Homeomorphisms of the open disk

by

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In admiration
of the distinguished contributions
of Professors
Mazur and Orlicz

The main concerns of this brief note are homeomorphisms of planar sets and their implications for a special fixed point theorem for measure invariant homeomorphisms. Although the restriction to homeomorphisms is a strong restriction, this is partially balanced by the fact that the space is not compact. The result is perhaps valid even if the condition of orientability preservation be dropped. However, this possibility though discussed is not settled. The bar to extension of the theorem to high-dimensional spaces is taken up.

First we explain our usage. A measure \( \mu \) needs merely be finite and positive for open bounded sets, but not necessarily completely additive. A simple line \( L \) is a non-self-intersecting homeomorph of the real axis such that the image of a non-Cauchy sequence of positive reals is non-Cauchy, and similarly for the negative reals (considered of order type \( \omega^\alpha \)). The open disk \( \{ |x| < 1 \} \) is denoted by \( D \) and a homeomorphism of \( D \) onto \( D \) by \( T \). The plane is indicated by \( P \) and Euclidean \( n \)-space by \( R^n \).

**Theorem.** Let \( T \) be a measure preserving and orientation preserving homeomorphism of \( D \) onto \( D \). Then \( T \) has a fixed point in \( D \).

Suppose the theorem false.

Let \( \lambda \) be a homeomorphism \( D \xrightarrow{\lambda} P \) of \( D \) onto the plane \( P \). Then

\[
(t = \lambda T k^{-1}, \quad \tau^n = \lambda T^n k^{-1}, \quad n = \pm 1, \pm 2, \ldots)
\]

are orientation preserving homeomorphisms on \( P \) onto \( P \).

With a view to possible generalization it may be remarked that the proof only requires existence of a set \( A \) with non-empty interior, a non-

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finite subcollection of whose transformation iterates, \( \tau^t A \), are disjunct, \( n = \pm 1, \pm 2, \ldots \). For the hypotheses at hand much more is available. Indeed, it is known [1] that since \( t \) is an orientation preserving homeomorphism, there is a simple line \( L_0 \) with the following properties: Let \( L_0 = \tau L_0 \). The four lines \( L_0, L_1, L_2, L_3 \) are simple and are pairwise disjunct. Let \( S_t \) be the open strip with boundaries \( L_0 \) and \( L_1 \). Let \( S_t = \tau t S_0, t = \pm 1, \pm 2, \ldots \).

(b) \[ S_0 \cap S_1 = \emptyset, \quad S_0 \cap S_{-1} = \emptyset, \quad S_1 \cap S_{-1} = \emptyset. \]

The strip \( S_t \) is referred to as a translation field. Remark for tacit use below that \( S_1 \) is connected as a homeomorph of \( S_2 \). \( S_2 \) is bounded by \( L_0 \) and \( tL_0 \). Note that

(c) \[ L_0 \cap L_1 = \emptyset, \quad S_1 \cap S_0 = \emptyset, \quad S_2 \cap S_1 = \emptyset. \]

Otherwise application of \( t^{-1} \) or \( t^{-1} \) yields the contradictions (with (b)):

\[ L_0 \cap L_1 \neq \emptyset, \quad S_1 \cap S_0 \neq \emptyset, \quad S_2 \cap S_1 \neq \emptyset. \]

From connectivity considerations

(d) \[ S_0 \cap S_{-1} = \emptyset. \]

Thus suppose (d) false. By the Jordan curve separation theorem \( L_0 \) separates the component \( K(-1) \) containing \( S_1 \) from that, \( K(0) \), containing \( S_0 \). \( L_0 \) is a common boundary for \( S_1 \) and \( S_2 \), so

(e) \[ S_2 \cap K(0) \neq \emptyset. \]

The negation of (d) implies

(f) \[ S_0 \cap L_0 \neq \emptyset. \]

However, \( L_0 \) is in the open part of \( \bar{S}_{-1} \cap \bar{S}_0 \) whence (e) and (f) imply

(g) \[ S_0 \cap S_0 \neq \emptyset \]

contrary to (d).

By induction, if \( S_{m+1} \cap S_n = \emptyset, m > n > -1, \) then

(h) \[ S_{m+1} \cap S_0 = \emptyset, \quad m+1 > j > -1. \]

Indeed, application of \( t^{-n} \) shows that

\( \tau^{n+1} S_2 \cap \tau^n S_0 = \emptyset. \)

A connectivity argument as in (e) and (f) then establishes (h). Moreover, similar arguments establish (h) for negative \( m \) or \( n \) and hence

(i) \[ S_m \cap S_n = \emptyset, \quad m \neq n. \]

Let \( J = h^{-1} S_0 \subset D \); whence by (a), \( T^* J = h^{-1} T S_0 \subset D \). Since \( S_0 \) is open, \( J \) is open. Accordingly

\[ \mu(T^* J) = \mu(J) > 0. \]

It follows that \( \mu(D) = \infty \), contradicting the assumption \( \mu(D) < \infty \). Hence \( T \) has a fixed point.

Corollary. If \( T \) is a measure preserving homeomorphism on \( D \to D \), then \( T^n \) has a fixed point.

If \( T \) is orientation preserving, this is a consequence of the theorem.

If not, \( T^n \) is orientation preserving.

Remark. For the validity of the theorem it would be enough to have

\[ \mu(T^m A) \geq \frac{1}{m} \mu(A). \]

In general (cf. Example 2),

\[ \bigcup_{n} S_n \neq P \quad \text{and} \quad \bigcup_{n} T^n J \neq D. \]

Of course, the fixed point guaranteed by the theorem need not be at the origin even if it is unique. An elementary illustration is afforded by the Möbius transformation. Thus let \( \alpha \) have modulus inferior to 1. Consider the circle \( C \) with center at the origin defined by

\[ \frac{z - \alpha}{z - \alpha^{-1}} = |\alpha|. \]

Let \( T \) be the homeomorphism represented by

\[ \frac{w - \alpha}{w - \alpha^{-1}} = e^{\theta} \frac{z - \alpha}{z - \alpha^{-1}} \]

for some \( \theta, 0 < \theta < 2\pi \). Then \( T \) has the unique fixed point \( \alpha \).

It is of interest to list some examples of translation fields with a view to possible weakening of the orientation preservation condition.

Example 1. Translation. For a translation along the \( x \)-axis of amount \( a, a > 0 \), an \( L_0 \) is a simple line cut only once by each line parallel to the \( x \)-axis. Then \( L_1 = (a \cdot L_0, L_0) \) is a displacement of magnitude \( a \) of \( L_0,

\[ \bigcup S_i = P. \]

Example 2. \( \bigcup S_i \neq P \). Let \( U \) be the strip \( 0 \leq y \leq 3 \).

Define \( t \) by

\[ t(x, y) = \begin{cases} (x+2(y-2), 2y-3) & \text{for } 2 \leq y \leq 3, \\ (x+1/3(y-2), y/2) & \text{for } 0 \leq y \leq 2. \end{cases} \]
Extend \( t \) to \( 3 < y < 6 \) by reflecting in \( y = 3 \) and similarly to all \( x \) and \( y \).

Measure angles in the counter-clockwise direction from the negative \( x \)-axis. Thus \( t(x, y) \) is a translation of magnitude \(-1\) and \( t(x, 3) \) a translation of magnitude \( 2 \).

The displacement vector
\[
v = t(x, y) - (x, y)
\]
rotates \( \pi \) in going from \( y = 0 \) to \( y = 3 \). Define \( L_1 \) in \( U \) by
\[
L_1 = \begin{cases} 
(x, 2) & \text{for } -\infty < x < 0, \\
(x, x+1) & \text{for } 0 < x < 2, \\
(x, 1) & \text{for } -\infty < x < 0.
\end{cases}
\]

Then \( L_1 = tL_0 \) is represented by
\[
L_1 = \begin{cases} 
(x, x+1) & \text{for } 0 < x < 2, \\
(x, 1) & \text{for } -\infty < x < 0.
\end{cases}
\]

Extend the definition of \( L_0 \) to \( 3 < y < 6 \) by replacing \( y \) by \( 6 - y \). Hence \( S_0 \) lies entirely in \( 0 < y < 6 \). Note that the segment
\[
A = \{(0, y) | 0 < y < 3\}
\]
meets \( t^nA \) for all \( n > 0 \). Note that
\[
\bigcup S_n = \{(x, y) | -\infty < x < \infty, 0 < y < 6\}.
\]

Another possible field is obtained by taking \( L_0 = c, a \) constant \( \neq 0 \) mod 3.

Suppose in fact \( 0 < c < 3 \). Then \( L_1 \) lies below \( L_0 \) and \( L_0 \) approaches \( y = 0 \) asymptotically for \( t \to \infty \) and \( y = 3 \) for \( t \to -\infty \),
\[
\bigcup S_n = U.
\]

For the next four examples the homeomorphisms are understood to be orientation preserving homeomorphisms denoted by \( t \) of the type \( t = at \), where \( t \) is an orientation preserving homeomorphism symmetrical with respect to the \( x \)-axis and \( a \) is reflection in the \( x \)-axis. Use \( L^+ \) and \( L^- \) to indicate lines in \( y > 0 \) and in \( y < 0 \) respectively.

**Example 3.** Let \( t \) be the unit translation as in Example 1. For instance, let \( L_0 = (0, y) \) and \( L_1 = (1, y) \). \( S_0 \) consists of one component. Evidently for an arbitrary point \( p \) there is a translation field \( S_0 \) including \( p \) as an interior point.

**Example 4.** Replace \( y \) by \( y - 3 \) in Example 2. Then \( L_0 \) and \( L_1 \) and \( S_0 \) are from the first translation field in Example 2, except that \( t \) maps \( L_0^+ \) into \( (tL_0)^- \) and \( L_0^- \) into \( (tL_0)^+ \).

**Example 5.** \( S_0 \) consists of two components. In Example 2 identify \( L_0, L_1 \) and \( S_0 \) with the new \( L_0^+ \), \( L_1^- \) and \( S_0^+ \) and \( L_0^- \), \( L_1^+ \), \( S_0^- \) are their reflections in the \( x \)-axis. Here
\[
(tL_0^+)^+ = (L_1^-)^-.
\]

Thus \( S_0 \) consists of two components \( S_0^+ \) and \( S_0^- \).

Similarly for the second translation field in Example 2 when \( L_0^+ \) is identified with \( L_0^+ \) and \( L_0^- \) with \( L_0^- \). Hence there are two components \( S_0 = S_0^+ \cup S_0^- \).

Note for either \( S_0^+ \) or \( S_0^- \) in this example the boundary of each component consists of the map of part of the boundary of the other.

**Example 6.** Sets of points admitting no containing translation field. Let \( A \) be a closed subset of \( P \) with
\[
A \cap P^+ = \emptyset
\]
and suppose that \( A \) constitutes the fixed point set of the homeomorphism \( t \) on \( P^+ \) onto \( P^+ \). The argument in the proof of the theorem indicates \( S_0 \cap S_0 = \emptyset \). Here no point of \( A \) can be interior to \( S_0 \) for from \( P^+ = A \) follows
\[
P^+(S_0) \cap S_0 \neq \emptyset
\]
if \( S_0 = a \times A \). The question of restrictions on an \( A \) so that it can comprise all of the fixed points of \( t \) is equivalent to that of existence of an extension of the identity map on \( A \) to a fixed point free homeomorphism on \( P^+ \) and it and its generalizations to other spaces constitutes a problem of independent interest particularly when \( A \) is not bounded.

The natural question is whether a similar theorem is true for open disks of arbitrary dimension. The following discussion answers this in the negative for certain composite \( a \) values and in particular for \( D^{15} \). Thus, it is known [2] that for every \( r \neq p^m \), \( p \) a prime, \( m > 1 \), there is a cyclic group of order \( r \) acting on \( P^+ \) with no fixed point. (The smallest odd \( r \) value is 15.) Let \( m \) be the Lebesgue measure on \( D \). Define a new measure \( \mu \) on \( D = D^{15} \) by
\[
\mu(A) = \frac{1}{15} (mA + \ldots mT^{14}A).
\]

Evidently \( \mu \) is measure preserving.

Let \( t, h, k \) be related analogously to \( (a) \).

By (1), \( T \) is \( \mu \) invariant yet has no fixed point.

If \( A \) is a set of zero Lebesgue measure \( m \), then \( TA \) may have non-zero Lebesgue measure. Accordingly though it is absolutely continuous in \( \mu \), the converse need not be true, so \( \mu \) may not be equivalent to Lebesgue measure. Accordingly the counterexample above...
has not been shown to apply to Lebesgue measure, but only to some measure (though conceivably detailed consideration of this counterexample may establish $\mu \sim \mu$).

In any case the question is still open as to whether the theorem is valid for low dimensions or for a prime, $n > 2$. Of course, the translation field theorem does not hold, but as pointed out this asserts a stronger conclusion than necessary. Another question is whether, when orientation invariance is dropped, there always exists a point interior to a translation field at least in the case that $i$ is obtained by reflecting $t$ in $y = 0$. If so, Theorem 1 would apply without the restriction of orientation preservation.

References


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