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On $C(S)$-subspaces of separable Banach spaces

by

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We recall that two Banach spaces X and X_1 are said to be [*isometrically*] *isomorphic* if there is a one-to-one bounded linear operator u from X onto X_1 [with $\|ux\| = \|x\|$ for all x in X], which is called an [*isometric*] *isomorphism*. A subspace (= closed linear subspace) Y of a Banach space X is called *complemented* if there is a projection (= bounded linear idempotent operator) from X onto Y. If S is a compact Hausdorff space, $C(S)$ will denote the Banach space of all complex-valued [or real-valued] continuous functions on S.

The main result of the present paper is the following

THEOREM 1. *Let S be a compact metric space. If a separable Banach space X contains a subspace Y isomorphic to $C(S)$, then there is a subspace Z of Y such that Z is isomorphic to $C(S)$ and Z is complemented in X.*

The proof of Theorem 1 bases upon the following result due to Mi ljutin [11]; cf. also [13], Theorem 8.5. We denote by Δ the Cantor discontinuum.

(M) *If S is an uncountable compact metric space, then the space $C(S)$ is isomorphic to $C(\Delta)$.*

Using (M) we will derive Theorem 1 from the following one:

THEOREM 1a. *Let S be a zero-dimensional compact metric space. If a separable Banach space X contains a subspace Y isometrically isomorphic to $C(S)$, then there are a subspace Z of Y and a projection $\pi\colon C(S) \underset{\text{onto}}{\rightarrow} Z$ such that Z is isometrically isomorphic to $C(S)$ and $\|\pi\| = 1$.*

It is quite possible that Theorem 1 will be useful for the problem of characterizing all isomorphic types of complemented subspaces of $C(S)$, S being compact and metric. We have

COROLLARY 1. *Let S be a compact metric space. If a complemented subspace Y of $C(S)$ contains a subspace Y_1 isomorphic to $C(S)$, then Y is isomorphic to $C(S)$.*

This Corollary is a simple consequence of Theorem 1 and the decomposition method (cf. [13], Proposition 8.3). First we show that for every infinite compact metric space S

(+) *$C(S)$ is isomorphic to* $(C(S)\times C(S)\times\ldots)_0$.

(For the definition of the c_0-product $(X\times X\times\ldots)_0$ of a Banach space X see [3], p. 31). Next we use the same computation as in [13], p. 41, replacing only the symbol D^n by S.

For S countable we verify (+) using [2], Theorem 1, and the fact that the space $(C(S)\times C(S)\times\ldots)_0$ is isomorphic to the space $C(S\times S_0)$, where S_0 denotes the one-point compactification of the set of integers. If S is an uncountable compact metric space, then, in view of (M) and the fact that the c_0-products of isomorphic Banach spaces are isomorphic, it is enough to verify (+) for $S = \Delta$. This is done in [13], p. 40.

The next two corollaries concern universal Banach spaces. We recall that a Banach space X is said to be *[isometrically] universal for all separable Banach spaces* if every separable Banach space is [isometrically] isomorphic to a subspace of X.

COROLLARY 2. *A separable Banach space X is [isometrically] universal for all separable Banach spaces if and only if X contains a complemented subspace Y [there is a projection $\pi: X \underset{\text{onto}}{\to} Y$ with $\|\pi\| = 1$] which is [isometrically] isomorphic to $C(\Delta)$.*

The part "if" follows from the fact that the space $C(\Delta)$ is isometrically universal for all separable Banach spaces ([3], p. 93, [6]). The part "only if" immediately follows from Theorem 1 (resp. Theorem 1a).

Combining Corollaries 1 and 2 we get

COROLLARY 3. *If a complemented subspace Y of $C(\Delta)$ is universal for all separable Banach spaces, then Y is isomorphic to $C(\Delta)$.*

Observe that, in view of (M), one may replace in Corollary 3 the space $C(\Delta)$ by $C(S)$, where S is an uncountable compact metric space. Corollaries 2 and 3 show that $C(\Delta)$, as well as the Banach-Mazur universal space $C([0;1])$ (cf. [1], p. 163), are in a certain sense the smallest possible Banach spaces universal for all separable Banach spaces.

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Preliminaries. By a *map* $\varphi: Q\to S$ we mean a continuous function from a topological space Q into a topological space S. If P is a closed subset of Q, then $i_P: P\to Q$ denotes the natural embedding of P into Q. If $\varphi: Q\to S$ is a map, then the restriction of φ to P is the composite map φi_P. We will say that the restriction of φ to P is a homeomorphism if φi_P regarded as a map from P onto $\varphi(P)$ is a homeomorphism. If $\varphi: Q\to S$ is a map, then $\varphi^0: C(S)\to C(Q)$, the *operator induced by* φ, is defined by

$$\big(\varphi^0(f)\big)(q) = f(\varphi q) \quad \text{for } f\in C(S) \text{ and for } q\in Q.$$

Observe that $(i_P)^0: C(Q)\to C(P)$ assigns to each f in $C(Q)$ its restriction to $C(P)$.

For a topological space S and for every ordinal number α (cf. [14] for the definition) we define (cf. [9], p. 150) the *α-th derived set* $S^{(\alpha)}$ by transfinite induction: $S^{(0)} = S$; $S^{(1)} =$ the set of all non-isolated points of S;

$$S^{(\alpha)} = \begin{cases} (S^{(\beta)})^{(1)} & \text{if } \alpha = \beta+1, \\ \bigcap_{\beta<\alpha} S^{(\beta)} & \text{otherwise.}\end{cases}$$

If F is a subset of a metric space with a distance function d, then

$$\operatorname{diam} F = \sup_{s\in F}\sup_{t\in F} d(t,s).$$

By *[isometrically] isomorphic embedding* from a Banach space Z into a Banach space X we mean an [isometric] isomorphism from Z onto a subspace of X.

If S is a compact Hausdorff space, then $[C(S)]^*$ denotes the dual of the space $C(S)$. We put

$$B_S^* = \{\mu\in[C(S)]^*: \|\mu\|\leqslant 1\}.$$

The w*-topology of $[C(S)]^*$ is the weakest topology on $[C(S)]^*$ such that for every f in $C(S)$ the function $\hat f(\cdot)$ defined by $\hat f(\mu) = \mu(f)$ for μ in $[C(S)]^*$ is continuous. In the sequel we will identify a point s of S with the evaluation functional at s defined by $\hat f(s) = f(s)$ for $f\in C(S)$. This identification determines the embedding of S into B_S^* which is a homeomorphism if B_S^* is regarded in the w*-topology. We recall that (cf. [5], p. 441)

$$\operatorname{ex} B_S^* = \bigcup_{|z|=1} z\cdot S,$$

where $\operatorname{ex} B_S^*$ denotes the set of all extreme points of B_S^* (cf. [5], p. 439, for the definition), and

$$z\cdot S = \{\mu\in B_S^*: z^{-1}\mu\in S\} \quad \text{for every complex } z\neq 0.$$

Proofs of the results. Our first proposition is due to Holsztyński [8]. A particular case of isotonic isometries was considered earlier in [6]. Our proof is different from that in [8].

PROPOSITION 1. *Let S and T be compact Hausdorff spaces and let $u: C(S)\to C(T)$ be an isometrically isomorphic embedding. Then there are a closed subset Q of T, a map $\varphi: Q \underset{\text{onto}}{\to} S$ and a continuous function $\varepsilon(\cdot)$ on Q such that $|\varepsilon(q)| = 1$ for q in Q and*

$$\varepsilon(q)(ug)(q) = g\big(\varphi(q)\big) \quad (q\in Q,\ g\in C(S)).$$

Proof. Let $u^*: [C(T)]^* \to [C(S)]^*$ denote the adjoint of u. Let $\Gamma = \{z: |z| = 1\}$ denote the unit circle on the complex plane. Let us set

$$Q_z = (u^*)^{-1}(z\cdot S) \cap T, \quad Q = \bigcup_{|z|=1} Q_z,$$

$$\varepsilon(q) = z^{-1} \quad \text{for } q \in Q_z \text{ and for } z \in \Gamma,$$

$$\varphi(q) = \varepsilon(q)u^*(q) \quad \text{for } q \in Q.$$

If $g \in C(S)$ and $q \in Q$, then, by the definition of the adjoint operator, we have

$$\varepsilon(q)(ug)(q) = \varepsilon(q)[u^*(q)](g) = [\varphi(q)](g) = g(\varphi(q)).$$

Hence to complete the proof it is enough to show that $\varepsilon(\cdot)$ and φ are continuous, Q is closed, and $\varphi(Q) = S$.

Let F be a closed subset of Γ. Then

$$\varepsilon^{-1}(F) = \bigcup_{z \in F} (u^*)^{-1}(z^{-1}\cdot S) \cap T = (u^*)^{-1}\Big(\bigcup_{z\in F} z^{-1}\cdot S\Big) \cap T.$$

Since the map $(z, s) \to z^{-1}s$ is a homeomorphism from $\Gamma \times S$ into B_S^* and since the operator $u^*: [C(T)]^* \to [C(S)]^*$ is w*-continuous (i. e. continuous if both spaces $[C(T)]^*$ and $[C(S)]^*$ carry their w*-topologies), the set $\varepsilon^{-1}(F)$ is closed. Hence $\varepsilon(\cdot)$ is a continuous function and $Q = \varepsilon^{-1}(\Gamma)$ is closed. Thus φ is continuous and maps Q into S. We will show that $\varphi(Q) = S$. To this end for each s in S let us consider the set

$$K(s) = (u^*)^{-1}(s) \cap B_T^*.$$

Since u^* is w*-continuous and since B_T^* is w*-compact, $K(s)$ is w*-compact. Clearly, by linearity of u^*, $K(s)$ is convex. Finally, $K(s)$ is non-empty because $\|s\| = 1$ and u^* maps B_T^* onto B_S^* (this follows from the Hahn-Banach extension theorem and the fact that u is an isometrically isomorphic embedding). Therefore, by the Krein-Milman theorem ([5], p. 440), $K(s)$ has at least one extreme point, say μ.

We will show that $\mu \in \operatorname{ex} B_T^*$. Let $\mu = \nu_1 + \nu_2$, where $\|\nu_1\| + \|\nu_2\| \leqslant 1$; $0 \neq \nu_1$; $0 \neq \nu_2$. Then $s = u^*\nu_1 + u^*\nu_2$. Since $\|u^*\| = 1$, we have

$$1 = \|s\| \leqslant \|u^*\nu_1\| + \|u^*\nu_2\| \leqslant \|u^*\|(\|\nu_1\| + \|\nu_2\|) \leqslant 1.$$

Hence

$$\|u^*\nu_1\| = \|\nu_1\|; \quad \|u^*\nu_2\| = \|\nu_2\|; \quad \|u^*\nu_1\| + \|u^*\nu_2\| = 1.$$

This formula implies that $u^*\nu_1 = as$ and $u^*\nu_2 = (1-a)s$ for some a with $0 < a < 1$, because $s = u^*\nu_1 + u^*\nu_2 \in \operatorname{ex} B_S^*$. Write $\lambda_1 = a^{-1}\nu_1$; $\lambda_2 = (1-a)^{-1}\nu_2$. Then $\lambda_1, \lambda_2 \in K(s)$ because $\|\lambda_i\| = 1$ and $u^*\lambda_i = s$ $(i = 1, 2)$. Since $\mu = a\lambda_1 + (1-a)\lambda_2$ and μ is an extreme point of $K(s)$, we get $\mu = \lambda_1 = \lambda_2$. This shows that μ is an extreme point of B_T^*. Hence

there are a t in T and a z in Γ such that $\mu = zt$. Since $\mu \in K(s)$, we get $u^*(t) = z^{-1}s$. Consequently, $t \in Q_{1/z}$ and $\varepsilon(t) = z$. Therefore $\varphi(t) = \varepsilon(t)u^*(t) = s$. Thus φ maps Q onto S. That completes the proof.

PROPOSITION 2. *Let Q and S be compact metric spaces. Let $\varphi: Q \to S$ be a map.*

(I) *If the set $\varphi(Q)$ is uncountable, then there is a closed subset P of Q such that P is homeomorphic to the Cantor discontinuum and the restriction of φ to P is a homeomorphism.*

(II) *If $\varphi(Q)$ is countable, then there is a closed subset P of Q such that P is homeomorphic to $\varphi(Q)$ and the restriction of φ to P is a homeomorphism. (In general, $\varphi(P)$ does not coincide with $\varphi(Q)$.)*

Proof. (I) has been proved by Kuratowski [9], p. 351.

(II) By a theorem of Mazurkiewicz and Sierpiński [10], the topological type of a countable metric space T is determined by the pair $(a(T), m(T))$, where $a(T)$ is a countable ordinal number such that $T^{(a(T))} \neq \emptyset$ but $T^{(a(T)+1)} = \emptyset$, and $m(T)$ is a positive integer such that the finite set $T^{(a(T))}$ has exactly $m(T)$ elements.

Observe that (II) is an easy consequence of the following fact:

(*) *If $\psi: Q \to S$ is a map (S, Q compact metric spaces) and if $\psi(Q)$ is countable, then there is a closed subset P of Q such that $a(P) = a(\psi(Q))$, $m(P) = 1$ and the restriction of ψ to P is a homeomorphism.*

Indeed, let $\varphi: Q \to S$ be an arbitrary map and let $\varphi(Q)$ be countable. Let

$$[\varphi(Q)]^{(a)} = \{s_1, s_2, \ldots, s_m\},$$

where $m = m(\varphi(Q))$ and $a = a(\varphi(Q))$. Since $\varphi(Q)$ is a countable compact metric space, it is zero-dimensional. Hence there are mutually disjoint closed-open sets S_i $(i = 1, 2, \ldots, m)$ such that $s_i \in S_i$ and

$$\bigcup_{i=1}^{m} S_i = S.$$

Let us set $Q_i = \varphi^{-1}(S_i)$ and let ψ_i denote the restriction of φ to Q_i. Clearly

$$[\psi_i(Q_i)]^{(a)} = [\varphi(Q_i)]^{(a)} = S_i^{(a)} = \{s_i\}$$

because S_i is a closed-open subset of $\varphi(Q)$ and $[\varphi(Q)]^{(a)} \cap S_i = \{s_i\}$ $(i = 1, 2, \ldots, m)$. Thus $a(\psi_i(Q_i)) = a(\varphi(Q)) = a$. Hence, by (*), there is a closed subset P_i of Q_i such that $m(P_i) = 1$; $a(P_i) = a(\varphi(Q)) = a$, and ψ_i restricted to P_i is a homeomorphism. Let us write

$$P = \bigcup_{i=1}^{m} P_i.$$

Since S_i are mutually disjoint components of $\varphi(Q)$ and $P_i \subset \varphi^{-1}(S_i) = Q$, the sets P_i are mutually disjoint components of P. Clearly φ restricted to P is a homeomorphism, because φ restricted to P_i is a homeomorphism and the closed sets $\varphi(P_i)$ are mutually disjoint (because $\varphi(P_i) \subset S_i$). Taking into account that P_i are components of P, we infer that

$$P^{(\alpha)} = \bigcup_{i=1}^{m} P_i^{(\alpha)} = \bigcup_{i=1}^{m} \{s_i\}.$$

Hence $\alpha(P) = \alpha(\varphi(Q))$ and $m(P) = m(\varphi(Q))$. Thus P is homeomorphic to $\varphi(Q)$.

We shall prove (*) by transfinite induction. If $\alpha(\psi(Q)) = 0$, then P may be any one-point subset of Q. Assume that (*) is true for every map $\psi: Q \to S$ (Q, S arbitrary compact metric spaces) such that $\alpha(\psi(Q)) < \alpha$, where $\alpha > 0$ is a fixed countable ordinal number. Let us consider a map $\varphi: Q \to S$ such that $\alpha(\varphi(Q)) = \alpha$. Fix $s \in [\varphi(Q)]^{(\alpha)}$. Choose a sequence of different points (s_n) in $\varphi(Q)$ so that

$$\lim s_n = s, \quad s_n \in [\varphi(Q)]^{(\beta_n)} \setminus [\varphi(Q)]^{(\alpha)} \quad \text{for } n = 1, 2, \ldots,$$

where $\beta_n = \beta$ if $\alpha = \beta+1$, and $\beta_1 < \beta_2 < \ldots$ with $\sup \beta_n = \alpha$ if α is a limit ordinal number (this is possible because for every countable compact metric space T and for arbitrary countable ordinals $0 \leqslant \beta < \alpha$, $T^{(\alpha)}$ is a nowhere-dense subset of $T^{(\beta)}$). Next choose a sequence (S_n) of mutually disjoint closed-open subsets of $\varphi(Q)$ such that

$$s_n \in S_n, \quad S_n \cap [\varphi(Q)]^{(\alpha)} = \emptyset \quad (n = 1, 2, \ldots), \quad \lim \operatorname{diam} S_n = 0$$

(this is possible because every countable compact space is zero-dimensional and $[\varphi(Q)]^{(\alpha)}$ is a finite set). Let $Q_n = \varphi^{-1}(S_n)$ and let φ_n denote the restriction of φ to Q_n. Let $\alpha(\varphi(Q_n)) = \gamma_n$. Then we have $\alpha > \gamma_n \geqslant \beta_n$, because $\varphi(Q_n) = S_n$ is a closed-open neighbourhood of $s_n \in [\varphi(Q)]^{(\beta_n)}$ and $S_n \cap [\varphi(Q)]^{(\alpha)} = \emptyset$. Therefore, by the inductive hypothesis, there is a closed subset P_n of Q_n such that

$$\alpha(\varphi_n(P_n)) = \alpha(\varphi(Q_n)) = \gamma_n;$$

$m(P_n) = 1$, and φ_n restricted to P_n is a homeomorphism. Let p_n be the unique point of $P_n^{(\gamma_n)}$. Observe that without loss of generality one may assume that $\operatorname{diam} P_n < n^{-1}$. To this end it is enough, if necessary, to replace P_n by a sufficiently small closed-open neighbourhood of p_n. Since Q is a compact metric space, one may also assume that the sequence (p_n) is convergent (we may replace (p_n) by a suitable subsequence). Let $p = \lim p_n$. Let us set

$$P = \{p\} \cup \bigcup_{n=1}^{\infty} P_n.$$

Observe that P is closed. Indeed, since P_n are mutually disjoint and closed, it is enough to show that if (p_n') is a convergent sequence such that $p_n' \in P_n$ for $n = 1, 2, \ldots$, then $\lim p_n' \in P$. In fact, $\lim p_n' = p$ because $\lim p_n = p$ and $\lim \operatorname{diam} P_n = 0$. Next we will show that φ restricted to P is a homeomorphism. Let p' and p'' be two different points in P. There are three possibilities:

1° There is an index n such that both p' and p'' belong to P_n. Then $\varphi(p') = \varphi_n(p') \neq \varphi_n(p'') = \varphi(p'')$ because φ_n is a homeomorphism.

2° $p' \in P_{n_1}$, $p'' \in P_{n_2}$, $n_1 \neq n_2$. Then $\varphi(p) \in S_{n_1}$ and $\varphi(p'') \in S_{n_2}$. Thus $\varphi(p') \neq \varphi(p'')$ because $S_{n_1} \cap S_{n_2} = \emptyset$.

3° $p' \in P_{n_0}$ for some n_0 and $p'' = p$. Then $\varphi(p') \in S_{n_0}$, while $\varphi(p'') = \varphi(p) = \lim \varphi(p_n) = s$ because $\varphi(p_n) \in S_n$, $\lim \operatorname{diam} S_n = 0$ and $\lim s_n = s$. Therefore $\varphi(p') \neq \varphi(p)$ because $\varphi(p) = s \in [\varphi(Q)]^{(\alpha)}$, whereas $S_{n_0} \cap [\varphi(Q)]^{(\alpha)} = \emptyset$.

Therefore the restriction of φ to P is a one-to-one map. Since P is compact, the restriction of φ to P is a homeomorphism.

Finally, we will show that $\alpha(P) = \alpha$ and $m(P) = 1$. Since the restriction of φ to P is a homeomorphism, $\alpha(\varphi(P)) = \alpha(P)$ and $m(\varphi(P)) = m(P)$. We have

$$\varphi(P) \cap [\varphi(Q)]^{(\alpha)} \subset \left(\{\varphi(p)\} \cap [\varphi(Q)]^{(\alpha)}\right) \cup \left(\left(\bigcup_{n=1}^{\infty} S_n\right) \cap [\varphi(Q)]^{(\alpha)}\right) = \{s\}.$$

Now if $\alpha = \beta+1$, then

$$P^{(\beta)} \supset \bigcup_{n=1}^{\infty} P_n^{(\beta)} = \bigcup_{n=1}^{\infty} \{p_n\}.$$

Hence $P^{(\beta)}$ is infinite. Therefore, by compactness of P, the set $P^{(\beta+1)} = P^{(\alpha)}$ is non-empty. Since the restriction of φ to P is a homeomorphism, the set $[\varphi(P)]^{(\alpha)}$ is non-empty. Hence $[\varphi(P)]^{(\alpha)} = \{s\}$. Similarly, if $\alpha = \sup \beta_n$ with $\beta_1 < \beta_2 < \ldots$, then $p_n \in P_n^{(\gamma_n)} \subset P^{(\beta_n)}$ for $n = 1, 2, \ldots$ Thus $p = \lim p_n \in P^{(\alpha)}$. Hence, also in this case, the set $[\varphi(P)]^{(\alpha)}$ is non-empty. Therefore $[\varphi(P)]^{(\alpha)} = \{s\}$. Hence, in either case, $[\varphi(P)]^{(\alpha)}$ is a one-point set, i.e., $\alpha(P) = \alpha(\varphi(P)) = \alpha$ and $m(P) = m(\varphi(P)) = 1$. That completes the induction and the proof of the Proposition.

Proposition 2a. *If φ is a map from a compact metric space Q onto a compact metric zero-dimensional space S, then there is a closed subset P of Q such that P is homeomorphic to S and φ restricted to P is a homeomorphism.*

Proof. If S is uncountable, then the desired conclusion follows from Proposition 2, (I) and the well-known fact that every zero-dimensional compact metric space is homeomorphic to a subset of Δ.

If S is countable, then we use Proposition 2, (II).

Remark. The assumption of metrizability of Q in Proposition 2 can not be removed. Indeed, let βN denote the Stone-Čech compactification of the set of integers. Since every compact metric space S contains a dense countable set, there is a map φ from βN onto S (cf. [7], p. 84). Since no infinite closed subset of βN is metrizable (because βN is extremally disconnected; [7], p. 96), the restriction of φ to a closed infinite subset of βN is not a homeomorphism. However, Proposition 2 is likely to hold for dyadic spaces (= continuous images of Cartesian products of two-point discrete spaces).

Proof of Theorem 1a. First assume that $X = C(T)$ for some compact metric space T. Let $u\colon C(S)\to C(T)$ be an isometrically isomorphic embedding such that $u[C(S)] = Y$. By Proposition 1, there are a closed subset Q of T, a map φ from Q onto S and a function $\varepsilon(\cdot)$ in $C(Q)$ with $|\varepsilon(q)| = 1$ for all q in Q such that

(1) $$\varepsilon(q)(ug)(q) = g(\varphi(q)) \quad \text{for } q\in Q \text{ and for } g\in C(S).$$

Restating (1) in terms of induced operators we obtain

(2) $$v_\varepsilon(i_Q)^0 u = \varphi^0,$$

where $v_\varepsilon(f) = \varepsilon(\cdot)f$ for $f\in C(Q)$.

Since S is a zero-dimensional compact metric space (by Proposition 2a), there is a closed subset P of Q which is homeomorphic to S and such that the restriction of φ to P is a homeomorphism. The fact that the restriction of φ to P is a homeomorphism may be reformulated in terms of induced operators as

(3) $$(i_{\varphi(P)})^0 = \Psi^0(i_P)^0\varphi^0,$$

where $\Psi\colon \varphi(P)\to P$ is the homeomorphism inverse to the restriction of φ to P. Let $L\colon C(\varphi(P))\to C(S)$ be a linear extension operator with $\|L\| = 1$, i.e.

(4) $$\big(L(f)\big)(s) = f(s),\quad \|L(f)\| = \|f\| \quad \big(s\in\varphi(P),\ f\in C(\varphi(P))\big).$$

The existence of L is a particular case of the Borsuk-Dugundji extension theorem (cf. [4] and [13]). Restating (4) in terms of induced operators we get

(5) $$(i_{\varphi(P)})^0 L = \mathrm{id}_{C(\varphi(P))},$$

where $\mathrm{id}_{C(\varphi(P))}$ denotes the identity on $C(\varphi(P))$. Let us define the operator $\pi\colon C(T)\to C(T)$ by

(6) $$\pi = uL\Psi^0(i_P)^0 v_\varepsilon(i_Q)^0.$$

(We regard i_Q as the natural embedding of Q into T and i_P as the natural embedding of P into Q.) Let us set

$$Z = uL\big(C(\varphi(P))\big).$$

By (4), u and L are isometrically isomorphic embeddings. Hence Z is isometrically isomorphic to $C(\varphi(P))$. Since P and $\varphi(P)$ are homeomorphic to S, Z is isometrically isomorphic to $C(S)$. Clearly, $Z\subset u(C(S)) = Y$. We will check that π is a projection from $C(T)$ onto Z with $\|\pi\| = 1$. Let $f\in C(T)$. Then

$$\pi(f) = uL\big[\big(\Psi^0(i_P)^0 v_\varepsilon(i_Q)^0\big)(f)\big]\in Z.$$

Thus $\pi(C(T))\subset Z$. Combining (2), (3), and (5) with (6) we get

$$\begin{aligned}\pi uL &= uL\Psi^0(i_P)^0 v_\varepsilon(i_Q)^0 uL\\ &= uL\Psi^0(i_P)^0\varphi^0 L\\ &= uL(i_{\varphi(P)})^0 L\\ &= uL.\end{aligned}$$

Hence

$$\pi^2 = \pi uL\Psi^0(i_P)^0 v_\varepsilon(i_Q)^0 = uL\Psi^0(i_P)^0 v_\varepsilon(i_Q)^0 = \pi.$$

Thus π is a (non-zero) projection and $\|\pi\|\geqslant 1$. Since the norms of all operators appearing in the left-hand side of (6) are equal to 1, we have $\|\pi\|\leqslant 1$. Thus $\|\pi\| = 1$. This completes the proof in the case where $X = C(T)$.

In the general case we may regard X as a subspace of $C(T)$ (actually for $T = \Delta$; cf. [3], p. 93). The desired projection is the restriction to X of the projection constructed for $C(T)$.

Proof of Theorem 1. We shall need the following facts:

(a) *If $u\colon Z\to X$ is an isomorphic embedding, then there exist a Banach space X_1 and an isomorphism v from X onto X_1 such that vu is an isometrically isomorphic embedding* ([12], Proposition 1).

(b) *If S is a compact metric space, then there is a zero-dimensional compact metric space S_1 such that $C(S)$ is isomorphic to $C(S_1)$.*

If S is countable, then $S_1 = S$. If S is uncountable, then, by (M), $C(S)$ is isomorphic to $C(\Delta)$.

We are now ready for the proof of Theorem 1. Let Y be a subspace of a separable Banach space X. If Y is isomorphic to a space $C(S)$, then, by (b), there are a zero-dimensional compact metric space S_1 and an isomorphic embedding $u\colon C(S_1)\to X$ such that $u(C(S_1)) = Y$. By (a), there exist a Banach space X_1 and an isomorphism $v\colon X\to X_1$ such that vu is an isometrically isomorphic embedding. Let us put $Y_1 = vu(C(S_1))$.

Since vu is an isometrically isomorphic embedding, Y_1 is isometrically isomorphic to $C(S_1)$. Hence, by Theorem 1a, there are a subspace Z_1 of Y_1 such that Z_1 is isometrically isomorphic to $C(S_1)$ and a projection π_1 from X_1 onto Z_1 such that $\|\pi_1\| = 1$. Let $Z = v^{-1}(Z_1)$ and let $\pi = v^{-1}\pi_1 v$. Clearly π is a projection from X onto Z. Since $Y_1 \supset Z_1$, we have $Y = v^{-1}(Y_1) \supset v^{-1}(Z_1) = Z$. Since v is an isomorphism, all spaces Z, Z_1, $C(S_1)$, and $C(S)$ are isomorphic each to other. This completes the proof.

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Composition of binary quadratic forms*

by

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1. Introduction. Gauss's complete discussion of the composition of binary quadratic forms over the integers ([6], sections 235 - 244 and several later sections) was a tour de force that makes remarkable reading to this day.

Several of the great mathematicians of the nineteenth and early twentieth centuries took up the theme and gave fresh accounts of the work. This material takes up twenty condensed pages in Dickson's history ([3], p. 60 - 79).

The idea of giving still another account of this venerable subject arose when I attempted to extend the theory to Bézout domains (integral domains where every finitely generated ideal is principal). Now the modern view of composition is that it is really just multiplication of suitable modules. (This idea is attributed by Dickson to Dedekind, quoting the eleventh supplement in [5]. A recent exposition is [1], p. 212 - 5.) But when one proceeds to a detailed execution, there are difficulties. The correspondence between quadratic forms and modules needs touching up. There is some trouble disentangling a module from its conjugate, overcome by "orienting" the module; there is also a need to use "strict" equivalence of modules, meaning multiplication by elements of *positive* norm. Both of these points seem to require an ordered integral domain, and on closer inspection one sees further obstacles if the base ring has units other than ± 1.

I might have concluded that ordering was indispensable for composition, had it not been for the existence of still another method, the technique of "united forms", also attributed by Dickson to Dedekind (tenth supplement in [5]; as late as 1929 Dickson [4], Ch. IX, thought this to be the best method to put in his book). It is a fact that this discussion is valid verbatim for any principal ideal domain of characteristic $\neq 2$. But I could not get it to work for Bézout domains (the difficulty comes

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