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Homeomorphisms of the open disk*

by

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In admiration
of the distinguished contributions
of Professors
Mazur and Orlicz

The main concerns of this brief note are homeomorphisms of planar sets and their implications for a special fixed point theorem for measure invariant homeomorphisms. Although the restriction to homeomorphisms is a strong restriction, this is partially balanced by the fact that the space is not compact. The result is perhaps valid even if the condition of orientability preservation be dropped. However, this possibility though discussed is not settled. The bar to extension of the theorem to high-dimensional spaces is taken up.

First we explain our usage. A *measure* μ needs merely be finite and positive for open bounded sets, but not necessarily completely additive. A *simple line* L is a non-self-intersecting homeomorph of the real axis such that the image of a non-Cauchy sequence of positive reals is non-Cauchy, and similarly for the negative reals (considered of order type ω^*). The open disk $\{x \mid \|x\| < 1\}$ is denoted by D and a homeomorphism of D onto D by T. The *plane* is indicated by P and Euclidean n-space by R^n.

THEOREM. *Let T be a measure preserving and orientation preserving homeomorphism of D onto D. Then T has a fixed point in D.*

Suppose the theorem false.

Let h be a homeomorphism $D \xrightarrow{h} P$ of D onto the plane P. Then

$$\text{(a)} \qquad t = hTh^{-1}, \qquad t^n = hT^n h^{-1}, \quad n = \pm 1, \pm 2, \ldots,$$

are orientation preserving homeomorphisms on P onto P.

With a view to possible generalization it may be remarked that the proof only requires existence of a set A with non-empty interior, a non-

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finite subcollection of whose transformation iterates, $t^n A$, are disjunct, $n = \pm 1, \pm 2, \ldots$ For the hypotheses at hand much more is available. Indeed, it is known [1] that since t is an orientation preserving homeomorphism, there is a simple line L_0 with the following properties: Let $L_i = t^i L_0$. The four lines L_0, L_1, L_2, L_{-1} are simple and are pairwise disjunct. Let S_0 be the open strip with boundaries L_0 and L_1. Let $S_i = t^i S_0$, $i = \pm 1, \pm 2, \ldots$; so

(b) $$S_0 \cap S_1 = \emptyset, \quad S_0 \cap S_{-1} = \emptyset, \quad S_1 \cap S_{-1} = \emptyset.$$

The strip S_0 is referred to as a *translation field*. Remark for tacit use below that S_i is connected as a homeomorph of S_0. S_2 is bounded by L_2 and tL_2. Note that

(c) $$L_2 \cap tL_2 = \emptyset, \quad S_2 \cap S_1 = \emptyset, \quad S_2 \cap S_0 = \emptyset.$$

Otherwise application of t^{-2} or t^{-1} yields the contradictions (with (b)):

$$L_0 \cap L_1 \neq \emptyset, \quad S_1 \cap S_0 \neq \emptyset, \quad S_1 \cap S_{-1} \neq \emptyset.$$

From connectivity considerations

(d) $$S_2 \cap S_{-1} = \emptyset.$$

Thus suppose (d) false. By the Jordan curve separation theorem L_0 separates the component $K(-1)$ containing S_{-1} from that, $K(0)$, containing S_0. L_2 is a common boundary for S_1 and S_2, so

(e) $$S_2 \cap K(0) \neq \emptyset.$$

The negation of (d) implies

(f) $$S_2 \cap L_0 \neq \emptyset.$$

However, L_0 is in the open part of $\bar{S}_{-1} \cap \bar{S}_0$ whence (e) and (f) imply

(g) $$S_2 \cap S_0 \neq \emptyset$$

contrary to (d).

By induction, if $S_m \cap S_n = \emptyset$, $m > n > -1$, then

(h) $$S_{m+1} \cap S_j = \emptyset, \quad m+1 > j > -1.$$

Indeed, application of t^{-m} shows that

$$t^{m+1} S_0 \cap t^m S_0 = \emptyset.$$

A connectivity argument as in (e) and (f) then establishes (h). Moreover, similar arguments establish (h) for negative m or n and hence

(i) $$S_m \cap S_n = \emptyset, \quad m \neq n.$$

Let $J = h^{-1} S_0 \subset D$; whence by (a), $T^i J = h^{-1} t^i S_0 \subset D$. Since S_0 is open, J is open. Accordingly

$$\mu(T^i J) = \mu(J) > 0.$$

It follows that $\mu(D) = \infty$, contradicting the assumption $\mu(D) < \infty$. Hence T has a fixed point.

COROLLARY. *If T is a measure preserving homeomorphism on $D \to D$, then T^2 has a fixed point.*

If T is orientation preserving, this is a consequence of the theorem. If not, T^2 is orientation preserving.

Remark. For the validity of the theorem it would be enough to have

$$\mu T^m A \geqslant \frac{1}{n} \mu(A).$$

In general (cf. Example 2),

$$\bigcup_{-\infty}^{\infty} \bar{S}_m \neq P \quad \text{and} \quad \bigcup_{-\infty}^{\infty} \overline{T^i J} \neq D.$$

Of course, the fixed point guaranteed by the theorem need not be at the origin even if it is unique. An elementary illustration is afforded by the Möbius transformation. Thus let a have modulus inferior to 1. Consider the circle C with center at the origin defined by

$$\left| \frac{z-a}{z-\bar{a}^{-1}} \right| = |a|.$$

Let T be the homeomorphism represented by

$$\frac{w-a}{w-\bar{a}^{-1}} = e^{i\theta} \frac{z-a}{z-\bar{a}^{-1}}$$

for some θ, $0 < \theta < 2\pi$. Then T has the unique fixed point a.

It is of interest to list some examples of translation fields with a view to possible weakening of the orientation preservation condition.

Example 1. *Translation.* For a translation along the x-axis of amount a, $a > 0$, an L_0 is a simple line cut only once by each line parallel to the x-axis. Then L_1 is a displacement of magnitude a of L_0,

$$\bigcup \bar{S}_i = P.$$

Example 2. $\bigcup \bar{S}_i \neq P$. Let U be the strip $0 \leqslant y \leqslant 3$. Define t by

$$t(x, y) = \begin{cases} (x+2(y-2),\, 2y-3) & \text{for } 2 \leqslant y \leqslant 3, \\ (x+1/2(y-2),\, y/2) & \text{for } 0 \leqslant y \leqslant 2. \end{cases}$$

Extend t to $3 \leqslant y \leqslant 6$ by reflecting in $y=3$ and similarly to all x and y.

Measure angles in the counter-clockwise direction from the negative x-axis. Thus $t(x,0)$ is a translation of magnitude -1 and $t(x,3)$ a translation of magnitude 2.

The displacement vector

$$v = t(x,y)-(x,y)$$

rotates through π in going from $y=0$ to $y=3$. Define L_0 in U by

$$L_0 = \begin{cases}(0,y) & \text{for } 2 \leqslant y \leqslant 3,\\ (x,2) & \text{for } -\infty < x \leqslant 0.\end{cases}$$

Then $L_1 = tL_0$ is represented by

$$L_1 = \begin{cases}(x,x+1) & \text{for } 0 \leqslant x \leqslant 2,\\ (x,1) & \text{for } -\infty < x \leqslant 0.\end{cases}$$

Extend the definition of L_0 to $3 \leqslant y \leqslant 6$ by replacing y by $6-y$. Hence S_0 lies entirely in $0 \leqslant y \leqslant 6$. Note that the segment

$$A = \{(0,y)\,|\,0 \leqslant y \leqslant 3\}$$

meets $t^n A$ for all $n \geqslant 0$. Note that

$$\bigcup \bar{S}_n = \{(x,y)\,|\,-\infty < x < \infty,\ 0 \leqslant y \leqslant 6\}.$$

Another possible field is obtained by taking $L_0 = c$, a constant $\neq 0$ mod 3.

Suppose in fact $0 < c < 3$. Then L_1 lies below L_0 and L_i approaches $y=0$ asymptotically for $i \to \infty$ and $y=3$ for $i \to -\infty$,

$$\bigcup_{-\infty}^{\infty} \bar{S}_n = U.$$

For the next four examples the homeomorphisms are understood to be orientation reversing homeomorphisms denoted by $\bar{t}$ of the type $\bar{t} = at$, where t is an orientation preserving homeomorphism symmetrical with recpect to the x-axis and a is reflection in the x-axis. Use L^+ and L^- to indicate lines in $y \geqslant 0$ and in $y \leqslant 0$ respectively.

Example 3. Let t be the unit translation as in Example 1. For instance, let $L_0 = (0,y)$ and $L_1 = (1,y)$. S_0 consists of one component. Evidently for an arbitrary point p there is a translation field S_0 including p as an interior point.

Example 4. Replace y by $y-3$ in Example 2. Then L_0 and L_1 and S_0 are from the first translation field in Example 2, except that t maps L_0^+ into $(tL_0)^-$ and L_0^- into $(tL_0)^+$.

Example 5. S_0 *consists of two components.* In Example 2 identify L_0, L_1 and S_0 with the new L_0^+, L_1^+ and S_0^+ and L_0^-, L_1^-, S_0^- are their reflections in the x-axis. Here

$$\bar{t}L_0^{\pm} = L_1^{\mp}.$$

Thus S_0 consists of two components S_0^+ and S_0^-.

Similarly for the second translation field in Example 2 when L_0^+ is identified with $L_0 = c$, $0 < c < 3$. Again there are two components $S_0 = S_0^+ \cup S_0^-$.

Note for either S_0^- or S_0^+ in this example the boundary of each component consists of the map of part of the boundary of the other.

Example 6. *Sets of points admitting no containing translation field.* Let A be a closed subset of P with

$$A \cap P^- = \emptyset$$

and suppose that A constitutes the fixed point set of the homeomorphism t on P^+ onto P^+. The argument in the proof of the theorem indicates $S_2 \cap S_0 = \emptyset$. Here no point a of A can be interior to S_0 for from $\bar{t}^2 a = a$ follows

$$\bar{t}^2(S_0) \cap S_0 \neq \emptyset$$

if $S_0 \supset a \in A$. The question of restrictions on an A so that it can comprise all of the fixed points of t is equivalent to that of existence of an extension of the identity map on A to a fixed point free homeomorphism on P^+ and it and its generalizations to other spaces constitutes a problem of independent interest particularly when A is not bounded.

The natural question is whether a similar theorem is true for open disks of arbitrary dimension. The following discussion answers this in the negative for certain composite n values and in particular for D^{135}. Thus, it is known [2] that for every $r \neq p^m$, p a prime, $m > 1$, there is a cyclic group of order r acting on R^{9r} with no fixed point. (The smallest odd $9r$ value is 135.) Let m be the Lebesgue measure on D. Define a new measure μ on $D = D^{135}$ by

$$\mu(A) = \frac{1}{15}(mA + \ldots mT^{14}A). \tag{1}$$

Evidently μ is measure preserving.

Let t, h and T be related analogously to (a).

By (1), T is μ invariant yet has no fixed point.

If A is a set of zero Lebesgue measure mA, its homeomorph TA may have non-zero Lebesgue measure. Accordingly though m is absolutely continuous in μ, the converse need not be true, so μ may not be equivalent to Lebesgue measure. Accordingly the counterexample above has not been shown to apply to Lebesgue measure, but only to some measure (though conceivably detailed consideration of this counterexample may establish $\mu \sim m$).

In any case the question is still open as to whether the theorem is valid for low dimensions or for n a prime, $n>2$. Of course, the translation field theorem does not hold, but as pointed out this asserts a stronger conclusion than necessary. Another question is whether, when orientation invariance is dropped, there always exists a point interior to a translation field at least in the case that $\bar{t}$ is obtained by reflecting t in $y=0$. If so, Theorem 1 would apply without the restriction of orientation preservation.

References

[1] L. E. J. Brouwer, *Beweis des ebenen Translationssatzes*, Math. Annalen 72 (1912), p. 37-54.

[2] J. M. Kister, *Examples of periodic maps on Euclidean spaces without fixed points*, Bull. Amer. Math. Soc. 67 (1961), p. 471-474.

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