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From triangular matrices to separated inductive limits

by

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Professor Mazur's prize-winning 1929 article, [5], contains some of the earliest and most successful applications of three fundamental tools of functional analysis, the closed graph and Hahn-Banach theorems and the principle of uniform boundedness. Subsequent developments in summability due to Mazur and Orlicz provided much of the impetus towards extending the Banach space theory to cover Fréchet spaces. In this article we shall show a thread of development which has its origin in the 1929 Mazur article. We shall also point out how the use of inductive limits simplifies, unifies, and generalizes some of the theory of interpolated and embedded spaces and of lattices of topologies found in articles by Steiner [11], [12], Schäffer [10], and Aronszajn and Gagliardo [1]. Also we recall the known fact that infima and quotients of linear topologies are special cases of inductive limits, which we shall use to set up their metrizability conditions. (Remarks 1, 2 following Theorem 17.)

If A is a *triangle* (a summation matrix with $a_{nk} = 0$ when $k > n$; $a_{nn} \neq 0$) the *convergence domain* c_A of A (set of sequences mapped by A into c, the convergent sequences) is made into a Banach space by the fact that $A: c_A \to c$ is an isomorphism onto. But the crucial fact about c_A, for Mazur's purpose, was this: if B is a *stronger* matrix ($c_B \supset c_A$), then $\lim_B$ is a continuous function on c_A ($\lim_B x = \lim_n \sum_k b_{nk} x_k$.) To prove this, it is sufficient to show that each x_k is continuous in x on c_A, for a standard argument about the pointwise limit of a sequence of continuous functions yields the continuity of $\lim_B$; see [14], Section 7.6, Theorem 3. The continuity of each x_k follows by the way that c_A is topologized; namely, for $x \in c_A$, $\|x\| = \|y\|_c = \sup |y_n|$ where $y = Ax \in c$. With $Z = A^{-1}$, we have

$$|x_k| = \left|\sum z_{ki} y_i\right| \leqslant \left(\sum |z_{ki}|\right) \cdot \|y\|.$$

The logical next step in this development, taken independently by Mazur-Orlicz [6], (these results were announced more than 20 years earlier!), and by Zeller [17], was to extend these ideas to non-triangular matrices by attempting to give c_A a topology making each x_k continuous in x, i.e. making coordinates continuous; and also such that the applications of functional analysis could be made, for example, to deduce the continuity of $\lim_B$ as above. Their outstanding achievement was to do this for an arbitrary matrix! An FK-space is a Fréchet space of sequences such that coordinates are continuous; every c_A can be made into an FK-space. See [14], Section 12.4, Theorem 2.

A third refinement of these ideas was given by Wilansky-Zeller in 1959, [16]. Let X be a sequence space; thus $X \subset s$, where s is the space of all sequences with its natural topology ([14], Section 4.1, Example 5). A topology for X makes coordinates continuous if and only if it is larger (stronger, finer) than the relative topology of s; this is true because s has the smallest topology which makes coordinates continuous. Thus, an FK-space is a linear subspace of s endowed with a Fréchet topology larger than that of s; in other words, it is a Fréchet space which is *continuously embedded* in s, in the sense that the inclusion map $i:X \to s$, given by $i(x) = x$, is continuous. The role of s in the theory of FK-spaces is two-fold. First, it gives a convenient way of specifying continuity of coordinates, as just mentioned. Secondly, it is a Hausdorff space, and so makes applications of the closed graph theorem possible (see the proof of Theorem 1, below, for details). The refinement consists of replacing s by an arbitrary Hausdorff space H. For convenience we shall take H to be a linear space with a Hausdorff topology; no assumption is made concerning continuity of the linear operations. An FH-space is a Fréchet space which is a linear subspace of H and is continuously embedded in H. Thus an FK-space is an FH-space with $H = s$. The following well-known theorem illustrates the theory of FH-spaces.

THEOREM 1. *Let H be given and let X, Y be FH-spaces with $X \subset Y$. Then the topology of X is larger than that of Y. In particular, we have the Uniqueness Theorem: A linear subspace of H has at most one FH-topology.*

Proof. The inclusion map $i:X \to Y$ is H-continuous, hence has closed graph since the H-topology is Hausdorff. Thus i has closed graph when X, Y have their own topologies since these topologies are larger than that of H (have more closed sets). By the closed graph theorem, i is continuous.

Just as the Mazur-Orlicz-Zeller replacement of Banach space by Fréchet space in the treatment of c_A extended the theory to cover all matrices, not just triangles; so the extension to FH-spaces allows non-matrix investigations, such, for example, as Abel summability, and integral transforms of function spaces. We shall not pursue these applications, but refer to [10], [3], [1], and [14], Sections 11.3 and 12.4. This small sample will suffice to indicate that the concept of continuously embedded space arises in many diverse branches of mathematics.

The following terminology will be useful: H is some (fixed) linear space which also has a Hausdorff topology (no a priori assumptions are made about continuity of the linear operations); an FH-space is a Fréchet space continuously embedded in H, a BH-space is a normed FH-space, a PH-space is a linear metric space (not necessarily complete) continuously embedded in H (called a *paranormed* space in [14]) and an NH-space is a normed space continuously embedded in H. A trivial example would be one in which H has finite or countably infinite dimension; every FH-space would then be finite dimensional.

It is possible that an NH-space may have no BH-completion; indeed (to avoid trivialities) an example can be given when H is an arbitrary infinite-dimensional Banach space: merely let X be the same as H but with a larger non-complete norm. More generally:

THEOREM 2. *Let H be a Fréchet space and X any infinite-dimensional linear subspace. Then X can be given a PH-topology with no FH-completion.*

Let p be the paranorm of H, ($p(x)$ is the distance from x to 0) and let f be a discontinuous linear functional on X. Let $q(x) = p(x)+|f(x)|$. Then $q \geqslant p$ so that (X, q) is a PH-space. Since f is not continuous, there exists a sequence $\{x_n\}$ in X with $f(x_n) = 1$, $p(x_n) \to 0$. Then $\{x_n\}$ is a q-Cauchy sequence; if (X, q) had an FH-completion Y we would have $x_n \to y$ in Y, hence $x_n \to y$ in H. But $x_n \to 0$ in H hence $y = 0$. This contradicts $q(x_n) \to 1$.

Good discussions of the completions of NH and BH-spaces may be found in [10], Section 2.2; [1] Section 3, [15], Section 5.3.

There are some spaces which cannot be given certain FH-topologies. An obvious example is a space of countably infinite dimension which, since it must be of first category in itself, cannot be completely metrized. Some less obvious examples will follow from the next result.

THEOREM 3. *Let X, Y be complementary linear subspaces of an FH-space Z. Then if X, Y can be given FH-topologies, they must be closed subspaces of Z.*

Let p, q be the paranorms which can be placed on X, Y. Then $Z = X + Y$ is an FH-space with $p+q$. By Theorem 1, this must be its original topology.

EXAMPLE 4. In c_0, the subspace l has an algebraic complement S. This space S cannot be made into an FK-space, since, by Theorem 3, this would force l to be a closed subspace of c_0.

EXAMPLE 5. In m, the subspace c has an algebraic complement S. This space S cannot be made into an FK-space, since, by Theorem 3, this would force S to be a closed subspace of m, and thus there would be a continuous projection of m onto c. This contradicts a result of R. S. Phillips (see [13]).

The problem arises whether a collection (X_a) of given linear metric spaces may, by suitable definition of H, be made into PH or FH-spaces. These may be partially overlapping spaces, the assumption being that where they overlap, the linear operations agree. To fix the ideas, we shall suppose all spaces are linear subspaces of a fixed linear space E (there is actually no loss of generality in this assumption, as explained in [1], Section 1, and [15], Section 2.6, Remark). Let H be the linear span of $\cup X_a$. There is an obvious candidate topology for H which will make each X_a continuously embedded in H; this is the inductive limit topology. We shall give a brief sketch of two forms of inductive limit, and then return to the question of whether we have succeeded in making the X_a PH-spaces, the answer being yes precisely if the inductive limit topology for H is separated (Hausdorff).

Let (X_a) be a family of linear topological spaces, E a linear space, and, for each a, $u^a: X_a \to E$ a linear map. Let H be the linear span of $\bigcup u^a(X_a)$. Then T_i, the *inductive limit topology* is the least upper bound of all locally convex topologies for H, each of which makes every u^a continuous, and T_u the *unrestricted inductive limit topology* is the least upper bound of all linear topologies each of which makes every u^a continuous. Discussions of T_i may be found in several texts, for example [2], Section 6.3; [9], Chapter 5; a discussion of T_u appears in [15], Chapter 2. For a finite number of locally convex spaces, T_u is locally convex, hence $T_u = T_i$. A special case is the *direct sum* $\sum X_a$. By definition, $\sum X_a$ is the set of those points in ΠX_a with only finitely many non-zero coordinates, D_i, D_u are the inductive and unrestricted inductive limit topologies on $\sum X_a$ for the spaces X_a and the injection maps $j^\beta: X_\beta \to \Sigma X_a$ for the spaces X_a and the injection maps $j^\beta: X_\beta \to \sum X_a$ ($j^\beta(x_\beta)$ is a point in the direct sum with exactly one non-zero coordinate, the β-th coordinate being x_β). We shall use the letter x to denote a member of $\sum X_a$ and for each β, x_β is the β-th coordinate of x.

THEOREM 6. *Let* $h: \sum X_a \to H$ *be defined by* $h(x) = \sum u^a(x_a)$. *Then* T_i *is the quotient topology of* D_i *by* h. *If there are only finitely many* X_a, T_u *is the quotient topology of* D_u *by* h.

The first half is standard; see [9], p. 94, Proposition 28. Next assume finitely many spaces $X_1, X_2, \ldots, X_n$. Since H is spanned by $\bigcup u^i(X_i)$, h is onto. Also h is continuous since it is a finite sum of continuous functions (each $u^\beta(x_\beta) = u^\beta \circ \pi^\beta(x)$, where π^β is the projection of $\sum X_i = \Pi X_i$ onto X_β). Finally, h is open, for let $V = \sum j^i[V_i]$ be a basic neighborhood of 0 in $\sum X_i = \Pi X_i$, where each V_i is a balanced neighborhood of 0 in X_i. Then $h[V] = \sum u^i(\pi^i[V]) = \sum u^i[W_i]$, say; each W_i being a balanced neighborhood of 0 in X_i. It remains to show that the latter set is a neighborhood of 0 in H. Let T be the linear topology for H generated by all sets of the form $\sum u^i[W_i]$, each W_i being a balanced neighborhood of 0 in X_i. (That T exists is easy to prove, using, for example, [14], Section 10.1, Theorem 3.) Each $u^a: X_i \to (H, T)$ is continuous since $(u^a)^{-1} \sum u^i[W_i] \supset W_a$ and so, by definition of D_u, we have $D_u \supset T$.

THEOREM 7. *The inductive limit topology of a family* (X_a) *of locally convex spaces is separated (Hausdorff) if and only if* $h^\perp$ *is a closed subspace of* $\sum X_a$, *where* h *is defined in Theorem* 6. *The same result is true for the unrestricted inductive limit of finitely many spaces without the assumption of local convexity.*

This is immediate from Theorem 6.

We can now solve the problem of making a collection of spaces into PH-spaces.

THEOREM 8. *Let* $\{X_a\}$ *be a family of locally convex linear metric spaces, each of which is a linear subspace of a linear space* E. *Then a space* H *may be defined making each* X_a *a* PH-*space if and only if* $\{x: \sum x_a = 0\}$ *is a closed subspace of* $\sum X_a$. *If the family is finite, we may drop the assumption of local convexity and express the condition as:* $\{x: x_1 + x_2 + \ldots + x_n = 0\}$ *is a closed subspace of* $\prod_{i=1}^{n} X_i$.

This is immediate from Theorem 7, taking the maps u^a to be inclusion.

The next 3 examples and 3 theorems deal with the intersection of topologies as a special case of inductive limit.

EXAMPLE 9. Let X be a linear space and let (S_a) be a family of linear topologies for X. Let $H = X$ and let each u^a be the identity map from X to itself. Then the inductive limit topology T_i is the least upper bound of all locally convex topologies smaller than each S_a; in other words, T_i is the locally convex infimum of the S_a. Similarly, T_u is the linear infimum of the S_a. Theorem 7 yields a criterion for the separation of these topologies.

EXAMPLE 10. Let X be a linear space and let S_1, S_2 be linear topologies for X. Let $S_1 \wedge S_2$ denote the linear infimum (see Example 11). Then Example 9 and Theorem 7 yield the following result: $S_1 \wedge S_2$ *is separated if and only if* $\{(x_1, x_2): x_1 + x_2 = 0\}$ *is a closed subset of* $(X, S_1) \oplus (X, S_2)$. This in turn leads immediately to the result: $S_1 \wedge S_2$ *is separated if and only if the identity map from* (X, S_1) *to* (X, S_2) *has closed graph*. This result leads us to the insight that the concept of closed graph should be considered with respect to the direct sum rather than the product when it is desired to extend the notion to more than two spaces; Theorem 7 is the natural generalization of the result of Example 10. Another proof of the result of this example can be based, for locally convex spaces, on a Lemma of V. Ptak, namely [7], Theorem 3.7 (this is also Section 1.1, Theorem 1.5 of [15]), for this implies the existence on X of a total family F of linear functionals which are both S_1- and S_2-continuous. The weak topology by F is separated and is smaller than both S_1 and S_2. It should also be noted that the condition stated in this example is equivalent to the requirement, when S_1, S_2 are metrizable, that (X, S_1), (X, S_2) can be made into PH-spaces as in Theorem 8.

EXAMPLE 11. It is possible that $S_1 \wedge S_2 \neq S_1 \frown S_2$, in other words, that the infimum of two linear topologies not be linear. Phrased in another way, this says that the lattice of linear topologies is not a sublattice of the lattice of all topologies. Let p, q be non-comparable complete norms for a linear space X ([14], Section 7.5, Example 6). Then $p \frown q$ is a T_1-topology, being the intersection of two T_1-topologies; but is not Hausdorff, for, as in Theorem 1, this would imply that the identity map has closed graph, hence is continuous. Thus $p \frown q$ is not a linear topology since it is T_1 but not Hausdorff. The lattice of linear topologies is discussed in [11], [12], where it is also proved constructively that $p \frown q$ is not linear if p, q are any non-comparable norms. Our example also shows two group topologies for a commutative group whose infimum is not a group topology; this is so since addition cannot be continuous in $S_1 \frown S_2$, a T_1-topological group being always Hausdorff. It is easy to check that the linear operations are separately continuous.

THEOREM 12. *Let S_1, S_2 be linear topologies for a linear space X. The following conditions are equivalent:*

(i) *The identity map from (X, S_1) to (X, S_2) has closed graph,*

(ii) *$S_1 \wedge S_2$ is separated,*

(iii) *$S_1 \frown S_2$ is Hausdorff.*

That (i) implies (ii) is given in Example 10. The rest is trivial.

Condition (iii) of Theorem 12 always implies (i); however, there exist examples of a set X, with topologies S_1, S_2, satisfying (i) but not (iii). If S_1, S_2 are first countable, then again (i) and (iii) are equivalent ([8], Theorem 6).

THEOREM 13. *Let S_1, S_2 be linear topologies for a linear space X. Then $S_1 \frown S_2$ is linear if and only if for every $S_1 \frown S_2$ neighborhood U of 0, there exist S_1, S_2 neighborhoods of 0, V, W respectively, with $V + W \subset U$.*

It is easy to check [14], Section 10.1, Theorem 1, using the fact that if V, W are S_1, S_2 open, respectively, then $V+W$ is both S_1 and S_2 open, hence is $S_1 \frown S_2$ open (since $V+W = \cup\{V+w : w \in W\}$, a union of translates of S_1 open sets).

THEOREM 14. *Let S_1, S_2 be locally convex linear topologies for a linear space X. Then $S_1 \frown S_2$ is linear if and only if it is locally convex.*

If $S_1 \frown S_2$ is linear it is an unrestricted inductive limit — as mentioned above, for finitely many spaces this is the same as the inductive limit, hence locally convex.

Conversely, if $S_1 \frown S_2$ has a local base at 0 of convex neighborhoods, the criterion of Theorem 13 applies, for if U is a convex $S_1 \frown S_2$ neighborhood of 0, we may take $V = W = \frac{1}{2}U$.

We return briefly to the question of when an inductive limit is separated. Clearly, if E has a separated linear topology T such that every $u^a: X_a \to (E, T)$ is continuous, it follows that T_u is separated, indeed $T_u \supset T$. It is a little less obvious that it is sufficient that T merely make addition continuous, as well as each u^a, to imply that $T_u \supset T$, hence, if T is Hausdorff, that T_u is separated. This is true because the map h given in Theorem 6 is continuous as a map to (E, T), and T_u is the quotient topology by h. We now attempt to relax the conditions on T.

THEOREM 15. *Let X, Y be linear topological spaces, E a linear space, and $u: X \to E$, $v: Y \to E$, linear maps. Suppose that E has a Hausdorff topology T making u, v continuous. Then T_u is separated.*

Applying Theorem 7 we shall show that $h^\perp$ is closed in $X \times Y$, where $h(x, y) = u(x) + v(y)$. Let (x_δ, y_δ) be a net in $h^\perp$ converging to (x, y). Then $-y_\delta \to -y$, hence $u(x_\delta) = -v(y_\delta) = v(-y_\delta) \to v(-y) = -v(y)$, also $u(x_\delta) \to u(x)$. Hence $u(x) = -v(y)$ and so $(x, y) \in h^\perp$.

Theorem 15 is false for 3 spaces as the next example shows.

EXAMPLE 16. Let L be a linear space and S_1, S_2 separated linear topologies for L such that $S_1 \frown S_2$ is not Hausdorff (see Example 11). Let $X = Y = (L, S_1)$, $Z = (L, S_2)$, $E = L \times L$, $u(x) = (x, 0)$, $v(y) = (0, y)$, $w(z) = (z, z)$. Let T be the topology for E which on $L \times \{0\}$ and $\{0\} \times L$ makes u, v homeomorphisms, on the diagonal $D = \{(t, t): t \in L\}$, makes w a homeomorphism, and is discrete elsewhere. Then T is Hausdorff and u, v, w are continuous maps to (E, T). We next note that $w: X \to (E, T_u)$ is continuous, for if x_δ is a net converging to 0 in X, then $x_\delta \to 0$ in Y (since $X = Y$) and so $w(x_\delta) = u(x_\delta) + v(x_\delta) \to 0$ in T_u. Now if T_u were separated, $w^{-1}[T_u]$ would also be separated, since w is one-to-one. But this is impossible since this latter topology is smaller than both S_1 and S_2.

A slightly different assumption will extend Theorem 15 to 3 spaces.

THEOREM 17. *Let X, Y, Z be linear topological spaces, E a linear space, and u, v, w linear maps from X, Y, Z to E, respectively. Suppose that E has a topology T such that addition is an operation with closed graph, and u, v, w are continuous. Then T_u is separated.*

We apply the criterion of Theorem 7. Let $(x_\delta, y_\delta, z_\delta)$ be a net in $h^\perp$ converging to (x, y, z). Then $u(x_\delta) + v(y_\delta) = -w(z_\delta) = w(-z_\delta) \to w(-z) = -w(z)$, also $[u(x_\delta), v(y_\delta)] \to [u(x), v(y)]$ hence $-w(z) = u(x) + v(y)$ and so $(x, y, z) \in h^\perp$.

Notice that there was no need to assume that T is separated, indeed we cannot conclude that $T_u \supset T$, as the next example shows.

EXAMPLE 18. Let E be the plane, and let T be the topology for E which is the Euclidean topology on the X and Y axes and is discrete elsewhere. Then addition is an operation with closed graph; one way to see this easily is that E has a smaller Hausdorff topology (the Euclidean topology) making addition continuous (Compare [14], Section 11.1, Lemma 1). Now if we let X and Y be the axes, and u, v the inclusion maps, we see that T_u is the Euclidean topology, which is separated, as predicted by Theorem 17, but is not larger than T.

THEOREM 19. *The unrestricted inductive limit T_u of a finite number of paranormed spaces $\{(X_i, p_i): i = 1, 2, \ldots, n\}$ is paranormed. If each p_i is locally convex or a seminorm, so also is T_u, and $T_u = T_i$. If each p_i is complete so is T_u.*

Recall the maps $u^i: X_i \to E$ and the definition of H given just before Theorem 6, above. For $h \in H$, define

$$q(h) = \inf\Big\{\sum_{i=1}^{n} p_i(x_i): x_i \in X_i \text{ for } i = 1, 2, \ldots n;\ h = \sum_{i=1}^{n} u^i(x_i)\Big\}.$$

We now show that the formula $d(h, k) = q(h-k)$ yields a semimetric d which induces T_u. Clearly $q(0) = 0$, $q(h) \geqslant 0$, $q(-h) = q(h)$. To prove the triangle inequality, let $h, k \in H$, $\varepsilon > 0$. Choose $x_i, y_i \in X_i$ with $h = \sum u^i(x_i)$, $\sum p^i(x_i) < q(h)+$, $k = \sum u^i(y_i)$, $\sum p^i(y_i) < q(k)+\varepsilon$. Then

$$q(h+k) \leqslant \sum p_i(x_i+y_i) \leqslant \sum p_i(x_i) + \sum p_i(y_i) < q(h)+q(k)+2\varepsilon.$$

This completes the proof that d is a semimetric for H; by its form d is invariant under translation hence provides H with a topology making addition continuous. Furthermore, each $u^i: X_i \to (H, q)$ is continuous since for any $x_i \in X_i$,

$$u^i(x_i) = u^1(0)+\ldots+u^{i-1}(0)+u^i(x_i)+u^{i+1}(0)+\ldots+u^n(0),$$

so that $q[u^i(x_i)] \leqslant p_i(x_i)$. Now we shall see that $q = T_u$ (i.e. that T_u is the topology induced by q *via* the semimetric d). First let V be a q-neighborhood of 0. Since f makes addition continuous, there is a balanced q-neighborhood W of 0 with $W+W+\ldots+W \subset V$ (there are n terms in the sum). Since each u^i is q-continuous, $(u^i)^{-1}[W]$ is a balanced neighborhood of 0 in X_i for each i. Now $V \supset \sum u^i\{(u^i)^{-1}[W]\}$ and, exactly as in the proof of Theorem 6, the latter set is seen to be a T_u-neighborhood of 0. This proves that T_u is larger than q. Conversely, it suffices to consider sequences since (H, q) is first countable. Let $q(h^j) \to 0$. For each j and each $i = 1, 2, \ldots, n$ choose x_i^j with $h^j = \sum u^i(x_i^j)$ and $\sum p_i(x_i^j) < q(h^j)+1/j$. Then $x_i^j \to 0$ in X_i for each i, and so $h^j \to 0$ in (H, T_u) since each u^i is T_u-continuous. Thus q is larger than T_u. It follows that q is a paranorm, the only missing property being continuity of multiplication; but this follows from $q = T_u$ and the fact that T_u is a linear topology.

If each p_i is locally convex, $T_u = T_i$ as mentioned earlier. If each p_i is a seminorm, it is clear that for scalar $t, h \in H$ we have $q(th) = |t| q(h)$ so that q is a seminorm. Finally, assume that each p_i is complete, and let $\{h^j\}$ be a Cauchy sequence in H. We may assume that $\sum q(h^j - h^{j-1}) < \infty$, $h^0 = 0$. For each j and $i = 1, 2, \ldots, n$ choose $x_i^j \in X_i$ with $\sum u^i\ (x_i^j) = h - h^{j-1}$,

$$\sum p_i(x_i^j) < q(h^j - h^{j-1}) + \frac{1}{j^2}.$$

For each i, let $y_i = \sum_{j=1}^{\infty} x_i^j$. Then

$$h^j = \sum_{r=1}^{j} \sum_{i=1}^{n} u^i(x_i^r) \to \sum_{i=1}^{n} u^i(y_i).$$

REMARKS. 1. A quotient is a special case of the inductive limit in which there is one space and one map. Theorem 19 gives the usual metrization and completeness theorems. This shows also that without metric assumptions the completeness part of Theorem 19 is false, even for $n = 1$, since G. Köthe has given an example of a complete space with an incomplete quotient (see [4], p. 195 # D, and, for a very elegant example, [7], p. 44).

2. Example 9 and Theorem 19 also give metrization and completeness theorems for the infimum of finitely many linear topologies on a linear space.

3. A more complicated proof of Theorem 19 was given in [15], Section 2.7. There continuity of multiplication for q was proved in detail. Here we managed to prove $q = T_u$ using only continuity of addition.

4. Theorem 19 generalizes [12], Theorem 2, which states that if S_1, S_2 are normed, $S_1 \wedge S_2$ is normed if and only if it is separated. Indeed, Theorem 19 implies that $S_1 \wedge S_2$ is always seminormed.

5. Theorem 5.12 (b), p. 67 of [1] can be taken as a special case of an inductive limit. The interested reader will notice that its proof inspired our proof of completeness in Theorem 19.

We now specialize Theorem 19 to obtain a known result. Let H be a linear space with a topology T making addition continuous, and suppose that $X_1, X_2, \ldots, X_n$ are FH-spaces. There is no loss of generality in assuming that $H = X_1+X_2+\ldots+X_n$. Apply Theorem 19 taking each u^i to be the inclusion map. This gives $X_1+X_2+\ldots+X_n$ a Fréchet topology F; we shall now prove that F is larger than T. The map h defined in Theorem 6 takes the form $h(x_1, x_2, \ldots, x_n) = \sum x_i$. Thus $h: \prod X_i \to (H, T)$ is continuous. But Theorem 6 says that F is the quotient topology by h, hence $F \supset T$. Thus we have:

THEOREM 20. *Let H be a linear space with a topology making addition continuous. Let $X_1, X_2, \ldots, X_n$ be PH (NH, FH or BH) spaces. Then $X_1+X_2+\ldots+X_n$ can be made into a PH (NH, FH or BH) space by the formula*

$$g(z) = \inf\left\{\sum p_i(x_i): z = \sum x_i\right\},$$

where p_i is the paranorm of X_i.

EXAMPLE 21. The most interesting special case is that of two FK-spaces X, Y; then $X+Y$ is an FK-space with the paranorm

$$r(z) = \inf\{p(x)+q(y): x \in X, y \in Y, z = x+y\},$$

p, q being the paranorms of X, Y. In case p, q are seminorms, r is the gauge of the sum of the two unit spheres.

Example 21 shows that, for each H, the collection of FH-spaces is a lattice; $X \vee Y$ is $X+Y$ with the topology T_u; $X \wedge Y$ is $X \cap Y$ with the sum of the paranorms of X, Y (see [14], Section 11.3, Theorem 3).

EXAMPLE 22. *Two non-equivalent norms for a linear space which agree on two complementary closed linear subspaces.* Let B be a Banach space and X, Y disjoint closed linear subspaces such that $X+Y$ is not closed. Let $H = X+Y$. Then X, Y are complementary subspaces of H. The norm n which H inherits from B, and the norm r given in Example 16 (in which $p = n/X$, $q = n/Y$,) are the two required norms. They are not equivalent since r is complete and n is not; they both coincide with n on X, Y; and X, Y are closed because complete.

We conclude this article with some remarks on the restriction of the topology T of H so that there shall be a meaningful FH-theory. Note first that Theorem 1 holds with the only restriction on T being that it is Hausdorff, as does the result that if X, Y are FH-spaces, $X \cap Y$ is an FH-space with the sum of the paranorms of X, Y. The Hausdorff separation of T is used crucially in the proof that $X \cap Y$ is complete (see [14], Section 11.3, Theorem 3). To see that it cannot be omitted note that in Example 11, $p+q$ cannot be complete, since, if it were, the closed graph theorem would make it equivalent to both p and q.

On the other hand, Theorem 20 does not require T to be Hausdorff, but instead includes a condition linking T to the linear operations. This condition cannot be omitted as is shown by Example 18, in which $X+Y$ has no FH-topology, its only possible topology being not larger than that of H.

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