does not imply that. On the other hand, all discrete groups have such invariant neighborhoods, $W = \{u\}$, so all discrete groups, whether they are amenable or not, have regular representations of finite type.

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Concerning extension of multiplicative linear functionals in Banach algebras

by

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A commutative complex Banach algebra A *has the ES-property* (Extension from Subalgebras) or *belongs to the class ES* (written as $A \in \mathrm{ES}$) if for every its (closed) subalgebra $A_0 \subset A$ and every multiplicative linear functional f defined on A_0 there exists a multiplicative linear functional F defined on A such that its restriction to A_0 equals f. In other words, $A \in \mathrm{ES}$ if and only if every multiplicative linear functional in any subalgebra of A is extentable to such a functional defined on A. Clearly, any subalgebra of a member of ES also belongs to this class. In this paper we characterize the class ES in terms of spectra of elements of algebras in this class. Our main result reads as follows:

THEOREM 1. *A Banach algebra A belongs to the class ES if and only if for every element $x \in A$ its spectrum $\sigma(x)$ is a totally disconnected subset of the complex plane.*

To illustrate this theorem we show that for any compact group G the group algebra $L_1(G)$ belongs to the class ES. (For related results see also [1] and [3].)

Let A be a commutative complex Banach algebra with unit e. We shall write $M(A)$ for the (compact) maximal ideal space of A provided with the Gelfand topology. The spectral (semi-) norm $\|x\|_s$ is defined as

$$\|x\|_s = \sup_{f \in M(A)} |f(x)| = \sup_{M(A)} |x^\wedge(f)| = \sup |(\sigma x)| \leqslant \|x\|,$$

where $x^\wedge(f) = f(x)$ is the Gelfand transform of $x \in A$. If p is any complex polynomial in one variable, then for any $x \in A$

$$\sigma(p(x)) = p(\sigma(x)),$$

and so

$$(1) \qquad \|p(x)\|_s = \sup_{t \in \sigma(x)} |p(t)|.$$

We shall write $\Gamma(A)$ for the Shilov boundary of A, i.e. for the smallest closed subset of $M(A)$ such that

$$\|x\|_s = \sup_{\Gamma(A)} |x^\wedge(f)|$$

for every $x \in A$. A theorem of Shilov states that any functional $f \in \Gamma(A)$ is extentable to an element of $M(A_1)$ for any superalgebra A_1 of A (having A as its closed subalgebra). A functional $f_0 \in M(A)$ is in $\Gamma(A)$ if and only if for any neighbourhood U of f_0 in $M(A)$ there exists an element $x \in A$ such that

$$\sup_{f \in U} |x^\wedge(f)| > \sup_{f \in M(A) \setminus U} |x^\wedge(f)|.$$

If A_0 is a subalgebra of A and $x \in A_0 \subset A$, then we write $\sigma(x)$ for the spectrum of x in A and $\sigma_0(x)$ for the spectrum of x in A_0. We have

(2) $$\sigma(x) \subset \sigma_0(x),$$

while

(3) $$\operatorname{bdry} \sigma_0(x) \subset \operatorname{bdry} \sigma(x),$$

so that if $\sigma(x)$ is a totally disconnected subset of the complex plane, then for any subalgebra A_0 containing the element x we have

(4) $$\sigma_0(x) = \sigma(x).$$

If A is an algebra without unit, then it may be imbedded in an algebra A' with unit element, as a maximal ideal. In this case the spectrum of any element x in A is defined as the spectrum of x in A_1.

The subalgebra A_0 generated by an element $x_0 \in A$ is defined as the closure in A of all the polynomials in x_0 with complex coefficients. The space $M(A_0)$ may be identified with the spectrum $\sigma_0(x)$.

If $M(A) = M_1 \cup M_2$, where M_1 and M_2 are open-closed disjoint subsets of $M(A)$, then there exists an idempotent $u \in A$ $(u^2 = u)$ such that

$$M_1 = \{f \in M(A) \colon u^\wedge(f) = 1\} \quad \text{and} \quad M_2 = \{f \in M(A) \colon u^\wedge(f) = 0\}.$$

For any $x \in A$ the element $\exp x$ is defined as

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!};$$

it is a well defined element in A and $(\exp x)^\wedge = \exp x^\wedge$.

The proofs of all these statements may be found e.g. in [4].

Lemma 1. *Let A be a commutative complex Banach algebra with unit element e. If for an element $x_0 \in A$ the spectrum $\sigma(x_0)$ contains a continuum, then there is a subalgebra $A_0 \subset A$ and a functional $f \in M(A_0)$ which is non-extentable to a member of $M(A)$.*

Proof. Suppose that there is a continuum K joining two points $t_1, t_2 \in \sigma(x_0)$, $t_1 \neq t_2$. We put

$$y = \frac{x_0 - t_1 e}{t_2 - t_1} \quad \text{and} \quad z = \exp \frac{\pi i}{2} y.$$

So z is an invertible element in A and there is a continuum $K_1 \subset \sigma(z)$ joining 1 and i. This implies that z^4 is an invertible element in A such that its spectrum $\sigma(z^4)$ separates the complex plane between 0 and ∞. Let A_0 be the subalgebra of A generated by z^4. We shall show that there is a functional f in $M(A_0)$ that cannot be extented to a multiplicative linear functional defined on A. To this end, for any complex polynomial p we put

$$f[p(z^4)] = p(0).$$

By the maximum principle and relations (1), (2) and (3) we have

$$|f[p(z^4)]| \leqslant \sup_{t \in \sigma_0(z^4)} |p(t)| = \sup_{t \in \operatorname{bdry} \sigma_0(z^4)} |p(t)| \leqslant \sup_{t \in \sigma(z^4)} |p(t)|$$
$$= \sup \left|\sigma\big(p(z^4)\big)\right| = \|p(z^4)\|_s \leqslant \|p(z^4)\|,$$

and so f may be extented by continuity onto the whole of A_0. Clearly, this extension belongs to $M(A_0)$. On the other hand, f cannot be extented to an element of $M(A)$ since $f(z^4) = 0$ and z^4 is an invertible element in A, q. e. d.

The following lemma is known (cf. [2], §42, II, 9):

Lemma 2. *A compact plane set is totally disconnected if and only if it contains no continuum.*

Lemma 3. *A commutative complex Banach algebra with unit element belongs to the class ES if and only if for every for subalgebra $A_0 \subset A$ its maximal ideal space $M(A_0)$ is totally disconnected.*

Proof. If for some subalgebra $A_0 \subset A$ the space $M(A_0)$ is not totally disconnected, then there are two points $f_1, f_2 \in M(A_0)$, $f_1 \neq f_2$, that cannot be separated by two disjoint closed-open subsets of $M(A_0)$. Take any element $x_0 \in A_0$ such that $x_0^\wedge(f_1) \neq x_0^\wedge(f_2)$. This implies that in the spectrum $\sigma_0(x_0)$ the points $x_0^\wedge(f_1)$ and $x_0^\wedge(f_2)$ cannot be separated by its two disjoint open-closed subsets, and so the spectrum $\sigma_0(x_0)$ is non-totally disconnected. By lemma 1 there is a subalgebra $A_1 \subset A_0$ and a functional $f \in M(A_1)$ that cannot be extented to a member of $M(A_0)$. This implies that f cannot be also extented to a member of $M(A)$ and so A is not in ES.

On the other hand, suppose that tor every subalgebra $A_0 \subset A$ the space $M(A_0)$ is totally disconnected. This means that for any functional

$f_0 \in M(A_0)$ there is a basis of neighbourhoods consisting of open-closed sets. The characteristic function of any such a neighbourhood is a Gelfand transform of some element of A_0. This implies that f_0 is in the Shilov boundary $\Gamma(A_0)$, and so f_0 may be extented to a member of $M(A)$. Since A_0 was an arbitrary subalgebra of A and f_0 an arbitrary element of $M(A_0)$, we infer that $A \in \mathrm{ES}$, q. e. d.

PROPOSITION 1. *A commutative complex Banach algebra A with unit element e belongs to the class ES if and only if for every $x \in A$ the spectrum $\sigma(x)$ is a totally disconnected subset of the complex plane.*

Proof. If for some $x_0 \in A$ the spectrum $\sigma(x_0)$ is non-totally disconnected, then taking as A_0 the subalgebra generated by x_0, we see by (2) that $\sigma_0(x_0)$ is non-totally disconnected. Since $M(A_0) = \sigma_0(x_0)$, we infer by lemma 3 that $A \notin \mathrm{ES}$.

On the other hand, suppose that $A \notin \mathrm{ES}$. By lemma 3 there is a subalgebra A_0 with non-totally disconnected maximal ideal space $M(A_0)$. Similarly as in proof of lemma 3, we obtain an element $x_0 \in A_0$ such that the spectrum $\sigma_0(x_0)$ is not totally disconnected. This implies that $\sigma(x_0)$ is also non-totally disconnected, otherwise, by formula (4) it would be $\sigma_0(x_0) = \sigma(x_0)$, q. e. d.

We have so obtained theorem 1 in the case of an algebra with unit. However, if A has no unit element, it may be imbedded as a subspace of codimension 1 in an algebra A_1 with unit element. Since the spectra of elements in A and in A_1 are the same and since any multiplicative linear functional defined on A may be uniquely extented to such a functional on A_1 by setting $f(e) = 1$ (and the same for any subalgebra $A_0 \subset A$), this implies that proposition 1 implies theorem 1.

COROLLARY. *If $C(X) \in ES$, and if $M(A) = X$, then $A \in ES$.*

Such a situation as above holds if X is a one-point compactification of a discrete space (or, more generally, a scattered space; cf. [1]). So applying the corollary to group algebras for compact groups we obtain

THEOREM 2. *If G is a compact abelian group, then $L_1(G)$ with convolution multiplication is a member of ES.*

Or equivalently

THEOREM 2'. *If G is an abelian compact group and if A is a subalgebra of the group algebra $L_1(G)$, then any multiplicative linear functional f defined on A is of the form*

$$f(x) = \int x(t)\chi(t)\,dt,$$

where $\chi(t)$ is a continuous character on G (which must not be determined uniquely by the functional f), and the integral is taken with respect to the Haar measure over the whole of G.

Suggested by the corollary and the results in [1], the author supposed that it is possible to obtain some stronger results than in Theorem 1, namely that

1° A commutative Banach algebra belongs to the class ES if and only if the spectrum of every its element is at most denumerable.

2° The fact that a Banach algebra A belongs to the class ES depends only upon its maximal ideal space, and $A \in \mathrm{ES}$ if and only if every continuous function (zero at infinity in the case of an algebra without unit) has at most denumerable range (i.e. if and only if the maximal ideal space of A (if A has a unit element) or one point compactification of this space (if A has no unit) is a scattered space; cf. [1] and [5]).

These conjectures, however, are disproved by J.-P. Kahane, who in a letter to the author proposed the following counterexample: Let Λ_a be the Banach algebra of all functions defined on the Cantor set E, which satisfy the Lipschitz condition with exponent a. Since for every $x \in \Lambda_a$ we have $\dim x(E) < (a-1)\dim E$ (Hausdorff dimension), this implies that $\Lambda_a \in \mathrm{ES}$. On the other hand, if $x(t) = t$, then $x \in \Lambda_a$, and $\sigma(x) = x(E)$ is non-denumerable, what disproves 1°, and E is not a scattered space, what disproves 2° (E is clearly the maximal ideal space for Λ_a).

Theorem 1, as well as the corollary, remains true if we replace the class of Banach algebras by more general class of locally bounded algebras (cf. [6]).

Added in proof. In paper [7] we have extended the concept of an ES-algebra onto non-commutative Banach algebras and in [8] we have generalized the results of this paper and paper [7] onto multiplicatively convex B_0-algebras.

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