minants and no special concern for characteristic 2; the only trouble is that the "product" is in general two-valued. On primitive pairs with a fixed discriminant the structure obtained is that of an abelian group in which every element has been identified with its inverse.

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The existence of the potential operator associated with an equicontinuous semigroup of class (C_0)

by

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Hunt [2] introduced the notion of potential operators V associated with transient Markov processes in a separable, locally compact, non-compact Hausdorff space. The present author gave an operator-theoretical treatment of Hunt's theory of potentials (see [4] and [5]). This treatment suggests us to give an *abstract definition of the potential operator which may be applied to transient as well as to some recurrent Markov processes.*

Let X be a locally convex, sequentially complete, linear topological Hausdorff space. Let a family $\{T_t; t \geqslant 0\}$ of continuous linear operators T_t on X into X satisfy the following three conditions:

(1) $T_t T_s = T_{t+s}$, $T_0 = I =$ the identity (the semigroup property);

(2) for any continuous seminorm $p(x)$ on X, there exists a continuous seminorm $q(x)$ on X such that $p(T_t x) \leqslant q(x)$ for all $t \geqslant 0$ and $x \in X$ (the equicontinuity);

(3) $\lim T_t x = T_{t_0} x$ for every $t_0 \geqslant 0$ and $x \in X$ (the class (C_0) property).

Thus $\{T_t; t \geqslant 0\}$ is an equicontinuous semigroup of class (C_0) in X (see [3]). We can prove the following existence theorem:

THEOREM. *The infinitesimal generator A of T_t defined through*

$$Ax = \lim_{h \downarrow 0} h^{-1}(T_t x - x) \tag{4}$$

admits a densely defined inverse A^{-1} if and only if

$$\lim_{\lambda \downarrow 0} \int_0^\infty \lambda e^{-\lambda t} T_t x\, dt = 0 \quad \text{for all } x \in X. \tag{5}$$

Moreover, (5) *is a consequence of an apparently weaker condition*

$$\text{weak-}\lim_{\lambda \downarrow 0} \int_0^\infty \lambda e^{-\lambda t} T_t x\, dt = 0 \quad \text{for all } x \in X. \tag{5'}$$

By virtue of this Theorem, we may give the

Definition. In case when (5) is satisfied, we shall call

(6) $$V = -A^{-1}$$

the *potential operator associated with the semigroup* $\{T_t; t \geqslant 0\}$.

Remark 1. The definition of the potential operator

(6′) $$Vx = \lim_{\lambda\downarrow 0}(\lambda I - A)^{-1}x$$

given in [4] well fits to (6).

Remark 2. Consider the case where X is the completion, with respect to the maximum norm, of the space $C_0(R^1)$ of real-valued continuous functions $x(\xi)$ with compact supports in the whole real line R^1. Then the semigroup

(7) $$(T_t x)(\xi) = (2\pi t)^{-1/2}\int_{-\infty}^{\infty} e^{-|\xi-\eta|^2/2t}x(\eta)\,d\eta, \quad x(\xi)\in X,$$

satisfies condition (5), although the Brownian motion in R^1 is recurrent.

Proof of the Theorem. It is known (see, e.g., [3]) that

(8) $$\lambda(\lambda I - A)^{-1}x = \int_0^{\infty} \lambda e^{-\lambda t}T_t x\, dt$$

and

(9) $$p\big(\lambda(\lambda I - A)^{-1}x\big) \leqslant q(x) \quad \text{for all } \lambda > 0 \text{ and } x\in X.$$

Hence the condition $Ax = 0$ is equivalent to $\lambda(\lambda I - A)^{-1}x = x$ (for all $\lambda > 0$). Thus condition (5) implies, by (8), the existence of the inverse A^{-1}.

On the other hand, we have

(10) $$A(\lambda I - A)^{-1}x = \big(\lambda(\lambda I - a)^{-1} - I\big)x$$

and hence

(11) the range $R(A)$ = the range $R\big(I - \lambda(\lambda I - A)^{-1}\big)$.

By virtue of condition (9), we can apply the ergodic theorem of the Hille type in [3] (cf. Theorem 18. 6. 2. in [1]) to the effect that

(12) $R\big(I - \lambda(\lambda I - A)^{-1}\big)$ is independent of $\lambda > 0$ and its closure in X is equal to the set $\{x\in X;\ \lim_{\lambda\downarrow 0}\lambda(\lambda I - A)^{-1}x = 0\}$.

Hence, condition (5) is equivalent to the condition that the range $R(A)$ is dense in X. That condition (5) may be replaced by a weaker condition (5′) is proved in the ergodic theorem of the Hille type mentioned above.

Therefore the Theorem is proved.

Added in proof. Professor H. Komatsu called the author's attention that another proof of the above Theorem may be obtained by applying Theorem 3.1 in his paper *Fractional powers of operators*, Pacific J. Math. 19 (1966), p. 285.

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