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Metrics on an arc

by

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Consider the following two questions:

(A) Does there exist a "smallest" continuous metric on a given arc A, i.e. a metric ϱ such that, for any continuous metric σ on A, there exist a homeomorphism $\varphi: A \to A$ and a number $a > 0$ such that $a\delta\langle\varphi x, \varphi y\rangle \geqslant \varrho\langle x, y\rangle$ for every $x, y \in A$? If not, does there exist a "minimal" metric with respect to the order just described?

(B) If ϱ is a continuous metric on an arc A, does there exist a normed linear space E and a distance-preserving mapping $f: \langle A, \varrho\rangle \to E$ such that the arc $f[A]$ admits of a "coordinatization"? (We say that an arc $B \subset E$ admits of a "coordinatization" if there are a point $a \in E$ and a continuous mapping f of $[0, 1]$ into a closed hyperplane $L \subset E$, $a \notin L$, such that B consists of all $ta+ft$, $0 \leqslant t \leqslant 1$.)

Both questions seem to be rather elementary. However, I have not found any answer in the literature. So the present note appears though the results may be already known.

In § 1 some definitions and lemmas are given; § 2 contains some auxiliary concepts and propositions. In § 3 the main results are stated and proved.

1.

1.1. The terminology and notation of [1] is used. Since it does not differ substantially from current terms and symbols, only two points of difference should be mentioned: an ordered pair a, b is denoted by $\langle a, b\rangle$; the value of a mapping f at an element x is usually denoted simply by fx. As usual we often denote, e.g., a space and the set of its points by the same symbol. The letters $\mathcal{N}$ and $\mathcal{R}$ respectively stand for the set of all natural numbers $0, 1, 2, \ldots$ and the set of all reals.

1.2. Definition. If X is a set, we denote by $M(X)$ the set of all bounded pseudometrics on Y. If X is a topological (or uniform) space, we denote by $M_t(X)$ (or $M_u(X)$) the set of all continuous (or uniformly continuous) bounded pseudometrics on X.

1.3. Definition. Let X be a set. If $\sigma \in M(X)$, $\sigma \in M(X)$, we put

$$d\langle \varrho, \sigma\rangle = \sup\{|\varrho\langle x, y\rangle - \sigma\langle x, y\rangle|\}.$$

Clearly, d is a metric on $M(X)$, and $\langle M(X), d\rangle$ is a complete metric space.

1.4. If X is a topological (or uniform) space, then $M_t(X)$ or $M_u(X)$ is closed in $M(X)$.

1.5. Convention. Every subset of $M(X)$ will be considered as a metric space (with the metric described in 1.3). If $H \subset M(X)$, $\bar{H} - H$ is meager in $\bar{H}$, and **P** is a property of elements ϱ of H, then "almost every $\varrho \in H$ possesses the property **P**" means that the set of all those $\varrho \in H$ which do not possess the property **P** is meager in H.

1.6. Proposition. *Let X be a metrizable uniform space. Then almost every uniformly continuous bounded pseudometric on X is a metric on X inducing its given uniformity.*

Proof. By M^* we denote the set of all bounded metrics inducing the uniformity of X. It is easy to see that M^* is dense in $M_u(X)$. Choose a metric τ inducing the uniformity of X. For $n = 1, 2, \ldots$ denote by H_n the set of all uniformly continuous bounded pseudometrics ϱ on X such that, for some $\varepsilon > 0$ depending on ϱ, $\varrho\langle x, y\rangle < \varepsilon$ implies $\tau\langle x, y\rangle < n^{-1}$. It is easy to show that H_n are open in $M_u(X)$ and $\bigcap H_n = M^*$.

Corollary. *Let X be a compact metrizable space. Then almost every continuous pseudometric on X is a metric inducing the topology of X.*

1.7. Proposition. *Let X be a separable metrizable topological space. Then almost every bounded continuous pseudometric on X is a metric inducing the given topology of X.*

Proof. Denote by M^* the set of all bounded metrics inducing the topology of X. Let $\{G_n\}$ be a countable open base of X; let B consist of all $\langle G_m, G_n\rangle$, $\bar{G}_m \subset G_n$. If $b \in B$, $n = \langle G_m, G_n\rangle$, let T_b consist of all $\varrho \in M_t(X)$ such that $\inf\{\varrho\langle x, y\rangle \mid x \in G_m, y \in X - G_n\} > 0$. Clearly, each set T_b is dense and open. It is not difficult to show that $M^* \supset \bigcap T_b$

Remark. It is easy to see that $M^* \neq \bigcap T_b$ in general.

2.

2.1. Conventions. A topological space homeomorphic to a compact non-degenerate interval of reals will be called an *arc*. If A is an arc, then every non-void connected $T \subset A$ will be termed an *interval* of A; sometimes, symbols $[a, b]$, etc., will be used to denote intervals of an arc. We shall say that two intervals T_1 and T_2 of an arc A *overlap* if $T_1 \cap T_2$ contains more than one point.

If A is an arc, $T \subset A$ is an interval with endpoints a, b, and ϱ is a pseudometric on A, we put $\varrho T = \varrho\langle a, b\rangle$. The set $\mathscr{R}$ and all of its subsets are considered (unless the contrary is stated or implied by the context) as spaces with the usual metric, say μ, defined by $\mu\langle x, y\rangle = |x - y|$; instead of μT we write $|T|$.

Clearly, every continuous metric on an arc A induces the given topology of A. If A is an arc and ϱ is a continuous metric on A, we shall say that $\langle A, \varrho\rangle$ is a *metric arc*.

2.2. Definition. If τ is a continuous pseudometric on an arc A and $T \subset A$ is an interval, then the least upper bound of numbers $\Sigma\{\tau T_i\}$ where $\{T_i\}$ is a finite family of non-overlapping intervals, $\bigcup T_i = T$, will be called the *τ-length* of T.

Clearly, if the τ-length of T is equal to a number $a < \infty$, then for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for any non-overlapping intervals T_i with $\bigcup T_i = T$, $\tau T_i < \delta$, we have $\Sigma \tau T_i > a - \varepsilon$.

2.3. Definition. Let ϱ be a continuous pseudometric on an arc A. If $x \in A$, $y \in A$, then

(1) $\varrho'\langle x, y\rangle$ will denote the greatest lower bound of numbers $\Sigma\{\varrho V_i\}$ where $\{V_i\}$ is a finite family of intervals, $\bigcup V_i = [x, y]$;

(2) $\varrho''\langle x, y\rangle$ will denote the greatest lower bound of numbers $\Sigma\{\varrho V_i\}$, where $\{V_i\}$ is a finite family of intervals of A, $\bigcup V_i \supset [x, y]$.

2.4. For any continuous pseudometric ϱ on an arc A, ϱ' and ϱ'' are continuous pseudometrics, $\varrho \geqslant \varrho' \geqslant \varrho''$. We have $\varrho = \varrho''$ if and only if, for any intervals T_1, T_2 of A, $T_1 \subset T_2$ implies $\varrho T_1 \leqslant \varrho T_2$.

Remark. If ϱ is a metric, it may happen that ϱ' is a pseudometric and even $\varrho'\langle x, y\rangle = 0$ for all $x \in A$, $y \in A$. See 2.5, Corollary, 2.10 and 2.13.

2.5. Proposition. *Let $\langle A, \varrho\rangle$ be a metric arc. Then the ϱ'-length and the ϱ''-length of every interval $T \subset A$ coincide.*

Proof. Denote the ϱ''-length of an interval T by $m''T$. We shall show that $m''T \geqslant \varrho'T$ for every non-degenerate closed interval T; from this, the assertion will follow at once. Let $\varepsilon > 0$. Clearly, due to the continuity of the pseudometric ϱ', there exists a closed interval $T^* \subset T$ strictly contained in T (i.e., with endpoints distinct from those of T) such that $\varrho'V > \varrho'T - \varepsilon$ for any interval V with $T^* \subset V \subset T$. Denote by γ the greatest lower bound of numbers $\varrho\langle x, y\rangle$, where $x \in T^*$, $y \in A - T$; clearly $\gamma > 0$. Choose non-overlapping closed intervals T_i, $i = 0, 1, \ldots, n$, with $\bigcup T_i = T^*$ such that $\varrho T_i < \gamma$, $i = 0, 1, \ldots, n$. For each $i = 0, 1, \ldots, n$ choose a finite family $\{T_{ij}\}$ of intervals in such a way that $\bigcup T_{ij} \supset T_i$, $T_{ij} \cap T_i \neq \emptyset$,

$$\sum_j \varrho T_{ij} < \varrho''T_i + \frac{1}{n+1}\varepsilon, \quad \sum_j \varrho T_{ij} < \gamma$$

(this is possible since $\varrho''T_i \leqslant \varrho T_i < \gamma$). Then all T_{ij} are contained in T. Clearly

$$\sum_{i,j} \varrho T_{ij} < \sum \varrho'' T_i + \varepsilon \leqslant m''T + \varepsilon.$$

Put $V = \bigcup T_{ij}$. Then V is an interval, $T^* \subset V \subset T$. We have

$$\varrho' V \leqslant \sum \{\varrho T_{ij}\} \leqslant \sum_{i,j} \varrho T_{ij} < m''T + \varepsilon.$$

Since $\varrho'T < \varrho'V + \varepsilon$, we obtain $\varrho'T < m''T + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\varrho'T \leqslant m''T$.

COROLLARY. *Let $\langle A, \varrho\rangle$ be a metric arc. For any interval $T \subset A$, the following four conditions are equivalent:*
(1) $\varrho'T = 0$;
(2) $\varrho''T = 0$;
(3) *the ϱ'-length of T is equal to 0;*
(4) *the ϱ''-length of T is equal to 0.*

This follows at once from the above proposition. E.g., if $\varrho''T = 0$, then $\varrho''V = 0$ for every interval $V \subset T$, hence the ϱ'-length, and therefore the ϱ''-length of T is equal to 0.

2.6. LEMMA. *Let ϱ be a continuous pseudometric on an arc A. Let $T \subset A$ be an interval. Then $\varrho'T + mT \geqslant 2\varrho T$, where mT is the ϱ-length of T.*

Proof. Let $\varepsilon > 0$. Choose intervals $T_i \subset T$, $i = 0, 1, \ldots, n$, such that $\bigcup T_i = T$ and $\Sigma \varrho T_i < \varrho'T + \varepsilon$. By an easy induction it can be shown that it is possible to select intervals $T_j^* = T_{i(j)}$, $j = 0, \ldots, m$, $m \leqslant n$, in such a way that, with $T_j^* = [a_j, b_j]$, we have $a_0 < a_1 \leqslant b_0 < a_2 \leqslant b_1 < a_3 \leqslant \ldots < a_n \leqslant b_{n-1} < b_n$, $\bigcup T_j^* = T$. Put

$$b_{-1} = a_0, \quad a_{n+1} = b_n,$$

$$\alpha = \sum \{\varrho\langle a_i, b_i\rangle | i \text{ even}\},$$

$$\beta = \sum \{\varrho\langle a_i, b_i\rangle | i \text{ odd}\},$$

$$\alpha' = \sum \{\varrho\langle b_i, a_{i+2}\rangle | i \text{ even}\},$$

$$\beta' = \sum \{\varrho\langle b_i, a_{i+2}\rangle | i \text{ odd}\}.$$

Then it is easy to show that $\varrho'T \leqslant \alpha + \beta < \varrho'T + \varepsilon$, $\alpha + \alpha' \geqslant \varrho T$, $\beta + \beta' \geqslant \varrho T$, $\alpha' + \beta' \leqslant mT$. Hence $\varrho'T + mT \geqslant 2\varrho T$.

Remark. Simple examples show that $\varrho''T + mT \geqslant 2\varrho T$ does not hold in general.

2.7. PROPOSITION. *Let ϱ be a continuous pseudometric on an arc A. Let $T \subset A$ be an interval. If the ϱ-length of T is finite, then it is equal to its ϱ'-length.*

This follows at once from 2.6.

2.8. LEMMA. *Let A be an arc and let σ be a continuous pseudometric on A. Let φ be a monotone real-valued function defined for $x \geqslant 0$, continuous at 0, and such that $\varphi x \geqslant 0$, $\varphi 0 = 0$. Then for any interval $T \subset A$ and any $\varepsilon > 0$ there exists a continuous metric σ on A such that $d\langle\varrho, \sigma\rangle < \varepsilon$ and for some intervals $T_i \subset T$ with $\bigcup T_i = T$ we have $\Sigma\varphi(\sigma T_i) < \varepsilon$.*

Proof. We consider the case $T = A$ only; the general case is quite similar. As it is well known, there exists an infinite-dimensional normed linear space E and a continuous mapping $f: A \to E$ such that $\varrho\langle x, y\rangle = |fx - fy|$. Now choose points $x_i \in A$, $i = 0, 1, \ldots, n$, in such a way that
(1) the intervals $T_i = [x_i, x_{i+1}]$ cover A and do not overlap;
(2) for any $x \in T_i$, $y \in T_i$, we have $\varrho\langle x, y\rangle < \frac{1}{8}\varepsilon$.

Find linearly independent points $b_i \in E$, $c_i \in E$, $|b_i| = 1$, $|c_i| = 1$, such that no element

$$\sum \lambda_i b_i + \sum \mu_i c_i \neq 0$$

is equal to a linear combination of the points fx_i. Choose a number $\delta > 0$ such that $2n\varphi\delta < \varepsilon$, $\delta < \frac{1}{8}\varepsilon$. Consider the linear segments $S_{i,0}, S_{i,1}, S_{i,2}$ joining the following pairs of points (with $i = 0, 1, \ldots, n-1$): fx_i and $fx_{i+1} + \delta b_i$, $fx_{i+1} + \delta b_i$ and $fx_i - \delta c_i$, $fx_i - \delta c_i$ and fx_{i+1}. Denote by B the union of these $3n$ segments. It is clear that B is an arc. Denote by τ the metric on B defined by $\tau\langle u, v\rangle = |u - v|$. Then

$$\sum_i \big(\varphi\tau(S_{i,0} \cup S_{i,1}) + \varphi\tau(S_{i,1} \cup S_{i,2})\big) \leqslant 2n\varphi\delta.$$

Clearly, the τ-diameter of $S_{i,0} \cup S_{i,1} \cup S_{i,2}$ is less than $\frac{1}{8}\varepsilon + 2\delta$. Now let g be a homeomorphism of A onto B such that $gx_i = fx_i$, $i = 0, 1, \ldots, n$; put $\varrho\langle x, y\rangle = \tau\langle gx, gy\rangle$, $S_{i,j}^* = g^{-1}[S_{i,j}]$. Then

$$\sum_i \big(\varphi\sigma(S_{i,0}^* \cup S_{i,1}^*) + \varphi\sigma(S_{i,1}^* \cup S_{i,2}^*)\big) \leqslant 2n\varphi\delta < \varepsilon.$$

It is easy to show that $d\langle\varrho, \sigma\rangle < \varepsilon$.

2.9. **Definition.** If $\langle P, \varrho\rangle$, $\langle Q, \sigma\rangle$ are metric spaces, then a mapping $f: \langle P, \varrho\rangle \to \langle Q, \sigma\rangle$ is called *contracting* if $\sigma\langle fx, fy\rangle \leqslant \varrho\langle x, y\rangle$.

2.10. THEOREM. *Let $\langle B, \sigma\rangle$ be a metric arc. Let A be an arc. Then almost every continuous metric ϱ on A has the following property: if $f: \langle A, \varrho\rangle \to \langle B, \sigma\rangle$ is monotone and contracting, then f is constant.*

Proof. By 1.5 and 1.7, the assertion is meaningful. Let $g: B \to [0, 1]$ be a homeomorphism. For any $\xi \geqslant 0$ let $\varphi\xi$ be equal to

$$\sup\{|gu - gv| \, \sigma\langle u, v\rangle \leqslant \xi\}.$$

Clearly, φ has the properties stated in 2.8, and, in addition, is upper semi-continuous. For any $\varepsilon > 0$ denote by $H(\varepsilon)$ the set of continuous pseudometrics ϱ on A such that for appropriate intervals T_i with $\bigcup T_i = A$ we have $\Sigma\varphi(\varrho T_i) < \varepsilon$. Clearly, $H(\varepsilon)$ is open in $M_t(A)$ because φ is upper semi-continuous. By 2.8, $H(\varepsilon)$ is dense.

If $\varrho \in H(\varepsilon)$ and $f: \langle A, \varrho\rangle \to \langle B, \sigma\rangle$ is monotone and contracting, choose intervals T_i with $\bigcup T_i = A$, $\Sigma\varphi(\varrho T_i) < \varepsilon$. Then $\sigma f[T_i] \leqslant \varrho T_i$, $\Sigma\varphi(\sigma f[T_i]) < \varepsilon$. Hence, $\Sigma|g\circ f[T_i]| < \varepsilon$ and therefore $|g\circ f[A]| < \varepsilon$.

Put

$$H = \bigcap_{n=1}^{\infty} \{H(n^{-1})\}.$$

Then H is a dense G_δ-set in $M_t(A)$. If $\sigma \in H$ and $f: \langle A, \varrho\rangle \to \langle B, \sigma\rangle$ is monotone and contracting, then $|g\circ f[A]| < n^{-1}$ for $n = 1, 2, \ldots$ Hence $|g\circ f[A]| = 0$, which implies that f is constant.

2.11. Proposition. *Let $\langle A, \varrho\rangle$ be a metric arc. If $f: \langle A, \varrho\rangle \to \mathscr{R}$ is monotone contracting, then also $f: \langle A, \varrho''\rangle \to \mathscr{R}$ is monotone contracting.*

The proof is easy and can be omitted.

2.12. Proposition. *Let $\langle A, \varrho\rangle$ be a metric arc. Let the ϱ'-length of A be finite. Let a be an endpoint of A; for any $x \in A$ let gx be equal to the ϱ'-length of $[a, x]$. Then*

(1) *if $f: \langle A, \varrho\rangle \to \mathscr{R}$ is monotone contracting, then $f = h\circ g$, where $h: \mathscr{R} \to \mathscr{R}$ is monotone contracting;*

(2) *if (1) holds with a function $g^*: \langle A, \varrho\rangle \to \mathscr{R}$ instead of g, then there exists a contracting monotone mapping ψ such that $g = \psi\circ g^*$.*

Proof. We put $ht = fg^{-1}t$. Then $|hu - hv| = |fg^{-1}u - gf^{-1}v|$, hence, by 2.11,

$$|hu - hv| \leqslant \varrho''\langle g^{-1}u, g^{-1}v\rangle;$$

clearly, $\varrho''\langle g^{-1}u, g^{-1}v\rangle$ does not exceed the ϱ''-length of $[g^{-1}u, g^{-1}v]$, which is equal to $|u - v|$. This proves the first assertion. The proof of the second one is left to the reader.

Remark. The function g itself need not be contracting, not even Lipschitzian.

2.13. Theorem. *Let $\langle A, \varrho\rangle$ be a metric arc. The existence of a non-constant monotone contracting function $f: \langle A, \varrho\rangle \to \mathscr{R}$ implies and is implied by the condition $\varrho' A > 0$. The existence of a contracting one-to-one function $f: \langle A, \varrho\rangle \to \mathscr{R}$ implies and is implied by the condition that $\varrho' T > 0$ (or $\varrho'' T > 0$) for every non-degenerate interval T of A.*

Proof. We prove only the "implied by" part of the second assertion. Both conditions ($\varrho' T > 0$ and $\varrho'' T > 0$) are equivalent (see 2.5, Corollary). Suppose that $\varrho'' T > 0$ for every non-degenerate interval $T \subset A$.

Let $g: [0, 1] \to \mathscr{R}$ be a homeomorphism. Choose numbers $\alpha_n, \beta_n, n \in \mathscr{N}$, so that $0 \leqslant \alpha_n < \beta_n \leqslant 1$ and $]\alpha_n, \beta_n[$ form an open base of A. Put $a = g0$, $b = g1$, $a_n = g\alpha_n$, $b_n = g\beta_n$. Let $f_n x$ be equal to the ϱ''-distance of x from $[a, a_n]$. Then f_n is monotone contracting, $f_n a_n = 0$, $f_n b_n = \varrho''\langle a_n, b_n\rangle > 0$.

Now put $f = \sum 2^{-n} f_n$. Clearly, f is monotone contracting, $fb_n - fa_n > 0$. Since the intervals $]a_n, b_n[$ form a base, f is one-to-one.

3.

3.1. Definition. If $\langle A, \varrho\rangle$, $\langle B, \sigma\rangle$ are metric arcs, we put $\varrho \precsim \sigma$ if there exists a homeomorphism g of A onto B and a number $c > 0$ such that $\varrho\langle x, y\rangle \leqslant c\sigma\langle gx, gy\rangle$ for every $x \in A$, $y \in A$. Clearly, the relation $\precsim$ is transitive and reflexive.

If $\varrho \precsim \sigma$ and $\sigma \precsim \varrho$, we put $\varrho \sim \sigma$ and call ϱ and σ, sometimes also $\langle A, \varrho\rangle$ and $\langle B, \sigma\rangle$, *equivalent.* We choose a fixed single-valued relation τ such that $\tau\varrho = \tau\sigma$ if and only if ϱ and σ are continuous metrics on arcs, $\varrho \sim \sigma$; the element $\tau\varrho$, where ϱ is a continuous metric on an arc A, will be called the *type of metric* ϱ or *of metric arc* $\langle A, \varrho\rangle$ and will be denoted by typ ϱ.

Remarks. 1. Clearly, if $\langle A, \varrho\rangle$ and $\langle B, \sigma\rangle$ are metric arcs and there exists an L-isomorphic (in the sense of [2], 1,7) mapping of $\langle A, \varrho\rangle$ onto $\langle B, \sigma\rangle$, then $\varrho \sim \sigma$; simple examples show that the converse does not hold.

2. It is clear that the cardinality of the set of all types of metric arcs does not exceed $\exp\aleph_0$.

3.2. Theorem. *Every metric arc of finite length admits of a homeomorphic contracting mapping onto an interval of reals endowed with the usual metric. The type of a compact interval of reals is a minimal element in the set of types of metric arcs.*

Proof. The first assertion follows at once from 2.7 and 2.13. The second assertion paraphrases the first.

3.3. Theorem. *Let $\langle B_n, \sigma_n\rangle$, $n \in \mathscr{N}$, be metric arcs. Let A be an arc. Then almost every continuous metric ϱ on A is such that $\sigma_n \precsim \varrho$ holds for no $n \in \mathscr{N}$.*

This follows at once from 2.10.

3.4. Theorem. *Let $\langle A, \varrho\rangle$ be a metric arc. Then the following properties are equivalent:*

(1) *$\langle A, \varrho\rangle$ admits of a distance-preserving embedding into a normed linear space E such that, for some continuous linear form φ on E, the restriction φ_A is one-to-one;*

(2) *there are no non-degenerate intervals $T \subset A$ with $\varrho' T = 0$.*

Proof. Clearly, (1) implies the existence of a contracting homeomorphism of A onto an interval of reals; this implies $\varrho''T > 0$ for every non-degenerate interval $T \subset A$.

If (2) holds, then, by 2.13, there exists a monotone contracting one-to-one function $f: \langle A, \varrho\rangle \to \mathscr{R}$. By well-known theorems, there exists a normed linear space P and a distance-preserving mapping $g: \langle A, \varrho\rangle \to P$. Define a normed linear space Q as follows: Q consists of pairs $\langle u, \eta\rangle$, $u \in P$, $\eta \in \mathscr{R}$; we put

$$|\langle u, \eta\rangle| = \max(|u|, |\eta|).$$

For $x \in A$ put $hx = \langle gx, fx\rangle$. Then h is a distance-preserving embedding and the linear form φ on Q defined by $\varphi\langle u, \eta\rangle = \eta$ has the required properties.

COROLLARY. *A metric arc $\langle A, \varrho\rangle$ of finite length admits of a distance-preserving embedding into a normed linear space E such that, for some continuous linear form φ on E, the function φ_A is one-to-one.*

3.5. Questions (A) and (B) stated in the introduction can now be answered. By 3.3, there is no "smallest" (with respect to the quasi-order described in 3.1) metric on an arc. By 3.2, the metric of a compact interval of $\mathscr{R}$ is minimal. It is clear that the property of an embedding $f: \langle A, \varrho\rangle \to E$ described in question (B) is equivalent to the following: there exists a continuous linear form φ on E such that the restriction of φ to the arc $f[A]$ is one-to-one. Therefore Theorem 3.4 provides an answer to question (B).

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