Consider the following two questions:

(A) Does there exist a "smallest" continuous metric on a given arc $A$, i.e., a metric $g$ such that, for any continuous metric $\sigma$ on $A$, there exist a homeomorphism $\varphi : A \to A$ and a number $a > 0$ such that $a\sigma(\varphi x, \varphi y) \geq g(x, y)$ for every $x, y \in A$? If not, does there exist a "minimal" metric with respect to the order just described?

(B) If $\varrho$ is a continuous metric on an arc $A$, does there exist a normed linear space $E$ and a distance-preserving mapping $f : (A, \varrho) \to E$ such that the arc $f(A)$ admits of a "coordinatization"? (We say that an arc $B \subset E$ admits of a "coordinatization" if there are a point $a \in E$ and a continuous mapping $f$ of $[0, 1]$ into a closed hyperplane $L \subset E$, $a \in L$, such that $B$ consists of all $ta + ft$, $0 \leq t \leq 1$.)

Both questions seem to be rather elementary. However, I have not found any answer in the literature. So the present note appears though the results may be already known.

In §1 some definitions and lemmas are given; §2 contains some auxiliary concepts and propositions. In §3 the main results are stated and proved.

1. The terminology and notation of [1] is used. Since it does not differ substantially from current terms and symbols, only two points of difference should be mentioned: an ordered pair $a, b$ is denoted by $\langle a, b \rangle$; the value of a mapping $f$ at an element $x$ is usually denoted simply by $fx$. As usual we often denote, e.g., a space and the set of its points by the same symbol. The letters $\mathcal{A}$ and $\mathcal{B}$ respectively stand for the set of all natural numbers $0, 1, 2, \ldots$ and the set of all reals.

1.2. Definition. If $X$ is a set, we denote by $\mathcal{M}(X)$ the set of all bounded pseudometrics on $Y$. If $X$ is a topological (or uniform) space, we denote by $\mathcal{M}(X)$ (or $\mathcal{M_0}(X)$) the set of all continuous (or uniformly continuous) bounded pseudometrics on $X$. 
1.3. Definition. Let \( X \) be a set. If \( \sigma \in \mathcal{M}(X) \), \( \sigma \in \mathcal{M}(X) \), we put
\[
d_\sigma(x, y) = \sup \{ |\sigma(x, y) - \sigma(x, y)| : y \in Y \}.
\]

Clearly, \( d \) is a metric on \( M(X) \), and \( \langle M(X), d \rangle \) is a complete metric space.

1.4. If \( X \) is a topological (or uniform) space, then \( M_t(X) \) or \( M_u(X) \) is closed in \( M(X) \).

1.5. Convention. Every subset of \( M(X) \) will be considered as a metric space (with the metric described in 1.3). If \( H \subset M(X) \), \( \overline{H} = H \) is meager in \( H \), and \( P \) is a property of elements \( \sigma \) of \( H \), then almost every \( \sigma \in H \) possesses the property \( P \) means that the set of all those \( \sigma \in H \) which do not possess the property \( P \) is meager in \( H \).

1.6. Proposition. Let \( X \) be a metrizable uniform space. Then almost every uniformly continuous bounded pseudometric on \( X \) is a metric on \( X \) inducing its given uniformity.

Proof. By \( M^* \) we denote the set of all bounded metrics inducing the uniformity of \( X \). It is easy to see that \( M^* \) is dense in \( M_u(X) \). Choose a metric \( \tau \) inducing the uniformity of \( X \). For \( n = 1, 2, \ldots \) denote by \( H_n \) the set of all uniformly continuous bounded pseudometrics \( \sigma \) on \( X \) such that,
\[
\text{for some } \epsilon > 0 \text{ depending on } \sigma, \quad \sigma(x, y) < \epsilon \text{ implies } \tau(x, y) < n^{-1}.
\]

It is easy to show that \( H_n \) are open in \( M_u(X) \) and \( \bigcap H_n = M^* \).

Corollary. Let \( X \) be a compact metrizable space. Then almost every continuous pseudometric on \( X \) is a metric inducing the topology of \( X \).

1.7. Proposition. Let \( X \) be a separable metrizable topological space. Then almost every bounded continuous pseudometric on \( X \) is a metric inducing the topology of \( X \).

Proof. Denote by \( M^* \) the set of all bounded metrics inducing the topology of \( X \). Let \( \{G_n \} \) be a countable open base of \( X \); let \( R \) consist of all \( \langle G_n, G_m \rangle, \overline{G_m} \subset G_m \). If \( b \in b, n = \langle G_m, G_n \rangle \), let \( T_b \) consist of all \( \sigma(x) \in M_t(X) \) such that \( \inf \{ |\sigma(x, y) - \sigma(x, y)| : y \in Y \} > 0 \). Clearly, each set \( T_b \) is dense and open. It is not difficult to show that \( M^* = \bigcap T_b \).

Remark. It is easy to see that \( M^* \neq \bigcap T_b \) in general.

2.

2.1. Conventions. A topological space homeomorphic to a compact non-degenerate interval of reals will be called an arc. If \( A \) is an arc, then every non-void connected \( T \subset A \) will be termed an interval of \( A \); sometimes, symbols \([a, b] \), etc., will be used to denote intervals of an arc. We shall say that two intervals \( T_1 \) and \( T_2 \) of an arc \( A \) overlap if \( T_1 \cap T_2 \) contains more than one point.

If \( A \) is an arc, \( T \subset A \) is an interval with endpoints \( a, b \), and \( \varphi \) is a pseudometric on \( A \), we put \( \varphi(T) = \varphi(a, b) \). The set \( \varphi \) and all of its subsets are considered (unless the contrary is stated or implied by the context) as spaces with the usual metric, say \( \mu \), defined by \( \mu(x, y) = |x - y| \); instead of \( \mu(T) \) we write \( |T| \).

Clearly, every continuous metric on an arc \( A \) induces the given topology of \( A \). If \( A \) is an arc and \( \varphi \) is a continuous metric on \( A \), we shall say that \( \langle A, \varphi \rangle \) is a metric arc.

2.2. Definition. If \( \tau \) is a continuous pseudometric on an arc \( A \) and \( T \subset A \) is an interval, then the least upper bound of numbers \( \Sigma \{ \sigma(T) \} \) where \( \{ T \} \) is a finite family of non-overlapping intervals, \( \bigcup T_i = T \), will be called the \( \tau \)-length of \( T \).

Clearly, if the \( \tau \)-length of \( T \) is equal to a number \( a < \infty \), then for any \( \epsilon > 0 \) there exists a number \( \delta > 0 \) such that for any non-overlapping intervals \( T_i \) with \( \bigcup T_i = T \), \( \tau(T_i) < \delta \), we have \( \Sigma \tau(T_i) > a - \epsilon \).

2.3. Definition. Let \( \varphi \) be a continuous pseudometric on an arc \( A \). If \( x \in A, \varphi(x) \in A \), then
\[
(1) \quad \varphi'(x, y) = \text{the greatest lower bound of numbers } \Sigma \{ \varphi(V_i) \}
\]
where \( \{ V_i \} \) is a finite family of intervals, \( \bigcup V_i = [x, y] \);
\[
(2) \quad \varphi''(x, y) = \text{the greatest lower bound of numbers } \Sigma \{ \varphi(V_i) \}
\]
where \( \{ V_i \} \) is a finite family of intervals of \( A \), \( \bigcup V_i = [x, y] \).

For any continuous pseudometric \( \varphi \) on an arc \( A \), \( \varphi' \) and \( \varphi'' \) are continuous pseudometrics, \( \varphi' \geq \varphi'' \). We have \( \varphi = \varphi'' \) if and only if, for any intervals \( T_i, T_2 \) of \( A \), \( T_i \subset T_2 \) implies \( \varphi(T_i) \leq \varphi(T_2) \).

Remark. If \( \varphi \) is a metric, it may happen that \( \varphi' \) is a pseudometric and even \( \varphi''(x, y) = 0 \) for all \( x \neq A, \varphi(x) \neq A \). See 2.5, Corollary, 2.10 and 2.13.

2.5. Proposition. Let \( \langle A, \varphi \rangle \) be a metric arc. Then the \( \varphi' \)-length and the \( \varphi'' \)-length of every interval \( T \subset A \) coincide.

Proof. Denote the \( \varphi'' \)-length of an interval \( T \) by \( \varphi''(T) \). We shall show that \( \varphi'(T) = \varphi''(T) \) for every non-degenerate closed interval \( T \); from this, the assertion will follow at once. Let \( \epsilon > 0 \). Clearly, due to the continuity of the pseudometric \( \varphi' \), there exists a closed interval \( T'' \subset T \) strictly contained in \( T \) (i.e., with endpoints distinct from those of \( T \)) such that \( \varphi(V) > \varphi'(T) - \epsilon \) for any interval \( V \) with \( T'' \subset V \subset T \). Denote by \( \gamma \) the greatest lower bound of numbers \( \varphi(x, y), x \in T'' \), \( y \in A - T \); clearly \( \gamma > 0 \). Choose non-overlapping closed intervals \( T_i, i = 0, 1, \ldots, n \), with \( \bigcup T_i = T'' \) such that \( \varphi(T_i) < \gamma, i = 0, 1, \ldots, n \). For each \( i = 0, 1, \ldots, n \) choose a finite family \( \{ T_{ij} \} \) of intervals in such a way that \( \bigcup T_{ij} = T_i, T_{ij} \cap T_i = \emptyset \),
\[
\sum_{i = 0} \varphi(T_{ij}) < \varphi'(T_i) + \frac{1}{n+1} \epsilon, \quad \sum_{i = 0} \varphi(T_{ij}) < \gamma
\]
2.7. Proposition. Let \( \varrho \) be a continuous pseudometric on an arc \( A \). Let \( T \subseteq A \) be an interval. If the \( \varrho \)-length of \( T \) is finite, then it is equal to its \( \varrho' \)-length.

This follows at once from 2.6.

2.8. Lemma. Let \( A \) be an arc and let \( \sigma \) be a continuous pseudometric on \( A \). Let \( \varphi \) be a monotone real-valued function defined for \( x \geq 0 \), continuous at 0, and such that \( \varphi x \geq 0 \), \( \varphi 0 = 0 \). Then for any interval \( T \subseteq A \) and any \( \varepsilon > 0 \) there exists a continuous metric \( \sigma \) on \( A \) such that \( d(\varrho, \sigma) < \varepsilon \) if and for some intervals \( T_1 \subseteq T \) with \( \bigcup T_i = T \) we have \( \Sigma \varphi(\sigma(T_i)) < \varepsilon \).

Proof. We consider the case \( T = A \) only; the general case is quite similar. As it is well known, there exists an infinite-dimensional normed linear space \( E \) and a continuous mapping \( f : A \to E \) such that \( \varrho(x, y) = |f(x) - f(y)| \). Now choose points \( x_i \in A, i = 0, 1, \ldots, n \), such that each interval \( T_i := [x_{i-1}, x_i] \) covers \( A \) and do not overlap; the intervals \( T_i \) are equal to 0.

2.9. Definition. If \( \langle P, \varrho \rangle \), \( \langle Q, \sigma \rangle \) are metric spaces, then a mapping \( f : \langle P, \varrho \rangle \to \langle Q, \sigma \rangle \) is called contracting if \( \sigma(f(x), f(y)) \leq \varrho(x, y) \). Then

\[
\sup_{\xi} \{\varrho(u, v) | \sigma(a, b) \leq \xi\}.
\]

2.10. Theorem. Let \( \langle B, \varrho \rangle \) be a metric arc. Let \( B \) be an arc. Then almost every continuous metric \( \rho \) on \( A \) has the following property: if \( f : \langle A, \varrho \rangle \to \langle B, \rho \rangle \) is monotone and contracting, then \( f \) is constant.

Proof. By 1.5 and 1.7, the assertion is meaningful. Let \( g : B \to [0, 1] \) be a homeomorphism. For any \( \xi \geq 0 \) let \( \varrho_\xi \) be equal to
Clearly, ϕ has the properties stated in 2.8, and, in addition, is upper semi-continuous. For any ε > 0 denote by H(ε) the set of continuous pseudometrics ϕ on A such that for appropriate intervals T with \( \bigcup T_i = A \) we have \( \sum \varphi(T_i) < \varepsilon \). Clearly, H(ε) is open in \( M_1(A) \) because ϕ is upper semi-continuous. By 2.8, H(ε) is dense.

If \( \varphi \in H(\varepsilon) \) and \( f: \langle A, \varphi \rangle \rightarrow \langle B, \sigma \rangle \) is monotone and contracting, choose intervals \( T_i \) with \( \bigcup T_i = A \), \( \sum \varphi(T_i) < \varepsilon \). Then \( \sigma(T_i) \leq \varepsilon \), \( \sum \varphi(T_i) < \varepsilon \). Hence, \( \sum \varphi(T_i) < \varepsilon \) and therefore \( |\varphi|_a \varphi(A) < \varepsilon \).

Put

\[
H = \bigcap_{n=1}^{\infty} \left( H(n^{-1}) \right).
\]

Then H is a dense \( G_\delta \)-set in \( M_1(A) \). If \( \varepsilon \in H \) and \( f: \langle A, \varphi \rangle \rightarrow \langle B, \sigma \rangle \) is monotone and contracting, then \( |\varphi|_a \varphi(A) < n^{-1} \) for \( n = 1, 2, \ldots \), \( n \rightarrow \infty \). Hence \( |\varphi|_a \varphi(A) = 0 \), which implies that \( f \) is constant.

**2.11. PROPOSITION.** Let \( \langle A, \varphi \rangle \) be a metric arc. If \( f: \langle A, \varphi \rangle \rightarrow \mathbb{R} \) is monotone contracting, then also \( f: \langle A, \varphi' \rangle \rightarrow \mathbb{R} \) is monotone contracting.

**2.12. PROPOSITION.** Let \( \langle A, \varphi \rangle \) be a metric arc. Let the \( \varphi' \)-length of \( A \) be finite. Let \( a \in A \) be an endpoint of \( A \); for any \( x \in A \) let \( gx \) be equal to the \( \varphi' \)-length of \( [a, x] \). Then

1. if \( f: \langle A, \varphi \rangle \rightarrow \mathbb{R} \) is monotone contracting, then \( f = h \circ g \), where \( h: A \rightarrow \mathbb{R} \) is monotone contracting;
2. if (1) holds with a function \( g': \langle A, \varphi \rangle \rightarrow \mathbb{R} \) instead of \( g \), then there exists a contracting monotone mapping \( g \) such that \( g = \varphi \circ g' \).

**Proof.** We put \( h(t) = \varphi^{-1} t \). Then \( |h(u) - h(v)| = |\varphi^{-1} u - \varphi^{-1} v| \), hence, by 2.11,

\[
|h(u) - h(v)| = |\varphi^{-1} u - \varphi^{-1} v|;
\]

clearly, \( \varphi^{-1} u, \varphi^{-1} v \) do not exceed the \( \varphi' \)-length of \( [\varphi^{-1} u, \varphi^{-1} v] \), which is equal to \( |u - v| \). This proves the first assertion. The proof of the second one is left to the reader.

**Remark.** The function \( g \) itself need not be contracting, not even Lipschitzian.

**2.13. THEOREM.** Let \( \langle A, \varphi \rangle \) be a metric arc. The existence of a non-constant monotone contracting function \( f: \langle A, \varphi \rangle \rightarrow \mathbb{R} \) implies and is implied by the condition \( \varphi A > 0 \). The existence of a contracting one-to-one function \( f: \langle A, \varphi \rangle \rightarrow \mathbb{R} \) implies and is implied by the condition \( \varphi T > 0 \) (or \( \varphi' T > 0 \)) for every non-degenerate interval \( T \in A \).

**Proof.** We prove only the "implied by" part of the second assertion. Both conditions \( \varphi T > 0 \) and \( \varphi' T > 0 \) are equivalent (see 2.5, Corollary). Suppose that \( \varphi' T > 0 \) for every non-degenerate interval \( T \in A \).

Let \( g: [0, 1] \rightarrow \mathbb{R} \) be a homeomorphism. Choose numbers \( a_n, b_n, n \in \mathbb{N} \), so that \( 0 < a_n < b_n \leq 1 \) and \( [a_n, b_n] \) form an open base of \( A \). Put \( a = 0, b = 1, a_n = g(a_n), b_n = g(b_n) \). Let \( f_n \) be equal to the \( \varphi' \)-distance of \( x \) from \( [a_n, b_n] \). Then \( f_n \) is monotone contracting, \( f_n(a_n) = 0, f_n(b_n) = \varphi T(a_n, b_n) > 0 \).

Now put \( f = \sum 2^{-n} f_n \). Clearly, \( f \) is monotone contracting, \( f(a_n) = f(b_n) > 0 \). Since the intervals \( [a_n, b_n] \) form a base, \( f \) is one-to-one.

### 3.

**3.1. Definition.** If \( \langle A, \varphi \rangle, \langle B, \sigma \rangle \) are metric arcs, we put \( \varphi 
\sigma \) if there exists a homeomorphism \( g \) of \( A \) onto \( B \) and a number \( c > 0 \) such that \( \varphi(x, y) \leq c \sigma(gx, gy) \) for every \( x \in A, y \in A \). Clearly, the relation \( \varphi \n\sigma \) is transitive and reflexive.

If \( \varphi \n\sigma \) and \( \sigma \n\varphi \), we put \( \varphi \sim \sigma \) and call \( \varphi \) and \( \sigma \) sometimes also \( \langle A, \varphi \rangle \) and \( \langle B, \sigma \rangle \), equivalent. We choose a fixed single-valued relation \( \tau \) such that \( \tau \varphi = \tau \sigma \) if and only if \( \varphi \) and \( \sigma \) are continuous metrics on arcs, \( \varphi \sim \sigma \); the element \( \tau \varphi \), where \( \varphi \) is a continuous metric on an arc \( A \), will be called the type of metric \( \varphi \) or of metric arc \( \langle A, \varphi \rangle \) and will be denoted by \( \text{typ} \varphi \).

**Remarks.** 1. Clearly, if \( \langle A, \varphi \rangle \) and \( \langle B, \sigma \rangle \) are metric arcs and there exists an \( L \)-isomorphic (in the sense of [2], 1.7) mapping of \( \langle A, \varphi \rangle \) onto \( \langle B, \sigma \rangle \), then \( \varphi \sim \sigma \); simple examples show that the converse does not hold.

2. It is clear that the cardinality of the set of all types of metric arcs does not exceed \( \exp \mathbb{N} \).

**3.2. THEOREM.** Every metric arc of finite length admits of a homeomorphic contracting mapping onto an interval of reals endowed with the usual metric. The type of a compact interval of reals is a minimal element in the set of types of metric arcs.

**Proof.** The first assertion follows at once from 2.7 and 2.13. The second assertion paraphrases the first.

**3.3. THEOREM.** Let \( \langle B_n, \sigma_n \rangle, n \in \mathbb{N} \), be metric arcs. Let \( A \) be an arc. Then almost every continuous metric \( \varphi \) on \( A \) is such that \( \sigma_n \varphi \) holds for no \( n \in \mathbb{N} \).

This follows at once from 2.10.

**3.4. THEOREM.** Let \( \langle A, \varphi \rangle \) be a metric arc. Then the following properties are equivalent:

1. \( \langle A, \varphi \rangle \) admits of a distance-preserving embedding into a normed linear space \( E \) such that, for some continuous linear form \( \varphi \) on \( E \), the restriction \( \varphi_A \) is one-to-one;
2. there are no non-degenerate intervals \( T \subseteq A \) with \( \varphi T = 0 \).
Proof. Clearly, (1) implies the existence of a contracting homeomorphism of $A$ onto an interval of reals; this implies $q''T > 0$ for every non-degenerate interval $T \subset A$.

If (2) holds, then, by 2.13, there exists a monotone contracting one-to-one function $f: \langle A, q \rangle \to \mathcal{R}$. By well-known theorems, there exists a normed linear space $P$ and a distance-preserving mapping $g: \langle A, q \rangle \to P$. Define a normed linear space $Q$ as follows: $Q$ consists of pairs $\langle u, \eta \rangle$, $u \in P$, $\eta \in \mathcal{R}$; we put

$$|\langle u, \eta \rangle| = \max(|u|, |\eta|).$$

For $x \in A$ put $hx = \langle gx, fx \rangle$. Then $h$ is a distance-preserving embedding and the linear form $\varphi$ on $Q$ defined by $\varphi \langle u, \eta \rangle = \eta$ has the required properties.

Corollary. A metric arc $\langle A, q \rangle$ of finite length admits of a distance-preserving embedding into a normed linear space $E$ such that, for some continuous linear form $\varphi$ on $E$, the function $\varphi_A$ is one-to-one.

3.5. Questions (A) and (B) stated in the introduction can now be answered. By 3.3, there is no "smallest" (with respect to the quasi-order described in 3.1) metric on an arc. By 3.2, the metric of a compact interval of $\mathcal{R}$ is minimal. It is clear that the property of an embedding $f: \langle A, q \rangle \to E$ described in question (B) is equivalent to the following: there exists a continuous linear form $\varphi$ on $E$ such that the restriction of $\varphi$ to the arc $f[A]$ is one-to-one. Therefore Theorem 3.4 provides an answer to question (B).

References


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