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On periodic solutions of linear differential-difference equations with constant coefficients

by

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The purpose of this paper is to give

1° simple conditions of the existence of periodic solutions of the differential-difference equation

$$\sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} x^{(k)}(t-\omega_j) = y(t), \tag{1}$$

where y is a given periodic function (which can be identically equal to zero), a_{k_j} and ω_j are real numbers and $\omega_0 = 0$, and $x^{(k)}$ denotes the k-th derivative of x;

2° effective formulae for these solutions.

All known methods of solving linear differential-difference equationt are very complicated, even in the case of constant coefficients and constans retardations (see for example Bellman and Cooke [1]). The reason is that the characteristic quasi-polynomial of equation (1) is a transcendental function. Then it is very difficult to find all the roots of that quasi-polynomial and their number is infinite. The usual method of determining periodic solutions of (1) is by using some asymptotic properties of roots of the characteristic quasi-polynomial (see Elsgolc [3], Halanay [5], Hahn [4] and Zvierkin [10] and [11]).

Our method will be different and much simpler. We shall show that in the class of periodic functions (with period ω not yet determined) equation (1) is equivalent to a finite number of ordinary linear differential equations with constant coefficients. Since the conditions of solvability of these equations are well known (see for example Coddington and Levinson [2] and Krasnoselskiĭ [6]), we shall not deal with them; we shall assume only that the required conditions are satisfied.

The proposed method is based on properties of involutions of order N which have been studied by the author ([7], [8] and also [9]).

1. Involutions of order N. We shall enumerate here without proofs those properties of involutions of order N which will be needed later. The reader can find the respective proofs in papers [7], [8] and in the book [9].

Let X be a linear space (over complex scalars). A linear operator S transforming X onto X is called an *involution of order* N if N is the smallest positive integer ($N \geqslant 2$) such that $S^N = I$, where I denotes the identity operator.

Let $\varepsilon = e^{2\pi i/N}$. We have

$$P_\nu = \frac{1}{N}(I + \varepsilon^{-\nu}S + \ldots + \varepsilon^{-\nu(N-1)}S^{N-1}), \quad \nu = 1, 2, \ldots, N.$$

If S is an involution of order N, then we have the following important properties of P:

$$\sum_{\nu=1}^{N} P_\nu = I,$$

(2)
$$P_\nu P_\mu = P_\mu P_\nu = \delta_{\mu\nu} P_\mu, \quad P_\nu S = SP_\nu \quad (\mu, \nu = 1, 2, \ldots, N),$$

where $\delta_{\mu\nu}$ is the Kronecker symbol;

(3)
$$SP_\nu = \varepsilon^\nu P_\nu \quad (\nu = 1, 2, \ldots, N).$$

This implies that X is a direct sum

(4)
$$X = \bigoplus_{\nu=1}^{N} X_{(\nu)}$$

of spaces $X_{(\nu)}$ such that $Sx_{(\nu)} = \varepsilon^\nu x_{(\nu)}$ for every $x_{(\nu)} \in X_{(\nu)}$ ($\nu = 1, 2, \ldots, N$).

Every element $x \in X$ can be written in a unique manner as a sum:

(5)
$$x = \sum_{\nu=1}^{N} x_{(\nu)}, \quad \text{where } x_{(\nu)} = P_\nu x \in X_{(\nu)} \quad (\nu = 1, 2, \ldots, N).$$

If a linear operator A acting in X commutes with an involution S of order N, then

(6)
$$A(D_A \cap X_{(\nu)}) \subset X_{(\nu)} \quad \textit{for} \quad \nu = 1, 2, \ldots, N,$$

where $D_A \subset X$ denotes the domain of A.

In fact, let $x \in D_A$. Then $x_{(\nu)} = P_\nu x \in X_{(\nu)}$ for $\nu = 1, 2, \ldots, N$ and

$$Ax_{(\nu)} = AP_\nu x = A\Big(\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k\Big)x$$
$$= \Big(\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k\Big)Ax = P_\nu(Ax) = (Ax)_{(\nu)} \in X_{(\nu)}.$$

Hence $A(D_A \cap X_\nu) \subset X_{(\nu)}$.

2. Solution of the problem. Let us write

$$Ax = \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} x^{(k)}(t - \omega_j), \quad \omega_0 = 0.$$

Equation (1) will be written further as

(7)
$$Ax = y.$$

Without loss of generality we can assume that $0 = \omega_0 < \omega_1 < \ldots < \omega_m$. Since we are looking for a periodic solution of (7), we assume that all numbers $\omega_1, \ldots, \omega_m$ are commensurable. Consequently, there is a number $r \neq 0$ and there are positive integers n_j such that

(8)
$$\omega_j = n_j r \quad \text{for} \quad j = 1, 2, \ldots, m.$$

THEOREM. *Let a real function $y(t)$ determined on the real line be periodic with period ω_{m+1} commensurable with real commensurable numbers $\omega_1, \ldots, \omega_m$. Then equation (7) has ω-periodic solutions if and only if all ordinary differential equations with constant coefficients*

(9)
$$A_\nu u = \sum_{k=0}^{n} b_{k\nu} u^{(k)} = y_{(\nu)} \quad (\nu = 1, 2, \ldots, N)$$

have ω-periodic solutions, where

(9′)
$$\begin{cases} y_{(\nu)} = \dfrac{1}{N} \displaystyle\sum_{k=0}^{N-1} \varepsilon^{-k\nu} y(t - kr), \\ b_{k\nu} = \displaystyle\sum_{j=0}^{m} a_{kj} \varepsilon^{-\nu n_j}, \quad \varepsilon = e^{2\pi i/N}; \end{cases} \quad \nu = 1, 2, \ldots, N,$$

$\omega_j = n_j r$ for $j = 1, 2, \ldots, m+1$, n_j are positive integers, $r \neq 0$, $n_0 = 0$, N is a common multiple of numbers $n_1, \ldots, n_{m+1}$ (not necessarily the smallest) and $\omega = Nr$.

The solutions are of the form

$$x = \frac{1}{N} \sum_{\nu=1}^{N} \sum_{k=0}^{N-1} \varepsilon^{-k(\nu)} x_{(\nu)}(t - kr),$$

where $x_{(\nu)}$ is an ω-periodic solution of the ν-th equation (9).

Proof. Let us consider the space X of all ω-periodic real functions $x(t)$ determined on the real line with the period ω described above. Let

(10)
$$Sx = x(t - r) \quad \text{for} \quad x \in X.$$

S is a linear operator transforming X onto X and, moreover, S is an involution of order N. In fact,

(11)
$$S^N x = x(t - Nr) = x(t - \omega) = x(t)$$

and N is the smallest number satisfying (11).

According to (4) we can decompose the space X into a direct sum of spaces $X_{(\nu)}$, where $X_{(\nu)} = P_\nu X$,

$$P_\nu x = \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} x(t-kr)$$

and $x_{(\nu)} = P_\nu x$ for every $x \in X$ $(\nu = 1, 2, \ldots, N)$.

Now we can write the operator A in the following form, if we remark that S, as a shift operator, commutes with derivation:

$$Ax = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} x^{(k)}(t-\omega_j) = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} S^{n_j} x^{(k)}(t)$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} S^{n_j}\Big[\sum_{\nu=1}^{N} x_{(\nu)}^{(k)}(t)\Big] = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj}\Big[\sum_{\nu=1}^{N} \varepsilon^{-\nu n_j} x_{(\nu)}^{(k)}(t)\Big]$$

$$= \sum_{\nu=1}^{N}\sum_{k=0}^{n}\Big[\sum_{j=0}^{m} a_{kj}\varepsilon^{-\nu n_j}\Big] x_{(\nu)}^{(k)}(t) = \sum_{\nu=1}^{N}\sum_{k=0}^{n} b_{k\nu} x_{(\nu)}^{(k)}(t)$$

$$= \Big[\sum_{\nu=1}^{N} A_\nu x_{(\nu)}\Big](t) = \Big[\sum_{\nu=1}^{N} A_\nu P_\nu\Big] x(t),$$

where $b_{k\nu}$ and A_ν are determined by (9) and (9′). Finally,

$$A = \sum_{\nu=1}^{N} A_\nu P_\nu .$$

Every operator A_ν commutes also with S. Hence the equation $Ax = y$ is equivalent to N equations

$$A_\nu x_{(\nu)} = y_{(\nu)}, \quad \text{where} \quad y_{(\nu)} = P_\nu y \ (\nu = 1, 2, \ldots, N).$$

Let us suppose that there is a solution $x \in X$ of the equation $Ax = y$. Then $x_{(\nu)} = P_\nu x$ is a solution of the equation $A_\nu u = y_{(\nu)}$, $\nu = 1, 2, \ldots, N$. and by definition $x_{(\nu)}$ belongs to X. Hence $x_{(\nu)}$ are ω-periodic functions.

Let us suppose that every equation $A_\nu u = y_{(\nu)}$ has a solution x_ν. By the assumptions, $x_\nu \in X$. Then

$$x = \sum_{\nu=1}^{N} P_\nu x_{(\nu)}$$

is a solution of $Ax = y$. In fact, (2) implies

$$x_{(\nu)} = P_\nu x = P_\nu\Big(\sum_{\mu=1}^{N} P_\mu x_\mu\Big) = P_\nu x_{(\nu)}$$

and

$$Ax = \sum_{\nu=1}^{N} A_\nu P_\nu x = \sum_{\nu=1}^{N} A_\nu x_{(\nu)} = \sum_{\nu=1}^{N} A_\nu P_\nu x_{(\nu)} = \sum_{\nu=1}^{N} P_\nu (A_\nu x_{(\nu)})$$

$$= \sum_{\nu=1}^{N} P_\nu y_{(\nu)} = \sum_{\nu=1}^{N} y_{(\nu)} = y .$$

From the preceding considerations it follows that the solution is of the form

$$x = \sum_{\nu=1}^{N} P_\nu x_{(\nu)} = \frac{1}{N}\sum_{\nu=1}^{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} S^k x_{(\nu)} = \frac{1}{N}\sum_{\nu=1}^{N}\sum_{k=0}^{N-1} \varepsilon^{-k\nu} x_{(\nu)}(t-kr),$$

where $x_{(\nu)}$ is an ω-periodic solution of the equation $A_\nu x_{(\nu)} = y_{(\nu)}$, which was to be proved.

This theorem is also true without any essential changes of the proof if $x(t)$ and $y(t)$ are vector functions. What is more, we can assume that the functions $x(t)$ and $y(t)$ determined on the real line belong to a Banach space and even to a linear metric space.

References

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