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Raikov systems and the pathology of $M(R)$

by

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Dedicated to
Professor Stanisław Mazur
and
Professor Władysław Orlicz

This paper may be regarded as a continuation of [8], where some preliminary results on Raikov systems and their applications were given. We shall assume here the basic results of that paper. The main result to be proved here (Theorem 2) was conjectured in [8] (Proposition 9′ of that paper) and generalises Proposition 9 of [8], which was stated without proof. In general terms, what we prove is that the pathological features of the measure algebra $M(R)$ of the real line R are in a certain sense uniformly spread throughout the algebra. If A and B are two subalgebras of $M(R)$, of a certain type, with A properly contained in B, then the phenomena associated with the names of Wiener and Pitt, which have been known for many years [4], [6] to occur between the atomic measures $M_a(R)$ and the whole measure algebra $M(R)$, occur also between A and B. The techniques used to apply to the measure algebra information available about Raikov systems are those described in [7], and we assume the results of § 1 of that paper. The main theorems of the present paper are generalisations (in the special case $G = R$) of Theorems 2.3 and 2.5 of [7].

In order to simplify the exposition we maintain the restriction (observed in [8]) of stating and proving results for the case of the real line R only. In many cases the extension to a general locally compact abelian group is straightforward, but there are some others where the difficulties are more substantial, and we reserve a full discussion of the general case for another occasion.

We begin by recalling the basic definition. A subset of R is of type F_σ if it is a countable union of compact sets. A collection $\mathscr{F}$ of subsets of R, of type F_σ, is a *Raikov system* if the following properties hold:

R1. If $A_1 \in \mathscr{F}$ and A_2 is a subset of A_1, of type F_σ, then $A_2 \in \mathscr{F}$;

R2. The union of a countable collection of sets in $\mathscr{F}$ is also in $\mathscr{F}$;

R3. If $A \in \mathscr{F}$ and $t \in R$, then $A - t \in \mathscr{F}$;

R4. If $A \in \mathscr{F}$, then $A + A \in \mathscr{F}$.

(In the above, $+$ and $-$ denote the group operations in R, not set-theoretic union and difference). If there holds also the condition:

R5. If $A \in \mathscr{F}$, then $-A \in \mathscr{F}$.

we shall say that the Raikov system $\mathscr{F}$ is *symmetric*.

For some elementary consequences of the definition see [1], §§ 31-33 and [8].

The lemma that follows is no doubt well known, but we state it explicitly, as we are unable to give a reference. The proof is straightforward.

LEMMA 1. *Let A be a compact subset of R; then either A is countable or we can write $A = A_1 \cup A_2$, where A_1 is perfect and A_2 is countable.*

COROLLARY. *Each generator of a Raikov system other than the minimal system $\mathscr{F}_0$ (consisting of all countable sets) may be assumed to be a perfect set.*

Proof. This follows at once by combining Lemma 1 with Proposition 4 (i) of [8].

We adopt the convention that a set A will be said to be *nowhere dense*, or *of the first category*, or *of the second category*, in a set B, even though A is not a subset of B, provided that $A \cap B$ has the property in question.

LEMMA 2. *The Raikov system $\mathscr{F}_1$ is properly contained in the system $\mathscr{F}_2$ if and only if there exists a compact set $B \in \mathscr{F}_2$ such that each set $A \in \mathscr{F}_1$ is of the first category in B.*

Proof. If $\mathscr{F}_1 = \mathscr{F}_2$ or $\mathscr{F}_1 \supset \mathscr{F}_2$, then each compact set B in $\mathscr{F}_2$ is also in $\mathscr{F}_1$. Since B is a compact subset of R, it is complete metric, and hence of the second category in itself. Thus there exists a set A $(= B)$ in $\mathscr{F}_1$ such that A is not of the first category in B.

If, on the other hand, $\mathscr{F}_1 \subset \mathscr{F}_2$ and $\mathscr{F}_1 \neq \mathscr{F}_2$, suppose that for each compact $B \in \mathscr{F}_2$ there exists $A \in \mathscr{F}_1$ such that A is of the second category in B. Fix such a set B, and let the corresponding set A be written as a countable union of compact (hence closed) sets A_r $(r = 1, 2, \ldots)$. At least one of these sets A_r is dense in some open subset of B; there exists an open interval $I_{r,A}$ such that $B \cap I_{r,A} \subset A_r$.

Let now B_0 be the subset of B consisting of points x which belong to no such interval $I_{r,A}$ for any possible choice of A. We may write

$$B_0 = B \setminus \bigcup_{r,A} (I_{r,A} \cap B),$$

and it follows that B_0 is closed in B, and hence compact. If B_0 is not empty, then, by the argument already applied to B, there must exist some set $A_0 \in \mathscr{F}_1$ with A_0 of the second category in B_0. There would then exist $x \in B_0$, with x contained in some interval I_{r,A_0} as above, so that B_0 would not be minimal. Thus B_0 is empty, and every $x \in B$ is contained in some $I_{r,A}$.

Since B is compact, it is covered by a finite set of the intervals $I_{r,A}$. Since $I_{r,A} \subset A_r$, B is covered by a finite collection of sets A_r, all of which are in $\mathscr{F}_1$; it follows that B itself is in $\mathscr{F}_1$. Thus an arbitrary compact set in $\mathscr{F}_2$ is necessarily in $\mathscr{F}_1$; but by Proposition 4 (i) of [8] each generator of $\mathscr{F}_2$ may be taken to be compact, hence each generator of $\mathscr{F}_2$ is in $\mathscr{F}_1$ and so $\mathscr{F}_2 \subset \mathscr{F}_1$. We have thus established a contradiction, and so there must exist $B \in \mathscr{F}_2$ such that no set $A \in \mathscr{F}_1$ is of the second category in B.

COROLLARY. *In Lemma 2, the set B may be assumed to be perfect.*

Proof. This follows at once from the Corollary to Lemma 1.

We now describe two generalisations of the basic definition of independence, for subsets of R. Suppose that E is a given subset of R; we say (as in [8]) that a subset X of R is *independent with respect to E*, or *E-independent*, if the relation

$$\sum_{r=1}^{N} n_r x_r \in E, \tag{1}$$

where $n_1, \ldots, n_N$ are integers and $x_1, \ldots, x_N$ are distinct elements of X, is possible only if $0 \in E$ and $n_1 = \ldots = n_N = 0$. This reduces to the standard definition of independence if $E = \{0\}$; it is clearly in general a more restrictive condition.

A less restrictive condition than independence is obtained if we require that the coefficients n_r should satisfy an inequality of the form

$$|n_r| \leqslant k \quad (1 \leqslant r \leqslant N). \tag{2}$$

We may say that X is *independent up to order k* if the relation

$$\sum_{r=1}^{N} n_r x_r = 0,$$

where $n_1, \ldots, n_N$ are integers satisfying (2) and $x_1, \ldots, x_N$ are distinct elements of X, is possible only if $n_1 = \ldots = n_n = 0$. This concept has already been used in the case $k = 2$ by Hewitt and Zuckerman [2], who call such a set *dissociate*.

These two generalisations may be combined; we shall call the set X *E-independent up to order k* or, more briefly, *(E, k)-independent*, if relation (1), where $n_1, \ldots, n_N$ are integers satisfying (2) and $x_1, \ldots, x_N$

are distinct points of X, is possible only if $0 \in E$ and $n_1 = \ldots = n_N = 0$. It is clear that in general (E, k)-independence neither implies nor is implied by ordinary independence; we have weakened the condition in one way and strengthened it in another.

PROPOSITION 1. *Let A be a given perfect subset of R, and E a set such that $E-x$ and $-E-x$ are of the first category in A for each $x \in R$. Then there exists a perfect $(E, 1)$-independent subset of A.*

Proof. Suppose that $E = E_1 \cup E_2 \cup \ldots$, where each set $\pm E_r - x$ is nowhere dense in A. We may assume without loss of generality that $E_1 \subset E_2 \subset \ldots$

We may now imitate the standard construction (see, e.g., [3], p. 20-21). At the first stage, choose two disjoint closed intervals $B_1^{(1)}$ and $B_2^{(1)}$, of length not exceeding 1, such that $B_1^{(1)} \cap A$ and $B_2^{(1)} \cap A$ are perfect, and such that $(B_1^{(1)} \cap A) \times (B_2^{(1)} \cap A)$ does not cut any line

$$n_1 x_1 + n_2 x_2 = a_1,$$

where $a_1 \in E_1$ and $|n_1| \leqslant 1$, $|n_2| \leqslant 1$.

At the j-th stage, if intervals $B_1^{(j-1)}, \ldots, B_{2^{j-1}}^{(j-1)}$ are initially present, choose $B_1^{(j)}$ and $B_2^{(j)}$ to be closed subintervals of $B_1^{(j-1)}, \ldots, B_{2^j-1}^{(j)}$ and $B_{2^j}^{(j)}$ to be closed subintervals of $B_{2^{j-1}}^{(j-1)}$, so that

(i) the intervals $B_r^{(j)}$ are disjoint;

(ii) each $B_r^{(j)}$ is of length not exceeding j^{-1};

(iii) each set $B_r^{(j)} \cap A$ is perfect;

(iv) the set $(B_1^{(j)} \cap A) \times \ldots \times (B_{2^j}^{(j)} \cap A)$ does not intersect any hyperplane

$$n_1 x_1 + \ldots + n_{2^j} x_{2^j} = a_j,$$

where $a_j \in E_j$ and $|n_r| \leqslant 1$ for $1 \leqslant r \leqslant 2^j$.

This is always possible. To show this, let S be a fixed selection of the integers $n_1, \ldots, n_{2^j}$, subject to $|n_r| \leqslant 1$ for $1 \leqslant r \leqslant 2^j$; let the map f_S of R^{2^j} to R be defined by

$$f_S(x_1, \ldots, x_{2^j}) = n_1 x_1 + \ldots + n_{2^j} x_{2^j}.$$

Then the inverse image under f_S of the set E_j is nowhere dense in A^{2^j}. For, if $f_S^{-1}(E_j)$ were dense in some neighbourhood N of $(a_1, \ldots, a_{2^j}) \in A^{2^j}$, we could suppose without loss of generality that N had the form

$$(J_1 \cap A) \times \ldots \times (J_{2^j} \cap A),$$

where each J_r is an interval such that $J_r \cap A$ is perfect. If then r_0 is chosen so that $n_{r_0} \neq 0$, and a_r is fixed in $J_r \cap A$ for $r \neq r_0$, the set

$$n_{r_0} E_j - \sum_{r \neq r_0} n_r a_r$$

is dense in $J_{r_0} \cap A$, which is a contradiction. It follows that the union of the finite collection of sets $f_S^{-1}(E_j)$, as S varies through the possible selections of the integers $n_1, \ldots, n_{2^j}$, subject to $|n_r| \leqslant 1$ for $1 \leqslant r \leqslant 2^j$, is nowhere dense in the set A^{2^j}. It follows that intervals $B_r^{(j)}$ can be chosen so as to satisfy (i)-(iv) above.

If we now write

$$A^{(j)} = \bigcup_{r=1}^{2^j} B_r^{(j)} \cap A \quad \text{and} \quad A' = \bigcap_{j=1}^{\infty} A^{(j)},$$

then clearly A' is a closed subset of A. It is perfect, since each open set that contains a point of A' must contain some set $B_r^{(j)} \cap A$, and hence infinitely many points of A'. Given N distinct points of A, since the lengths of the intervals $B_r^{(j)}$ tend to zero as j tends to infinity, it follows that if j is large enough these N points will be in N distinct sets of the form $B_r^{(j)} \cap A$. Hence no linear relation of the form

$$\sum_{r=1}^{N} n_r x_r = a$$

can hold if $a \in E_j$ (for j sufficiently large) and $|n_r| \leqslant 1$ $(1 \leqslant r \leqslant N)$. But this implies that no linear relation of the form indicated can hold if $a \in E$ and $|n_r| \leqslant 1$ $(1 \leqslant r \leqslant N)$, which is what was required.

Remark. We could obviously secure by the same construction an E-independent subset of A if we made the assumption that, for each $x \in R$ and positive integer n, the sets $n^{-1}E - x$ and $-n^{-1}E - x$ are of the first category in A.

We now turn to the application of Raikov systems to the algebra $M(R)$, that is to say, the bounded complex Borel measures of finite total mass on R. For an account of the basic definitions and properties of $M(R)$ see, for example [5], § 1.3 (where the case of a general locally compact abelian group is treated), or [1], Chapter V (the ring V^b treated there is in all respects equivalent to $M(R)$). We also require the basic results on Banach algebra elements with independent powers, as given in [7], and we use the notation and terminology of that paper without further explanation. We denote by $M(\mathscr{F})$ the closed sub-algebra of $M(R)$ consisting of measures that are concentrated on the Raikov system $\mathscr{F}$. We denote by $M(\mathscr{F})^{\perp}$ the complementary ideal of $M(\mathscr{F})$, consisting of all measures that are singular with respect to all measures in $M(\mathscr{F})$. As a consequence of the existence of a complementary ideal, it follows that if a measure $\mu \in M(\mathscr{F})$ has an inverse $\mu^{-1} \in M(R)$, then in fact $\mu^{-1} \in M(\mathscr{F})$. We may therefore apply the results of [7], in particular Proposition 1.7, with $\mathscr{F}$ the class of translations and inversions.

PROPOSITION 2. *Let $\mathscr{F}$ be a proper symmetric Raikov system with a single generator; let H be a group that generates $\mathscr{F}$. Let $\{P_i\}$ be a disjoint* *collection of subsets of R, with $P = \bigcup P_i$ $(H, 1)$-independent, and for each i let μ_i be a continuous measure concentrated on $Q_i = P_i \cup (-P_i)$. Then the set of measures $\{\mu_i\}$ has independent powers with respect to $M(\mathscr{F})$ and is completely decomposable.*

Proof. We may assume without loss of generality that the index-set is finite, say $i = 1, 2, \ldots, N$. Write $\mu = \mu_1 + \ldots + \mu_N$ and $Q = P \cup (-P)$ let $(\nu)_z$ denote the translate of the measure ν by the real number z. We wish to show first that if $a_m, a_n \in M(\mathscr{F})$, then the measures $a_n\mu^n$ and $(a_m\mu^m)_z$ are mutually singular in the two cases (i) $m \neq n$, or (ii) $m = n$, $z \neq 0$. We may clearly assume for (i) that $m < n$.

Suppose first that a_n and a_m are concentrated on H, and that $\mu_1, \ldots, \ldots, \mu_N$ are all non-negative. Then the measures $a_n\mu^n$, $(a_m\mu^m)_z$ are concentrated on the sets $H+(n)Q$, $H+(m)Q-z$ respectively. Evidently if these two sets are disjoint, the two measures are mutually singular. If the sets are not disjoint, we have

$$h_1 + x_1 + \ldots + x_n = h_2 + y_1 + \ldots + y_m - z$$

for some $h_1, h_2 \in H$ and $x_1, \ldots, x_n, y_1, \ldots, y_m \in Q$. Denote by S the set of points $(x_1, \ldots, x_n) \in R^n$ such that, for some $h \in H$, we have

$$h + x_1 + \ldots + x_n \in H + (m)Q - z;$$

since H is a group, this is the set of points $(x_1, \ldots, x_n)$ such that

$$x_1 + \ldots + x_n \in H + (m)Q - z.$$

If we can show that $(\mu\times\ldots\times\mu)(S) = 0$, then it will follow that $[a_n \times (\mu\times\ldots\times\mu)](H\times S) = 0$, which implies that $(a_n\mu^n)(H+(m)Q-z) = 0$.

Let $z = h' + y'_1 + \ldots + y'_m - x'_1 - \ldots - x'_n$; each x'_i, y'_j is of the form $\pm p'_k$, with $p'_k \in P$. Let also $x_i, y_j = \pm p_k$, with $p_k \in P$. Then, if $p_1, \ldots, p_{m+n}$ were all different, and different also from $p'_1, \ldots, p'_{m+n}$, there would be in either of the cases (i) or (ii) a non-trivial linear relation among the p_i, p'_i of the form

$$\sum \pm p_k + \sum \pm p'_k \in H,$$

which is not possible. Thus S is contained in a finite union of sets of the form

$$\begin{array}{ll} \{x\colon x_i + x_j = 0\} & (i \neq j) \\ \{x\colon x_i - x_j = 0\} & (i \neq j) \\ \{x\colon x_i = x'_j\} & (\text{any } i, j) \\ \{x\colon x_i = -x'_j\} & (\text{any } i, j) \\ \{x\colon x_i = y'_j\} & (\text{any } i, j) \\ \{x\colon x_i = -y'_j\} & (\text{any } i, j), \end{array}$$

and it is clear that these are all of $(\mu\times\ldots\times\mu)$-measure zero, since μ is continuous. It follows that $(a_n\mu^n)(H+(m)Q-z) = 0$, and so $a_n\mu^n$ and $(a_m\mu^m)_z$ are mutually singular.

We show next that if a_1 and a_2 are concentrated on H, and $r_1, \ldots, \ldots, r_N, s_1, \ldots, s_N$ are positive integers or zero, and we write

$$\lambda_1 = a_1\mu_1^{r_1}\ldots\mu_N^{r_N}, \quad \lambda_2 = a_2\mu_1^{s_1}\ldots\mu_N^{s_N},$$

then the measures λ_1 and $(\lambda_2)_z$ are mutually singular unless $(r_1, \ldots, r_N) = (s_1, \ldots, s_N)$ and $z = 0$. Let $r = r_1 + \ldots + r_N$ and $s = s_1 + \ldots + s_N$; then λ_1, λ_2 are absolutely continuous with respect to $a_1\mu^r$, $a_2\mu^s$ respectively. The mutual singularity of λ_1 and $(\lambda_2)_z$ follows from the mutual singularity of $a_1\mu^r$, $(a_2\mu^s)_z$ in the cases $r \neq s$ or $r = s$, $z \neq 0$, by what we have just proved. If $r = s$ but $(r_1, \ldots, r_N) \neq (s_1, \ldots, s_N)$ we can use a very similar argument. Since λ_1 is concerntrated on $H+(r_1)Q_1+\ldots \ldots+(r_N)Q_N$ and λ_2 on $H+(s_1)Q_1+\ldots+(s_N)Q_N$, we wish to show that if S is the subset of R^r consisting of points $(x_1, \ldots, x_r)$ such that

$$x_1, \ldots, x_{r_1} \in Q_1, \quad \ldots, \quad x_{r-r_N+1}, \ldots, x_r \in Q_N$$

and

$$x_1 + \ldots + x_r \in H + (s_1)Q_1 + \ldots + (s_N)Q_N - z,$$

then

$$(\mu_1\times\ldots\times\mu_1\times\mu_2\times\ldots\times\mu_N)(S) = 0$$

(where we have r_1 factors $\mu_1, \ldots, r_N$ factors μ_N). Since we must have $r_k > s_k$ for some k, we may apply the argument already used, but this time with Q_k in place of Q, to show that S must be contained in a finite union of sets of the kind specified. Each of these sets has $(\mu_1\times\ldots\times\mu_1\times \times\mu_2\times\ldots\times\mu_N)$-measure zero, and the required result follows.

We next relax the condition that the measures a_1 and a_2 should be concentrated on H. Since H is a group that generates $\mathscr{F}$, a_1 and a_2 must be concentrated on F_σ-subsets of countable unions of translates of H. Suppose that for $i = 1, 2$ we have

$$a_i = \sum_{k=1}^{\infty} a_{ik},$$

where a_{ik} is concentrated on $H - z_{ik}$. Then $(a_{1k})_{-z_{1k}}$ and $(a_{2j})_{-z_{2j}}$ are concentrated on H, and the measures

$$(a_{1k})_{-z_{1k}}\mu_1^{r_1}\ldots\mu_N^{r_N} \quad \text{and} \quad [(a_{2j})_{-z_{2j}}\mu_1^{s_1}\ldots\mu_N^{s_N}]_{z_{2j}-z_{1k}}$$

are mutually singular if $(r_1, \ldots, r_N) \neq (s_1, \ldots, s_N)$, which is the same as saying that $a_{1k}\mu_1^{r_1}\ldots\mu_N^{r_N}$ and $a_{2j}\mu_1^{s_1}\ldots\mu_N^{s_N}$ are mutually singular for all j and k. It follows then that the measures $a_1\mu_1^{r_1}\ldots\mu_N^{r_N}$ and $a_2\mu_1^{s_1}\ldots\mu_N^{s_N}$

are mutually singular. There is now of course no need to include explicitly the possibility of one of the measures being translated through z, since this is covered by the assumptions we have made about α_1 and α_2.

Suppose that $\alpha = \sum_{j=1}^{\infty} \alpha_j$, where α_j is concentrated on $H-z_j$ ($z_j \neq z_k$ if $j \neq k$). Then, by what we have proved above,

$$\|\alpha\mu_1^{r_1} \ldots \mu_N^{r_N}\| = \sum_{j=1}^{\infty} \|\alpha_j \mu_1^{r_1} \ldots \mu_N^{r_N}\|, \tag{3}$$

since $\alpha_j \mu_1^{r_1} \ldots \mu_N^{r_N}$ and $(\alpha_k \mu_1^{r_1} \ldots \mu_N^{r_N})_{z_k - z_j}$ are mutually singular. Moreover, if $\alpha = \sum_{j=1}^{\infty} \alpha_j$, where the α_j are concentrated on disjoint subsets of H, then the measures $\alpha_j \mu_1^{r_1} \ldots \mu_N^{r_N}$ and $\alpha_k \mu_1^{r_1} \ldots \mu_N^{r_N}$ are concentrated on disjoint sets if $j \neq k$, because of the $(H, 1)$-independence of P, so that once again (3) holds. So, if we can write $\alpha = \sum_{j=1}^{\infty} c_j \alpha_j$, where the c_j are (in general complex) constants and the measures α_j are non-negative and have disjoint supports, we have

$$\|\alpha\mu_1^{r_1} \ldots \mu_N^{r_N}\| = \Big\| \sum_{j=1}^{\infty} c_j \alpha_j \mu_1^{r_1} \ldots \mu_N^{r_N} \Big\| = \sum_{j=1}^{\infty} |c_j| \, \|\alpha_j\| \, \|\mu_1\|^{r_1} \ldots \|\mu_N\|^{r_N}$$
$$= \|\alpha\| \, \|\mu_1\|^{r_1} \ldots \|\mu_N\|^{r_N}.$$

We have used here the elementary fact that the norm of any product of non-negative measures is equal to the product of the norms of the measures. Since any measure $\alpha \in M(\mathscr{F})$ can be approximated arbitrarily closely by a sum $\sum_{j=1}^{\infty} c_j \alpha_j$, as above, it follows that the relation that we have just proved holds for and $\alpha \in M(\mathscr{F})$ and non-negative $\mu_1, \ldots, \mu_N$. It follows that for such measures we have

$$\Big\| \sum \alpha_{r_1 \ldots r_N} \mu_1^{r_1} \ldots \mu_N^{r_N} \Big\| = \sum \|\alpha_{r_1 \ldots r_N}\| \, \|\mu_1\|^{r_1} \ldots \|\mu_N\|^{r_N}$$

for any polynomial in the μ_i with coefficients in $M(\mathscr{F})$; that is, the set $\{\mu_i\}$ has independent powers with respect to $M(\mathscr{F})$ if the μ_i are non-negative.

Finally we remove the restriction that the measures $\mu_1, \ldots, \mu_N$ should be non-negative; the argument is that used in Proposition 2.2 of [7] (of which the present Proposition is a generalisation). We can write

$$\mu_i = \int f d|\mu_i|,$$

where f is a function of absolute value 1. By approximating to f by suitably chosen step-functions, we can approximate to μ_i by sums $\sum c_{ij}\mu_{ij}$, where the μ_{ij} are non-negative measures concentrated on sets P_{ij} with $P_i = \bigcup P_{ij}$ and $P_{ij} \cap P_{ik} = \emptyset$ if $j \neq k$. The set of all such $\{\mu_{ij}\}$ (as both i and j run through their respective index-sets) has independent powers with respect to $M(\mathscr{F})$, by what has just been proved, and the set of all sums $\{\sum c_{ij}\mu_{ij}\}$ has independent powers with respect to $M(\mathscr{F})$ also. By Proposition 1.2 (iv) of [7], the set $\{\mu_i\}$ has then independent powers with respect to $M(\mathscr{F})$, as required.

The complete decomposability of the set $\{\mu_i\}$ is clear since each μ_i is continuous.

THEOREM 1. *Let $\mathscr{F}_1$ be a symmetric Raikov system with a single generator, and let $\mathscr{F}_2$ be a strictly larger symmetric system; let $\varepsilon > 0$ be given. Then there exists a measure $\mu \in M(\mathscr{F}_2)$ such that*

(i) $\|\mu\| = 1$;

(ii) $0 \leqslant h(\mu) < \varepsilon$ *for all symmetric homomorphisms* h;

(iii) μ *has independent powers with respect to* $M(\mathscr{F}_1)$.

Proof. Let H be a group that generates $\mathscr{F}_1$, and let A be a perfect set in $\mathscr{F}_2$ such that $H-x$ is of the first category in A for each $x \in R$; such sets exist, by the Corollary to Lemma 2. Now apply Proposition 1; let P be a perfect $(H, 1)$-independent subset of A, and write $P = P_1 \cup P_2$, where P_1 and P_2 are disjoint perfect sets. Let λ_1 and λ_2 be continuous Hermitian measures concentrated on $P_1 \cup (-P_1)$ and $P_2 \cup (-P_2)$ respectively. By Proposition 2 the set $\{\lambda_1, \lambda_2\}$ has independent powers with respect to $M(\mathscr{F}_1)$; the rest of the proof now follows as in Theorem 2.3 of [7]. For any positive integer n, the measure

$$\mu_n = (\|\lambda_1\|^2 \delta_0 - \lambda_1^2)^n \lambda_2^2$$

has independent powers with respect to $M(\mathscr{F}_1)$, since the set $\{\lambda_1, \lambda_2\}$ has this property; evidently if $\mu = \mu_n/\|\mu_n\|$, then μ satisfies (i) and (iii). If n is chosen so large that $2^{-n} < \varepsilon$, then (ii) holds also; for since λ_1 is Hermitian we have

$$-\|\lambda_1\| \leqslant h(\lambda_1) \leqslant \|\lambda_1\|$$

if h is a symmetric homomorphism. Then

$$0 \leqslant h(\mu_n) \leqslant \|\lambda_1\|^{2n} \|\lambda_2\|^2$$

and since

$$\|\mu_n\| = 2^n \|\lambda_1\|^{2n} \|\lambda_2\|^2,$$

(ii) follows.

THEOREM 2. *Let $\mathcal{F}_1$ be a symmetric Raikov system with a single generator, and $\mathcal{F}_2$ a strictly larger symmetric system; let $\varepsilon > 0$ be given. Then there exists a continuous Hermitian measure $\mu \in M(\mathcal{F}_2)$ such that*

(i) $\|\mu\| = 1$;

(ii) $0 \leqslant h(\mu) < \varepsilon$ *for all symmetric homomorphisms* h;

(iii) *if* $a \in M(\mathcal{F}_1)$ *is such that* $\inf_h |h(a)| \leqslant 1$ *and* $\nu \in M(\mathcal{F}_1)^{\perp}$ *is such that* $h(\nu) = 0$ *for all non-symmetric homomorphisms* h, *then* $(a+\mu+\nu)$ *has no inverse in* $M(R)$.

Proof. Let P_1 and P_2 be as in Theorem 1; let μ_1 and μ_2 be continuous Hermitian measures concentrated on $P_1 \cup (-P_1)$, $P_2 \cup (-P_2)$ respectively, such that

$$\|\mu_1\| = \|\mu_2\| = 1$$

and $0 \leqslant h(\mu_1) < \min(\frac{1}{2}, \frac{1}{2}\varepsilon)$, $0 \leqslant h(\mu_2) < \min(\frac{1}{2}, \frac{1}{2}\varepsilon)$ if h is symmetric. Write $\mu = \mu_1 + \mu_2$. Since $\{\mu_1, \mu_2\}$ has independent powers with respect to $M(\mathcal{F}_1)$, by Proposition 2, we have $\|\mu\| = 1$, so that (i) holds; clearly (ii) holds also.

We now prove (iii); as in the case of Theorem 1, the proof is essentially the same as that of Theorem 2.5 of [7], with minor variations. There are two possibilities: either a^{-1} exists or it fails to exist. If the latter, then $(a+\mu+\nu)^{-1}$ fails to exist also. For, if $(a+\mu+\nu)^{-1}$ were to exist, we could write it as $\beta+\gamma$, where $\beta \in M(\mathcal{F}_1)$ and $\gamma \in M(\mathcal{F}_1)^{\perp}$. Since $M(\mathcal{F}_1)^{\perp}$ is an ideal and $\mu, \nu \in M(\mathcal{F}_1)^{\perp}$, it follows that $\beta = a^{-1}$, a contradiction.

If a^{-1} exists, suppose that

$$c \in \sigma(a) \quad \text{and} \quad |c| = \inf_h |h(a)| \leqslant 1.$$

Since $M(\mathcal{F}_1)$ is closed under translations and inversions (this last fact is easily established by the argument used in the preceding paragraph), we can apply Proposition 1.7 of [7] to show that there exists a homomorphism h such that

$$h(a) = c, \quad |h(\mu_1)| = |h(\mu_2)| = \tfrac{1}{2}, \quad h(\mu_1+\mu_2) = -c.$$

This h cannot be symmetric, in view of (ii). It follows that $h(\nu) = 0$, and we then have

$$h(a+\mu+\nu) = c-c+0 = 0,$$

so that $(a+\mu+\nu)$ has no inverse in $M(R)$.

We have thus succeeded in showing that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are symmetric Raikov systems, $\mathcal{F}_1$ having a single generator and $\mathcal{F}_2$ strictly larger than $\mathcal{F}_1$, all the Wiener-Pitt pathology that occurs in $M(R)$ as a whole occurs also between $M(\mathcal{F}_1)$ and $M(\mathcal{F}_2)$. While it would certainly be of interest to refine this result by showing that the pathology occurs between any two similarly related algebras of a larger class (for example, the class obtained by dropping the qualification "symmetric" for the Raikov systems in question), perhaps the problem of specifying some *non*-pathological subalgebras of $M(R)$ offers a greater challenge. Very little is known about this, and it seems likely that quite new ideas and techniques will have to be devised in order to deal with it.

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