has not been shown to apply to Lebesgue measure, but only to some measure (though conceivably detailed consideration of this counterexample may establish $\mu \sim m$).

In any case the question is still open as to whether the theorem is valid for low dimensions or for n a prime, $n > 2$. Of course, the translation field theorem does not hold, but as pointed out this asserts a stronger conclusion than necessary. Another question is whether, when orientation invariance is dropped, there always exists a point interior to a translation field at least in the case that $\bar{t}$ is obtained by reflecting t in $y = 0$. If so, Theorem 1 would apply without the restriction of orientation preservation.

References

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Generalized modular spaces

by

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The modular space was originally discovered in [2] as a special case of the Banach space. For instance we can discuss the L_p-space as a Banach space, but if p is a function, then the theory of the Banach space is not available. However, considering the L_p-space as a modular space, we can discuss the case of function p as well as the case of constant p. The definition of the modular formally includes the Banach norm, but the conjugate of the modular is quite different from that of the Banach norm. In this paper we attempt to unify the two theories of the Banach space and the modular space in one, considering that a modular has two kinds of conjugates: the conjugate and the polar.

1. Modulars. The convex modular is defined in [2], and the concave modular is defined in [4]. A generalized definition of modulars is given in [1]. Here we define modulars more generally to include the Fréchet norm too.

A function m on a linear space S is called a *modular* on S, if for any $x \in S$ we have

1) $0 \leqslant m(x) \leqslant +\infty$,
2) $m(-x) = m(x)$,
3) $\inf\limits_{\xi>0} m(\xi x) = 0$,
4) $m(\xi x) \leqslant m(\eta x)$ for $0 \leqslant \xi < \eta$,
5) there is $\chi > 0$ such that

$$m(x+y) \leqslant \chi\big(m(\chi x)+m(\chi y)\big) \quad \text{for all } x, y \in S.$$

Such χ is called a *character* of m. It is obvious by 3) and 4) that $m(0) = 0$.

Let m be a modular on a linear space S. An element $x \in S$ is said to be *finite*, if $m(\xi x) < +\infty$ for all $\xi > 0$. All finite elements form a linear manifold of S, which is called the *finite manifold*. A modular m is said to be *finite*, if S is the finite manifold.

An element $a \in S$ is said to be *null*, if $m(\xi a) = 0$ for all $\xi > 0$. All of null elements form a linear manifold of S, which is called the *null manifold*. A modular m is said to be *pure*, if the null manifold consists only of 0.

A modular m is said to be *normal*, if

$$m(x) = \sup_{0<\xi<1} m(\xi x) \quad \text{for all } x \in S.$$

For any modular m, setting

$$m_0(x) = \sup_{0<\xi<1} m(\xi x) \quad \text{for } x \in S,$$

we obtain a normal modular m_0, which is called the *normalization* of m.

A modular m is said to be *simple*, if $m(x) = 0$ implies $m(\xi x) = 0$ for all $\xi > 0$, that is, if $m(x) = 0$ implies that x is null.

A modular q on S is called a *quasi-norm*, if 1 is a character of q, that is, if

$$q(x+y) \leqslant q(x)+q(y) \quad \text{for all } x, y \in S.$$

It is obvious by the definition that every quasi-norm is finite, simple and normal.

A quasi-norm n on S is called a *norm*, if $n(\xi x) = \xi n(x)$ for all $\xi > 0$ and $x \in S$. For a quasi-norm q on S, setting

$$n_q(x) = \inf_{\xi \geqslant 1} \frac{1}{\xi} q(\xi x) \quad \text{for all } x \in S,$$

we obtain a norm n_q, which is called the *associated norm* of q.

A modular m is said to be *convex*, if $\alpha+\beta = 1$, $\alpha, \beta > 0$, implies

$$m(x+y) \leqslant \alpha m\left(\frac{1}{\alpha} x\right) + \beta m\left(\frac{1}{\beta} y\right) \quad \text{for all } x, y \in S.$$

It is obvious by the definition that 2 is a character of any convex modular. We can easily show that a quasi-norm q is convex if and only if q is a norm.

A modular m is said to be *singular*, if $m(x) = 0$ or $+\infty$ for all $x \in S$. For a convex modular m, setting

$$s_m(x) = \begin{cases} +\infty & \text{if} \quad m(x) > 1, \\ 0 & \text{if} \quad m(x) \leqslant 1, \end{cases}$$

we obtain a singular convex modular s_m, which is called the *singularity* of m.

2. Convergence. A sequence $x_\nu \in S$ $(\nu = 1, 2, \ldots)$ is said to be *convergent* to $x \in S$ for a modular m and we write $x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x$, if

$$\lim_{\nu\to\infty} m\big(\xi(x_\nu - x)\big) = 0 \quad \text{for all } \xi > 0.$$

Such x is called a *limit* of a convergent sequence $x_\nu \in S$ $(\nu = 1, 2, \ldots)$. We can prove that $x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x$ and $y_\nu \underset{\nu}{\overset{m}{\rightarrow}} y$ imply

$$\alpha x_\nu + \beta y_\nu \underset{\nu}{\overset{m}{\rightarrow}} \alpha x + \beta y \quad \text{for all real numbers } \alpha \text{ and } \beta,$$

and that $\lim_{\nu\to\infty} \alpha_\nu = \alpha$ implies $\alpha_\nu x \underset{\nu}{\overset{m}{\rightarrow}} \alpha x$ for all $x \in S$. We can also prove that $x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x$ implies $x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x+z$ if and only if z is null. Thus the limit of a convergent sequence is uniquely determined if and only if m is pure.

A manifold $A \subset S$ is said to be *closed* if $A \ni x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x$ implies $x \in A$. A manifold $A \subset S$ is said to be *complete* if for any sequence $x_\nu \in A$ $(\nu = 1, 2, \ldots)$ such that $x_\mu - x_\nu \underset{\mu,\nu}{\overset{m}{\rightarrow}} 0$, there is $x \in A$ such that $x_\nu \underset{\nu}{\overset{m}{\rightarrow}} x$. It is obvious by the definition that if A is complete and B is closed, then the intersection $A \frown B$ is complete. We can easily prove that the null manifold is complete and closed. We say that a modular m on S is *complete* if S is complete.

3. Quotient modulars. Let A be a linear manifold of S. The elements of the quotient space S/A are denoted by $x+A$ $(x \in S)$, that is,

$$(x+A)+(y+A) = (x+y)+A \quad \text{for } x, y \in S,$$

and we have $x+A = y+A$ if and only if $x-y \in A$. For a modular m on S, setting

$$m_A(x+A) = \inf_{u \in A} m(x+u) \quad \text{for } x \in S,$$

we obtain a modular m_A on S/A which is called the *quotient modular* of m by A. It is obvious by the definition that if m is a quasi-norm, then m_A also is a quasi-norm; if m is a norm, then m_A also is a norm; if m is convex, then m_A also is convex; and if m is singular, then m_A also is singular.

As a generalization of the so-called Hausdorff's theorem about norms, we can prove

COMPLETENESS THEOREM. *If a modular m is complete, then the quotient modular m_A also is complete.*

We can also prove

PURITY THEOREM. *If a linear manifold A is closed for a modular m, then the quotient modular m_A is pure.*

Let N be the null manifold for a modular m on S. Since N is closed, the quotient modular m_N is pure by the Purity Theorem. This pure modular m_N is called the *purification* of m. For the purification m_N of m, we have by the definition

$$m_N(x+N) = m(x) \quad \text{for all } x \in S.$$

4. Conjugates. All of linear functionals on S form a linear space, which is called the *free dual* of S and denoted by $\tilde{S}$. Let m be a modular on S. A $\varphi \in \tilde{S}$ is said to be *bounded* for m, if there is $\gamma > 0$ such that

$$\varphi(x) \leqslant \gamma\big(m(x)+1\big) \quad \text{for all } x \in S.$$

It is obvious by the definition that all bounded linear functionals form a linear manifold of $\tilde{S}$, which is called the *dual* of S for m and denoted by $\bar{S}$.

We define the *conjugate* $\overline{m}$ of a modular m as

(1) $$\overline{m}(\varphi) = \sup_{x \in S}\big(\varphi(x) - m(x)\big) \quad \text{for } \varphi \in \tilde{S}.$$

Then 1) $0 \leqslant \overline{m}(\varphi) \leqslant +\infty$ for $\varphi \in \tilde{S}$, as $\varphi(0) = m(0) = 0$. It is obvious that 2) $\overline{m}(-\varphi) = \overline{m}(\varphi)$ for all $\varphi \in \tilde{S}$. If $\alpha+\beta = 1$, $\alpha, \beta > 0$, then

$$\overline{m}(\alpha\varphi+\beta\psi) \leqslant \alpha\overline{m}(\varphi) + \beta\overline{m}(\psi) \quad \text{for all } \varphi, \psi \in \tilde{S}.$$

Thus the conjugate $\overline{m}$ satisfies all conditions of the convex modulars except 3).

We have by the definition

(2) $$\varphi(x) \leqslant \overline{m}(\varphi) + m(x) \quad \text{for all } \varphi \in \tilde{S} \text{ and } x \in S.$$

We can easily prove that $\varphi \in \tilde{S}$ is bounded if and only if there is $\xi > 0$ such that $\overline{m}(\xi\varphi) < +\infty$, that is,

(3) $$\bar{S} = \{\varphi : \overline{m}(\xi\varphi) < +\infty \quad \text{for some } \xi > 0\}.$$

For any $\varphi \in \bar{S}$ there is $\gamma > 0$ such that $\varphi(x) \leqslant \gamma\big(m(x)+1\big)$ for all $x \in S$. and if $0 < \xi < 1/\gamma$, then

$$\xi\varphi(x) - m(x) \leqslant (\xi\gamma - 1)m(x) + \xi\gamma \leqslant \xi\gamma$$

for all $x \in S$. Thus $\overline{m}(\xi) \leqslant \xi\gamma$ for $0 < \xi < 1/\gamma$, and we conclude that $\inf_{\xi>0} \overline{m}(\xi\varphi) = 0$ for all $\varphi \in \bar{S}$. Therefore $\overline{m}$ is a convex modular on $\bar{S}$.

If $\overline{m}(\xi\varphi) = 0$ for all $\xi > 0$, then $\xi\varphi(\eta x) \leqslant m(\eta x)$ by (2), and we have

$$\varphi(x) \leqslant \frac{1}{\xi\eta} m(\eta x) \quad \text{for all } \xi, \eta > 0.$$

Since there is $\eta > 0$ such that $m(\eta x) < +\infty$, we obtain $\varphi(x) \leqslant 0$ for all $x \in S$, that is, $\varphi = 0$. Therefore $\overline{m}$ is pure on $\bar{S}$.

Since

$$\sup_{0<\xi<1} \overline{m}(\xi\varphi) = \sup_{x \in S, 0<\xi<1}\big(\xi\varphi(x) - m(x)\big) = \overline{m}(\varphi),$$

$\overline{m}$ is normal on $\bar{S}$.

For a sequence $\varphi_\nu \in \bar{S}$ $(\nu = 1, 2, \ldots)$ if $\varphi_\mu - \varphi_\nu \xrightarrow[\mu,\nu]{\overline{m}} 0$, then, since we have by (2)

$$|\varphi_\mu(x) - \varphi_\nu(x)| \leqslant \frac{1}{\xi\eta}\Big(\overline{m}\big(\xi(\varphi_\mu - \varphi_\nu)\big) + m(\eta x)\Big) \quad \text{for } x \in S \text{ and } \xi, \eta > 0,$$

we obtain

$$\lim_{\mu,\nu\to\infty} |\varphi_\mu(x) - \varphi_\nu(x)| = 0.$$

Thus, setting

$$\varphi(x) = \lim_{\nu\to\infty} \varphi_\nu(x) \quad \text{for } x \in S,$$

we have $\varphi \in \tilde{S}$. For any $\xi > 0$ there is ϱ, by the assumption, such that $\overline{m}\big(\xi(\varphi_\mu - \varphi_\nu)\big) \leqslant 1$ for $\mu, \nu \geqslant \varrho$, and we have

$$|\varphi_\mu(x) - \varphi_\nu(x)| \leqslant \frac{1}{\xi}\big(m(x)+1\big) \quad \text{for all } x \in S;\ \mu, \nu \geqslant \varrho.$$

Thus

$$|\varphi_\mu(x) - \varphi(x)| \leqslant \frac{1}{\xi}\big(m(x)+1\big)$$

for $x \in S$, $\mu \geqslant \varrho$, and $\varphi_\mu - \varphi \in \bar{S}$ for $\mu \geqslant \varrho$ by the definition. As $\varphi_\mu \in \bar{S}$, we have $\varphi \in \bar{S}$. Furthermore, for any $\xi > 0$ we have by the definition

$$\overline{m}\big(\xi(\varphi_\mu - \varphi)\big) = \sup_{x \in S}\Big(\xi\big(\varphi_\mu(x) - \varphi(x)\big) - m(x)\Big)$$

$$\leqslant \sup_{x \in S, \nu \geqslant \mu}\Big(\xi\big(\varphi_\mu(x) - \varphi_\nu(x)\big) - m(x)\Big) = \sup_{\nu \geqslant \mu} \overline{m}\big(\xi(\varphi_\mu - \varphi_\nu)\big),$$

and we obtain

$$\lim_{\mu\to\infty} \overline{m}\big(\xi(\varphi_\mu - \varphi)\big) = 0 \quad \text{for all } \xi > 0.$$

Therefore $\overline{m}$ is complete on $\bar{S}$.

Now we can state

Conjugate Theorem. *The conjugate $\overline{m}$ of a modular m is a normal, complete and convex modular on the dual $\bar{S}$ of S for m.*

For the dual $\bar{S}$ of S for a modular m, setting $x(\varphi) = \varphi(x)$ for $x \in S$, $\varphi \in \bar{S}$, every $x \in S$ is considered as a linear functional on $\bar{S}$, bounded for $\overline{m}$, by (2). A modular m on S is said to be *reflexive*, if m coincides with the conjugate $\overline{\overline{m}}$ of $\overline{m}$ on S, that is, if

$$m(x) = \sup_{\varphi \in \bar{S}} \left(\varphi(x) - \overline{m}(\varphi)\right) \quad \text{for all } x \in S.$$

Acccording to § 80, Theorem 2 in [3], if m is convex and normal, then m is reflexive. Thus we have by the Conjugate Theorem

REFLEXIVITY THEOREM. *A modular m on S is reflexive if and only if m is convex and normal.*

The conjugate $\overline{\overline{m}}$ of the conjugate $\overline{m}$ is a convex and normal modular on S by the Conjugate Theorem, and reflexive by the Reflexivity Theorem. We have by (2)

$$\overline{\overline{m}}(x) = \sup_{\varphi \in \bar{S}} \left(\varphi(x) - \overline{m}(\varphi)\right) \leqslant m(x) \quad \text{for } x \in S.$$

Thus, setting $M(x) = \overline{\overline{m}}(x)$ for $x \in S$, we have

$$\overline{M}(\varphi) = \sup_{x \in S} \left(\varphi(x) - M(x)\right) \geqslant \overline{m}(\varphi)$$

for $\varphi \in \tilde{S}$. If $\overline{m}(x) < +\infty$, then $\varphi \in \bar{S}$ by (3), and

$$\overline{m}(\varphi) = \sup_{\Phi \in \bar{\bar{S}}} \left(\Phi(\varphi) - \overline{\overline{m}}(\Phi)\right)$$

for the dual $\bar{\bar{S}}$ of $\bar{S}$ for $\overline{m}$, because $\overline{m}$ is reflexive by the Reflexivity Theorem. Thus we obtain

$$\overline{m}(\varphi) \geqslant \sup_{x \in S} \left(\varphi(x) - M(x)\right) = \overline{M}(\varphi),$$

and we conclude that $\overline{M}(\varphi) = \overline{m}(\varphi)$ for all $\varphi \in \tilde{S}$ and that the dual $\bar{S}$ of S for m is the dual of S for M.

Conversely, for a reflexive modular M on S, if $\overline{M}(\varphi) = \overline{m}(\varphi)$ for all $\varphi \in \tilde{S}$, then

$$M(x) = \sup_{\varphi \in \bar{S}} \left(\varphi(x) - \overline{m}(\varphi)\right) = \overline{\overline{m}}(x) \quad \text{for all } x \in S.$$

Thus for any modular m on S there exists a unique reflexive modular M on S such that $\overline{m}(\varphi) = \overline{M}(\varphi)$ for all $\varphi \in \tilde{S}$. Such M is called the *associate* of m. If m is convex, then the associate of m is the normalized of m, because the normalized of m also is convex and reflexive by the Reflexivity Theorem, and has the same conjugate with m by the definition.

For a quasi-norm q on S we can easily prove that $\bar{q}(\varphi) = \bar{n}_q(\varphi)$ for all $\varphi \in \tilde{S}$ for the associated norm n_q of q. Since every norm is a reflexive modular by the Reflexivity Theorem, the associated norm n_q is the associate of q.

According to § 82, Theorem 4 in [3], we have

PROXIMITY THEOREM. *Let $\overline{m}$ be the conjugate of a convex modular m on S, and let $\bar{S}$ be the dual of S for m. Given $\gamma > 0$ and a finite system of real numbers a_ν and $\varphi_\nu \in \bar{S}$ $(\nu = 1, 2, \ldots, n)$, then for any $0 < \varepsilon < 1$ we can find $x \in S$ such that $\varphi_\nu(x) = a_\nu$ $(\nu = 1, 2, \ldots, n)$ and $m(\varepsilon x) \leqslant \gamma$ if and only if*

$$\sum_{\nu=1}^{n} \xi_\nu a_\nu \leqslant \gamma + \overline{m}\left(\sum_{\nu=1}^{n} \xi_\nu \varphi_\nu\right)$$

for any finite number of real numbers ξ_ν $(\nu = 1, 2, \ldots, n)$.

5. Polars. For a modular m on S we define the *polar* P_m of m as

$$P_m(\varphi) = \sup_{m(x) \leqslant 1} \varphi(x) \quad \text{for } \varphi \in \tilde{S}.$$

It is obvious by this definition that $0 \leqslant P_m(\varphi) \leqslant +\infty$ for $\varphi \in \tilde{S}$, and

$$P_m(\xi\varphi) = \xi P_m(\varphi) \quad \text{for } \xi > 0 \text{ and } \varphi \in \tilde{S}.$$

$$P_m(\varphi + \psi) \leqslant P_m(\varphi) + P_m(\psi) \quad \text{for } \varphi, \psi \in \tilde{S}.$$

Since $\xi\varphi(x) \leqslant m(x) + \overline{m}(\xi\varphi)$ by (2), we obtain by the definition

$$P_m(\varphi) \leqslant \inf_{\xi > 0} \frac{1}{\xi}\left(1 + \overline{m}(\xi\varphi)\right) \quad \text{for } \varphi \in \tilde{S}. \tag{4}$$

Thus we obtain by (3) that $P_m(\varphi) < +\infty$ for all $\varphi \in \bar{S}$, and we conclude that P_m is a norm on the dual $\bar{S}$. In addition, P_m is pure, because if $P_m(\varphi) = 0$, then $m(x) \leqslant 1$ implies $\varphi(x) = 0$, and since for any $x \in S$ there is $\xi > 0$ such that $m(\xi x) < 1$, we obtain $\varphi(x) = 0$ for all $x \in S$.

POLAR THEOREM. *If m is convex, then we have*

$$P_m(\varphi) = \inf_{\xi > 0} \frac{1}{\xi}\left(1 + \overline{m}(\xi\varphi)\right) \quad \text{for } \varphi \in \tilde{S}, \tag{5}$$

$$P_m(\varphi) \leqslant 1 \text{ implies } \overline{m}(\varphi) \leqslant P_m(\varphi), \tag{6}$$

$$\bar{s}_m = P_m \text{ for the singularity } s_m \text{ of } m. \tag{7}$$

Proof. We set

$$a = \inf_{\xi > 0} \frac{1}{\xi}\left(1 + \overline{m}(\xi\varphi)\right),$$

and we suppose $P_m(\varphi) < +\infty$. Then we can find $\varrho > 0$ such that $P_m(\varrho\varphi) < 1$. If $+\infty > m(\gamma x) > 1$ for some $0 < \gamma < 1$, then we can find $\xi > 0$ such that $m(\xi x) = 1$, $\xi < 1$, and we have

$$\varrho\varphi(x) = \frac{1}{\xi}\varrho\varphi(\xi x) \leqslant \frac{1}{\xi} \leqslant m(x),$$

because m is convex and $1 = m(\xi x) \leqslant \xi m(x)$. Thus we obtain

$$\varrho\varphi(x) \leqslant m(x)+1 \quad \text{for all } x \in S,$$

and we have $\overline{m}(\varrho\varphi) \leqslant 1$ by the definition. Therefore $a < +\infty$ and we have

$$\xi a \leqslant 1+\overline{m}(\xi\varphi) \quad \text{for all real numbers } \xi.$$

Thus for any $0 < \varepsilon < 1$ there is $x_\varepsilon \in S$ by the Proximity Theorem such that $\varphi(x_\varepsilon) = a$ and $m(\varepsilon x_\varepsilon) \leqslant 1$. Then we have by the definition

$$P_m(\varphi) = \sup_{m(x)\leqslant 1} \varphi(x) \geqslant \varepsilon a \quad \text{for } 0 < \varepsilon < 1,$$

and we conclude that $P_m(\varphi) \geqslant a$. Thus $P_m(\varphi) = a$ by (4).

Since

$$\frac{1}{\xi}\big(1+\overline{m}(\xi\varphi)\big) \geqslant \frac{1}{\xi} > 1$$

for $0 < \xi < 1$, if $P_m(\varphi) \leqslant 1$, we have by (5)

$$P_m(\varphi) = \inf_{\xi\geqslant 1}\frac{1}{\xi}\big(1+\overline{m}(\xi\varphi)\big) \geqslant \overline{m}(\varphi),$$

because $\overline{m}$ is convex and $\overline{m}(\xi\varphi) \geqslant \xi\overline{m}(\varphi)$ for $\xi \geqslant 1$.

We have by the definition

$$\overline{s}_m(\varphi) = \sup_{s_m(x)=0} \varphi(x) = \sup_{m(x)\leqslant 1} \varphi(x) = P_m(\varphi) \quad \text{for } \varphi \in \tilde{S}.$$

If m is convex, then we have by (5)

(8) $$\overline{S} = \{\varphi\colon P_m(\varphi) < +\infty\}.$$

Since $P_m = \overline{s}_m$, we conclude by the Conjugate Theorem that P_m is complete on the dual $\overline{S}$.

SINGULARITY THEOREM. *For a norm n on S the conjugate $\overline{n}$ is the singularity of the polar P_n on $\overline{S}$.*

Proof. Since by the definition

$$\overline{n}(\varphi) = \sup_{\xi>0, x\in S}\big(\varphi(\xi x)-n(\xi x)\big) = \sup_{\xi>0} \xi \sup_{x\in S}\big(\varphi(x)-n(x)\big),$$

we have $\overline{n}(\varphi) = 0$ if and only if $\varphi(x) \leqslant n(x)$ for all $x \in S$. If there is $x \in S$ such that $\varphi(x) > n(x)$, then $\overline{n}(\varphi) = +\infty$. Thus $\overline{n}$ is singular by the definition. Since $\varphi(x) \leqslant n(x)$ for all $x \in S$ if and only if $P_n(\varphi) \leqslant 1$, we conclude by the definition that $\overline{n}$ is the singularity of P_n.

According to the Singularity Theorem, for a norm n on S we have

$$\sum_{\nu=1}^{h} \xi_\nu a_\nu \leqslant \gamma + \overline{n}\Big(\sum_{\nu=1}^{h} \xi_\nu \varphi_\nu\Big)$$

if and only if

$$\sum_{\nu=1}^{h} \xi_\nu a_\nu \leqslant \gamma P_n\Big(\sum_{\nu=1}^{h} \xi_\nu \varphi_\nu\Big).$$

Thus the Proximity Theorem is a generalization of the so-called Helly's theorem.

6. Modular norms. For a modular m on S we defined the associate M such that $\overline{m} = \overline{M}$ and

$$M(x) = \sup_{\varphi\in\overline{S}}\big(\varphi(x)-\overline{m}(\varphi)\big) \quad \text{for all } x \in S.$$

Considering every $x \in S$ as a linear functional on $\overline{S}$, we set

$$N_1(x) = P_{\overline{m}}(x) \quad \text{for } x \in S.$$

Then N_1 is a norm on S and by (5) we have

(9) $$N_1(x) = \inf_{\xi>0}\frac{1}{\xi}\big(1+M(\xi x)\big) \quad \text{for } x \in S.$$

This norm N_1 is called the *first modular norm* of m.

For the singularity s_M of M we have $P_M(\varphi) = \overline{s}_M(\varphi)$ for $\varphi \in \overline{S}$ by (7). Since s_M is convex and normal, we have by the Reflexivity Theorem

$$s_M(x) = \sup_{\varphi\in\overline{S}}\big(\varphi(x)-P_M(\varphi)\big) \quad \text{for } x \in S.$$

According to the Singularity Theorem, setting

$$N_2(x) = P_{P_M}(x) \quad \text{for } x \in S$$

we obtain a norm N_2 on S such that s_M is the singularity of N_2, that is, N_2 and M have the same singularity. Thus we have

(10) $$M(x) \leqslant 1 \quad \text{if and only if } N_2(x) \leqslant 1.$$

Since by (5)

$$N_2(x) = \inf_{\xi>0}\frac{1}{\xi}\big(1+s_M(\xi x)\big) = \inf_{s_M(\xi x)=0, \xi>0}\frac{1}{\xi} \quad \text{for } x \in S,$$

we obtain

(11) $$N_2(x) = \inf_{M(\xi x)\leqslant 1, \xi>0}\frac{1}{\xi} \quad \text{for } x \in S.$$

This norm N_2 is called the *second modular norm* of m.

If $M(x) \leqslant 1$, then by (11)

$$N_2(x) = \inf_{M(\xi x)\leqslant 1, \xi\geqslant 1}\frac{1}{\xi}.$$

Since $\xi \geqslant 1$ implies $M(\xi x) \geqslant \xi M(x)$, we have $1/\xi \geqslant M(x)$ if $M(\xi x) \leqslant 1$ and $\xi \geqslant 1$. Thus we obtain

$$M(x) \leqslant N_2(x) \quad \text{if } M(x) \leqslant 1. \tag{12}$$

If $1 < M(x) < +\infty$, then there is $\eta > 0$ such that $M(\eta x) = 1$, because $M(\xi x)$ is a convex function of ξ. As $0 < \eta < 1$, we have

$$\frac{1}{\eta} = \frac{1}{\eta} M(\eta x) \leqslant M(x).$$

Thus by (11) we obtain

$$M(x) \geqslant N_2(x) \quad \text{if } M(x) > 1. \tag{13}$$

If $M(\xi x) \leqslant 1$, $\xi > 0$, then $1/\xi \geqslant N_2(x)$ by (10), and we have

$$\frac{1}{\xi}\big(1+M(\xi x)\big) \geqslant \frac{1}{\xi} \geqslant N_2(x).$$

If $M(\xi x) > 1$, $\xi > 0$, then $M(\xi x) \geqslant N_2(\xi x)$ by (13), and we have

$$\frac{1}{\xi}\big(1+M(\xi x)\big) \geqslant \frac{1}{\xi} N_2(\xi x) = N_2(x).$$

Thus $N_1(x) \geqslant N_2(x)$ for all $x \in S$ by (9). On the other hand, by (9) and (11) we have

$$N_1(x) \leqslant \inf_{M(\xi x)\leqslant 1, \xi > 0} \frac{1}{\xi}\big(1+M(\xi x)\big) \leqslant \inf_{M(\xi x)\leqslant 1, \xi>0} \frac{2}{\xi} = 2N_2(x).$$

Therefore

$$N_2(x) \leqslant N_1(x) \leqslant 2N_2(x) \quad \text{for } x \in S. \tag{14}$$

Convergence Theorem. *$x_\nu \xrightarrow[\nu]{M} x$ is equivalent to each one of $x_\nu \xrightarrow[\nu]{N_1} x$ and $x_\nu \xrightarrow[\nu]{N_2} x$.*

Proof. If $x_\nu \xrightarrow[\nu]{M} 0$, then for any $\xi > 0$ there is ϱ such that $M(\xi x_\nu) \leqslant 1$ for $\nu \geqslant \varrho$, and we have $N_2(x_\nu) \leqslant 1/\xi$ for $\nu \geqslant \varrho$ by (10), that is, $x_\nu \xrightarrow[\nu]{N_2} 0$. Conversely if $x_\nu \xrightarrow[\nu]{N_2} 0$, then for any $\xi > 1$ there is $\varrho > 0$ such that $N_2(\xi^2 x_\nu) \leqslant 1$ for $\nu \geqslant \varrho$, and we have $M(\xi^2 x_\nu) \leqslant 1$ for $\nu \geqslant \varrho$ by (10). Since $\xi M(\xi x_\nu) \leqslant M(\xi^2 x_\nu)$, we obtain $x_\nu \xrightarrow[\nu]{M} 0$. It is obvious by (14) that $x_\nu \xrightarrow[\nu]{N_1} x$ is equivalent to $x_\nu \xrightarrow[\nu]{N_2} x$.

According to the Convergence Theorem, M is complete if and only if one of N_1 and N_2 is complete.

Now we consider the modular norms of the conjugate $\overline{m}$. The associate of $\overline{m}$ is $\overline{m}$ itself by the Conjugate and Reflexivity Theorems. Referring to (10), we have $P_M = P_{N_2}$ by the definition, and P_{N_2} is the first modular norm of $\overline{m}$ by (5) and (9).

The conjugate $\overline{N}_1$ of the first norm N_1 is the singularity of $\overline{m}$ on $\overline{S}$. Because for $\varphi \in \overline{S}$ if $\overline{m}(\varphi) \leqslant 1$, then $\varphi(x) \leqslant 1+M(x)$ for all $x \in S$ by (2), as $\overline{m} = \overline{\overline{M}}$. Thus by (9) we have

$$\varphi(x) \leqslant \inf_{\xi>0} \frac{1}{\xi}\big(1+M(\xi x)\big) = N_1(x) \quad \text{for all } x \in S,$$

and we obtain

$$\overline{N}_1(\varphi) = \sup_{n \in S}\big(\varphi(x) - N_1(x)\big) = 0.$$

If $\overline{m}(\varphi) > 1$, then there is $a \in S$ by the definition such that $\varphi(a) - M(a) > 1$, and by (9) we obtain

$$\varphi(a) > 1+M(a) \geqslant N_1(a).$$

Thus

$$\overline{N}_1(\varphi) \geqslant \sup_{\xi>0}\big(\varphi(\xi a) - N_1(\xi a)\big) = +\infty.$$

Therefore $\overline{N}_1$ is the singularity of $\overline{m}$.

According to the Singularity Theorem, $\overline{N}_1$ is the singularity of P_{N_1}. Thus $\overline{m}$ and P_{N_1} have the same singularity on $\overline{S}$, and we see that $\overline{m}(\varphi) \leqslant 1$ if and only if $P_{N_1}(\varphi) \leqslant 1$. Thus we conclude that

$$P_{N_1}(\varphi) = \inf_{\overline{m}(\xi\varphi)\leqslant 1, \xi>0} \frac{1}{\xi} \quad \text{for } \varphi \in \overline{S}.$$

Therefore P_{N_1} is the second modular norm of $\overline{m}$ by (11).

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