omega_0$$

has a solution ω *iff* $\omega_0 z_j = 0$ *for* $j = 1, \ldots, r$. *The general form of the solution of* $(*)$ *is given by*

$$x = (S-B^*)x_0 + a_1 z_1 + \ldots + a_r z_r$$

and the general form of the solution of $(**)$ *is*

$$\omega = \omega_0(S-B^*) + \omega_1 \xi_1 + \ldots + b_{r+d} \xi_{r+d},$$

where S *is a quasi-inverse of* U *and* $a_1, \ldots, a_r$, $b_1, \ldots, b_{r+d}$ *are arbitrary constants.*

The formulae for $\xi_1, \ldots, \xi_{r+d}$ and $z_1, \ldots, z_r$ can be obtained immediately from (2) and (3) by application of (12), which form complete systems of solutions of $\omega(U+T) = 0$ and $(U+T)x = 0$, respectively. The formulae for solutions of $(*)$ and $(**)$ can be obtained by use of the identities

$$D_{n+1} \begin{pmatrix} \omega_0, \ldots\ldots\ldots, \omega_n \\ (U+T)x_0, x_1, \ldots, x_{n+d} \end{pmatrix} = \sum_{i=0}^{n} (-1)^i \omega_i x_0 D_n \begin{pmatrix} \omega_0, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n \\ x_1, \ldots\ldots\ldots\ldots, x_{n+d} \end{pmatrix},$$

$$D_{n+1} \begin{pmatrix} \omega_0(U+T), \omega_1, \ldots, \omega_n \\ x_0, x_1, \ldots, x_{n+d} \end{pmatrix} = \sum_{i=0}^{n+d} (-1)^i \omega_0 x_i D_n \begin{pmatrix} \omega_1, \ldots\ldots\ldots\ldots, \omega_{r+d} \\ x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{r+d} \end{pmatrix}$$

for $n = r$, so that, by virtue of identity (11),

$$\omega B^*(U+T)x = \omega STx - \sum_{i=1}^{r} \omega z_i \cdot \eta_i STx,$$

$$\omega(U+T)B^* x = \omega(US+TS-I)x - \sum_{i=1}^{r+d} \omega(US+TS-I)y_i \cdot \xi_i x$$

or equivalently

$$(S-B^*)(U+T) = I + \sum_{i=1}^{r} z_i \cdot \eta_i ST,$$

$$(U+T)(S-B^*) = I + \sum_{i=1}^{r+d} (US+TS-I)y_i \cdot \xi_i.$$

Multiplying the first equation by ω_0 on the left, and the second equation by x_0 on the right and assuming that $\omega_0 z_j = 0$, $\xi_i x_0 = 0$ $(i = 1, \ldots, r+d,\ j = 1, \ldots, r)$ we obtain

$$(U+T)(S-B^*)x_0 = x_0 \quad \text{and} \quad \omega_0(S-B^*)(U+T) = \omega_0.$$

This completes the proof.

References

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On functions and distributions with a vanishing derivative

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1. The main purpose of this note is to give some existence and unicity theorems for the equation $f^{(m)} = 0$, where f is a distribution or function of q real variables, and $f^{(m)}$ denotes the mixed derivative of order $m = (\mu_1, \ldots, \mu_q)$. The results presented here are closely related to papers [3] and [4].

We shall first fix the notation. If $x = (\xi_1, \ldots, \xi_q)$ and $s = (\sigma_1, \ldots, \sigma_q)$, where ξ_j are real numbers and σ_j are non-negative integers, then we use the notation $x^s = \xi_1^{\sigma_1} \ldots \xi_q^{\sigma_q}$ (if $\xi_j = 0$ and $\sigma_j = 0$, then we read $\xi_j^{\sigma_j} = 1$); thus the "power" of the vector x to the vector exponent s is a real number. By a *polynomial* of x of degree m we understand $\sum_{0 \leqslant s \leqslant m} a_s x^s$, where the coefficients a_s are real numbers.

Let $I = (A, B)$; in other terms, we assume that $A = (A_1, \ldots, A_q)$ and $B = (B_1, \ldots, B_q)$ are given points of the q-dimensional Euclidean space $\boldsymbol{R}^q$, such that $A_j < B_j$, and I is the set of points x satisfying $A < x < B$, i.e., $A_j < \xi_j < B_j$ $(j = 1, \ldots, q)$. Given the order $m = (\mu_1, \ldots, \mu_q)$, we assume that, for every $j = 1, \ldots, q$, the interval I is cut by μ_j different hyperplanes $\xi_j = \xi_{j1}, \ldots, \xi_j = \xi_{j\mu_j}$; the intersection of the hyperplane $\xi_j = \xi_{jk}$ with I will be denoted by H_{jk}. Throughout this section, we assume that the interval I, the order $m = (\mu_1, \ldots, \mu_q)$ and the numbers ξ_{jk} $(j = 1, \ldots, q;\ k = 1, \ldots, \mu_j)$ are fixed. If $\mu_j = 0$ for some index j, then we understand that no number ξ_{jk} with that index j is given. The union of all H_{jk} will be denoted by U. Thus we may say that U is the intersection of I with the union of all hyperplanes $\xi = \xi_{jk}$.

By x_s $(0 \leqslant s \leqslant m)$ we shall understand $x_s = (\xi_1\sigma_1, \ldots, \xi_q\sigma_q)$, where ξ_{j0} denotes ξ_j. We see that the set of points $x = x_s$ is a hyperplane whose number of dimensions is $q - \operatorname{sgn}\sigma_1 - \ldots - \operatorname{sgn}\sigma_q$, where $\operatorname{sgn}\sigma_j = 0$, if $\sigma_j = 0$, and $\operatorname{sgn}\sigma_j = 1$, if $\sigma_j \geqslant 1$. Thus, in particular, x_0 denotes the variable x. The intersection of the hyperplane $x = x_s$ with I will be denoted by K_s. In particular, $K_0 = I$. Evidently, if $s \neq 0$, then K_s is included in some H_{jk}. This implies that the union of all K_s is U.