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Critical modulars

by

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Let m be a modular on a linear space S, defined in [1]. An element $x \in S$ is said to be *homogeneous*, if $m(\xi x) = \xi m(x)$ for all $\xi > 0$. Every homogeneous element is finite by the definition. A modular m is said to be *homogeneous*, if every element of S is homogeneous. If m is convex and homogeneous, then m is a norm by the definition.

An element $x \in S$ is said to be *singular*, if $m(\xi x) = 0$ or $+\infty$ for all $\xi > 0$. A modular m is said to be *singular*, if every element of S is singular. It is obvious by the definition that an element $x \in S$ is homogeneous and singular simultaneously if and only if x is *null*, that is, $m(\xi x) = 0$ for all $\xi > 0$.

An element $x \in S$ is said to be *critical*, if x is homogeneous or singular. A modular m is said to be *critical*, if every element of S is critical. In this paper we consider structure and characterization of critical modulars.

If a modular m is homogeneous or singular, then m is critical by the definition. Conversely we have

STRUCTURE THEOREM. *If a modular m is critical, then m is homogeneous or singular.*

Proof. We suppose that m is critical and χ is a character of m. We have by the definition

$$(1) \qquad m(\xi(x+y)) \leqslant \chi m(\chi\xi x) + \chi m(\chi\xi y) \quad \text{for } x, y \in S \text{ and } \xi > 0.$$

If both x and y are homogeneous, then $x+y$ also is homogeneous, because $m(\xi(x+y)) < +\infty$ for all $\xi > 0$ by (1). Thus if $x+y$ is singular, then $m(\xi(x+y)) = 0$ for all $\xi > 0$, that is, $x+y$ is null. If both x and y are singular, then $x+y$ also is singular, because we can find $\xi > 0$ such that $m(\chi\xi x) = m(\chi\xi y) = 0$, and we have $m(\xi(x+y)) = 0$ for such $\xi > 0$ by (1). Thus if $x+y$ is homogeneous, then $m(\xi(x+y)) = 0$ for all $\xi > 0$, that is, $x+y$ is null.

If there is no homogeneous element which is not null, then m is singular by the definition. Now we suppose that there is a homogeneous element $x \in S$, which is not null. For any other element $y \in S$, if y is singular, then $x-y$ is homogeneous, because if $x-y$ is singular, then $x=(x-y)+y$ is singular, as proved above. Thus $y-x$ also is homogeneous, and $y=(y-x)+x$ is homogeneous, as proved above. Therefore m is homogeneous.

For a modular m we have defined in [1] the first and second modular norms N_1 and N_2.

CRITICAL THEOREM. *For a convex normal modular m we have $N_1(x) = N_2(x)$ if and only if x is critical.*

Proof. For a convex normal modular m, the associate of m is m itself by the Reflexivity Theorem in [1], and by (9) and (11) in [1] we have

$$N_1(x) = \inf_{\xi>0} \frac{1}{\xi}\big(1+m(\xi x)\big), \tag{2}$$

$$N_2(x) = \inf_{m(\xi x)\leqslant 1,\xi>0} \frac{1}{\xi} \tag{3}$$

for all $x \in S$. If x is homogeneous, then by (2) and (3) we have

$$N_1(x) = \inf_{\xi>0}\left(\frac{1}{\xi}+M(x)\right) = m(x),$$

$$N_2(x) = \inf_{m(\xi x)\leqslant 1,\xi>0} \frac{1}{\xi} = m(x).$$

If x is singular, then by (2) and (3)

$$N_1(x) = \inf_{m(\xi x)=0,\xi>0} \frac{1}{\xi} = N_2(x).$$

Conversely, we suppose $N_1(x) = N_2(x)$ and set $a = \sup_{m(\xi x)\leqslant 1} \xi$. Then $N_2(x) = 1/a$ by (3) and $m(ax)\leqslant 1$, as m is normal. If $m(ax)<1$, then $m(\xi x) = +\infty$ for $\xi > a$, because if there is $\beta > a$ such that $m(\beta x) < +\infty$, then $m(\xi x)$ is a continuous function on $0\leqslant \xi \leqslant \beta$ and $m(ax)=1$. Thus if $m(ax)=0$, then x is singular.

Now we suppose $m(ax)>0$. It is obvious that

$$\frac{1+m(\xi x)}{\xi} > \frac{1}{a} \quad \text{for } 0<\xi\leqslant a.$$

If $m(ax)<1$, then $m(\xi x) = +\infty$ for $\xi > a$. If $m(ax) = 1$, then

$$\frac{1+m(\xi x)}{\xi} > \frac{m(\xi x)}{\xi} \geqslant \frac{m(ax)}{a} = \frac{1}{a} \quad \text{for } \xi > a,$$

since $m(\xi x)$ is a convex function of $\xi>0$. Therefore

$$\frac{1+m(\xi x)}{\xi} > \frac{1}{a} = N_2(x) \quad \text{for all } \xi > 0.$$

Thus if $N_1(x) = N_2(x)$, then

$$\lim_{\xi\to\infty} \frac{m(\xi x)}{\xi} = \frac{1}{a} \quad \text{and} \quad m(ax) = 1.$$

Since

$$\frac{m(\xi x)}{\xi} \geqslant \frac{m(ax)}{a} = \frac{1}{a} \quad \text{for } \xi\geqslant a,$$

we obtain $m(\xi x)/\xi = 1/a$ for $\xi \geqslant a$, that is, $m(\xi x) = \xi/a$ for $\xi\geqslant a$. In addition, we have

$$\frac{1}{a} \leqslant \frac{m(ax)-m(\xi x)}{a-\xi} \leqslant \frac{m(\eta x)-m(ax)}{\eta - a} \quad \text{for } 0<\xi<a<\eta.$$

Since

$$\lim_{\eta\to\infty} \frac{m(\eta x)-m(ax)}{\eta-a} = \lim_{\eta\to\infty}\frac{m(\eta x)}{\eta} = \frac{1}{a},$$

we obtain

$$\frac{1-m(\xi x)}{a-\xi} = \frac{1}{a} \quad \text{for } 0<\xi\leqslant a.$$

Thus $m(\xi x) = \xi/a$ for all $\xi>0$, that is, x is homogeneous.

As an immediate consequence of the Critical Theorem we have

CHARACTERIZATION THEOREM. *A convex normal modular m is critical if and only if $N_1 = N_2$.*

This Characterization Theorem is proved in [2], when S is a linear lattice and m is convex, normal and additive:

$$m(x+y) = m(x)+m(y) \quad \text{for } x\wedge y = 0.$$

References

[1] H. Nakano, *Generalized modular spaces*, this fasc., p. 439-449.
[2] S. Yamamuro, *On linear modulars*, Proc. Japan Acad. 29 (1951), p. 623-624.

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