STUDIA MATHEMATICA, T. XXXI. (1968)

On the spectra of certain Laurent operators on Orlicz spaces l^{Φ}

by

W. A. J. LUXEMBURG (Pasadena)*

Dedicated to Professors
Stanisław Mazur and Władysław Orlicz
on the occasion of the 40th anniversary
of their scientific research

1. Introduction. Let $l^p(Z)$ $(1 \leqslant p \leqslant \infty)$, where $Z = \{0, \pm 1, \pm 2, \ldots\}$, denote as usual the Banach spaces of all doubly infinite sequences x $x = \{x(n) : n \in Z\}$ of complex numbers such that

$$\|x\|_p = \Big(\sum_{-\infty}^{+\infty} |x(n)|^p\Big)^{1/p} < \infty \qquad (1 \leqslant p < \infty)$$

and such that

$$\|x\|_\infty = \sup\big(|x(n)| : n \in Z\big) < \infty$$

when $p = \infty$.

If $a \in l^1$, then it is well-known that the doubly infinite matrix $\big(a(n-m) : n, m \in Z\big)$ defines a bounded operator, which is called a *Laurent operator*, $L_a : x \to a * x$ of l^p into l^p for all $1 \leqslant p \leqslant \infty$, where $*$ denotes the convolution operation. The spectra of the Laurent operators L_a, $a \in l^1$, were shown to be equal to the sets $\{\sum_{-\infty}^{\infty} a(n) e^{int} : |t| \leqslant \pi\}$ by Toeplitz [13] for the case $p = 2$ and by Krabbe [5] for the case $1 \leqslant p \leqslant \infty$. Krabbe based his proof on Wiener's celebrated theorem (see [6], p. 71-72) concerning continuous functions whose Fourier series converge absolutely and the fact that every Laurent form can be approximated by so-called regular Laurent forms. Recently, Eisen and Gindler [2] obtained Krabbe's result without using Wiener's theorem. They based their proof on the operational calculus for bounded operators on normed spaces and the approximation theorem referred to above, and so obtained as an application another proof of Wiener's theorem.

The purpose of the present note is to show that the same results hold for the Laurent operators of the form L_a, $a \in l^1$, on Orlicz spaces l^{Φ}. We

* Work on this paper was supported in part by NSF Grant GP — 7691.

shall obtain them by using some elementary results from classical Fourier analysis and some results of the theory of commutative Banach algebras. We shall be able to avoid the approximation theorem referred to above and consequently we shall not need to use the Newburgh convergence theorem (see [11], p. 36-37) for the spectra of certain convergent sequences of operators. Furthermore, we shall also determine the spectral radius of the Laurent operator L_a, $a \in l^1$. At the end of Section 4 we shall present another proof of Krabbe's theorem for the l^p-spaces ($1 \leqslant p \leqslant \infty$), the essential ingredients being a simple form of the well-known M. Riesz convexity theorem and Theorems 4.3 and 5.3.

In the last section we shall consider Laurent operators

$$L_f = \big(f(n-m) : n, m \in Z\big),$$

where $\{f(n) : n \in Z\}$ is the sequence of Fourier coefficients of a complex function of bounded variation F on $|t| \leqslant \pi$, where F is supposed to be normalized as to be right continuous for $-\pi \leqslant t < \pi$ and left continuous at $t = \pi$. It was shown by Stečkin [12] that such Laurent operators are bounded on l^p ($1 < p < \infty$) and Hartman [3] showed that the spectrum of L_f is equal to the closure of the range of F. We shall show that such Laurent operators are bounded on reflexive Orlicz spaces l^Φ and show that Hartman's result also holds in this case. We shall also determine the spectral radius of L_f, and as an application we can show that the norm of the Hilbert transform on l^p ($1 < p < \infty$) is bounded below by π.

2. Orlicz spaces l^Φ. In 1932 W. Orlicz (see [10]) introduced a new class of Banach spaces of measurable functions which contained the classical Lebesgue spaces as special cases. To this class of spaces we justly refer to nowadays as the class of *Orlicz spaces*. It is beyond the scope of the present paper to describe here the various new developments in functional analysis which were prompted by the theory of Orlicz spaces. Let me suffice by stating that the work of Professor W. Orlicz in the theory of spaces of measurable functions has been an important source of inspiration to the present author in his work on the theory of Banach function spaces.

In this section we shall briefly recall the definition and elementary properties of the Orlicz sequence spaces l^Φ. For a more detailed account of the theory of Orlicz spaces we refer the reader to [7] and [14].

Let $v = \varphi(u)$, $u \geqslant 0$, be a non-decreasing real function of u such that $\varphi(0) = 0$, φ does not vanish identically, and φ is left continuous for $u > 0$. Furthermore, let ψ be the left continuous inverse of φ, i.e., $\psi(0) = 0$ and $\psi(v) = \sup\big(u : \varphi(u) < v\big)$ for $v > 0$, and so $\psi\big(\varphi(u)\big) \leqslant u$ ($u \geqslant 0$) and $\psi\big(\varphi(u)+\big) \geqslant u$ ($u \geqslant 0$).

If φ and ψ satisfy the above conditions, then Φ and Ψ defined for $u, v \geqslant 0$ by

$$\Phi(u) = \int_0^u \varphi(t)\,dt, \qquad \Psi(v) = \int_0^v \psi(t)\,dt$$

are called *complimentary Young functions* (W. H. Young, 1912).

For example, $\Phi(u) = u$ ($u \geqslant 0$) and $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = \infty$ for $v > 1$ is a pair of complimentary Young functions. If $1 < p < \infty$ and q is defined by $p^{-1} + q^{-1} = 1$, and if $\varphi(u) = u^{p-1}$, $\psi(v) = v^{q-1}$ ($u, v \geqslant 0$), then $\Phi(u) = p^{-1}u^p$, $\Psi(v) = q^{-1}v^q$ ($u, v \geqslant 0$) is also a pair of complimentary Young functions.

A pair (Φ, Ψ) of complimentary Young functions satisfy the following important inequality, known as Young's inequality:

$$uv \leqslant \Phi(u) + \Psi(v) \text{ holds for all } u, v \geqslant 0;$$

equality occurs if and only if one at least of the equalities $v = \varphi(u)$, $u = \psi(v)$ holds.

In this generality Young's inequality was first proved by A. C. Zaanen (see [14], p. 77). Zaanen's proof is somewhat geometric in nature. We shall present now a new and purely analytic proof of Young's inequality. It will be based on the notion of the *spectral measure* determined by a measurable function (see [8], sections 2 and 4). From Lemma 4.4 of [8] it follows that the spectral measure determined by ψ is the Radon measure $d\varphi$. With this observation in mind using integration by parts and the left continuity of φ and ψ, we obtain the formula

$$\begin{aligned}\Phi(u) &= \int_{(0,u)} \varphi(t)\,dt = u\varphi(u) - \int_{(0,u)} t\,d\varphi = u\varphi(u) - \int_{(0,\infty)} t\chi_{(0,u)}(t)\,d\varphi \\ &= u\varphi(u) - \int_{(0,\infty)} \psi(t)\,\chi_{(0,u)}\big(\psi(t)\big)\,dt,\end{aligned}$$

where $\chi_{(0,u)}$ denotes the characteristic function of the open interval $(0, u)$. Since for all $u > 0$ we have $\varphi(u) = \sup\big(v : \psi(v) < u\big)$, we obtain $\Phi(u) = u\varphi(u) - \Psi\big(\varphi(u)\big)$ for all $u \geqslant 0$. Then

$$\begin{aligned}0 \leqslant uv &= u\varphi(u) + u\big(v - \varphi(u)\big) = \Phi(u) + \Psi(v) + \Psi\big(\varphi(u)\big) - \Psi(v) + u\big(v - \varphi(u)\big) \\ &= \Phi(u) + \Psi(v) + a,\end{aligned}$$

where

$$a = \int_{(v,\varphi(u))} \psi(t)\,dt + u\big(v - \varphi(u)\big).$$

Now observe that if $v > \varphi(u)$, then

$$\int_{(\varphi(u),v)} \psi(t)\,dt \geqslant \psi(\varphi(u)+)(v-\varphi(u)) \geqslant u(v-\varphi(u)),$$

i.e., $a \leqslant 0$, if, on the other hand, $v < \varphi(u)$, then

$$\int_{(v,\varphi(u))} \psi(t)\,dt \leqslant (\varphi(u)-v)\psi(\varphi(u)) \leqslant u(\varphi(u)-v),$$

and so again $a \leqslant 0$. This proves Young's inequality. We have equality if and only if $a = 0$, and so, by interchanging the role of φ and ψ we obtain that equality holds if and only if either $v = \varphi(u)$ or $u = \psi(v)$. This finishes the proof.

Let (Φ, Ψ) be a pair of complimentary Young functions and let $P = P(Z)$ denote the set of all doubly infinite sequences x of real numbers such that $x(n) \geqslant 0$ for all $n \in Z$.

To every $x \in P$ we assign the numbers

$$\varrho_{M\Phi}(x) = \inf\Big(k^{-1}: k \geqslant 0 \text{ and } \sum_{-\infty}^{+\infty} \Phi(kx(n)) \leqslant 1\Big),$$

$$\varrho_{\Phi}(x) = \sup\Big(\sum x(n)y(n) : y \in P \text{ and } \varrho_{M\Psi}(y) \leqslant 1\Big).$$

It is well-known (see [7]) that ϱ_Φ and $\varrho_{M\Phi}$ are *saturated function norms* in the sense of [9]. Furthermore, ϱ_Φ as well as $\varrho_{M\Phi}$ has the *Fatou property*. The norms $\varrho_{M\Phi}$ and ϱ_Φ are *equivalent*; more precisely, we have (see [7])

$$\varrho_{M\Phi} \leqslant \varrho_\Phi \leqslant 2\varrho_{M\Phi}.$$

The corresponding Orlicz spaces $l^{M\Phi}(Z)$ and $l^{\Phi}(Z)$ are the spaces of doubly infinite sequences f of complex numbers such that $\varrho_{M\Phi}(|f|) < \infty$ and with the norms $\|f\|_{M\Phi} = \varrho_{M\Phi}(|f|)$ and $\|f\|_\Phi = \varrho_\Phi(|f|)$ respectively. Since $\varrho_{M\Phi}$ and ϱ_Φ are equivalent, the spaces $l^{M\Phi}$ and l^Φ contain the same elements and are homeomorphic as normed spaces. The elements of these spaces can be characterized independently of the norms. We have $f \in l^{M\Phi}$, and so also $f \in l^\Phi$, if and only if

$$\sum_{-\infty}^{+\infty} \Phi(k|f(n)|) < \infty$$

for some $k = k(f) > 0$. For the sake of convenience we shall denote the linear space of sequences f satisfying the latter condition by $l(\Phi)$.

Since the norms $\varrho_{M\Phi}$ and ϱ_Φ have the Fatou property, the Orlicz spaces $l^{M\Phi}$ and l^Φ are *complete*. Furthermore, the associate space of $l^{M\Phi}$ in the sense of [7] and [8] is l^Ψ and the associate space of l^Φ is $l^{M\Psi}$. Thus the Orlicz spaces are *perfect*, i.e., they are identical with their second associate space. For the norm $\|\cdot\|_{M\Phi}$ this means, in particular, that

$$\|f\|_{M\Phi} = \sup\Big(\sum_{-\infty}^{+\infty} |f(n)g(n)| : \|g\|_\Psi \leqslant 1\Big).$$

Hölder's inequality takes the form

$$\sum_{-\infty}^{+\infty} |f(n)g(n)| \leqslant \|f\|_{M\Phi}\|g\|_\Psi.$$

Observe that the spaces l^Φ ($l^{M\Phi}$) are *intermediate spaces*, i.e., $l^1 \subset l(\Phi) \subset l^\infty$, and $\|a\|_\Phi \leqslant K_1\|a\|_1$ and $\|a\|_\infty \leqslant K_2\|a\|_\Phi$ for some constants K_1, K_2, and similarly for the norm $\|\cdot\|_{M\Phi}$.

Finally, we recall (see [7]) that the elements of $l(\Phi)$ can also be characterized as follows: $f \in l(\Phi)$ if and only if

$$\sum_{-\infty}^{+\infty} |f(n)g(n)| < \infty$$

for all $g \in l(\Psi)$.

3. The imbedding of l^1 in $[l^{M\Phi}]$ and $[l^\Phi]$. It is well-known that l^1 in its usual norm

$$\|a\|_1 = \sum_{-\infty}^{+\infty} |a(n)|$$

is a commutative Banach algebra under convolution. The Banach algebra l^1 has a unit e determined by $e(n) = 0$ for $n \neq 0$ and $e(0) = 1$. By e_k we shall denote the element satisfying $e_k(n) = \delta_{k,n}$, $n \in Z$, where δ is the Kronecker delta. To every $a \in l^1$ we assign its Fourier transform

$$A(t) = \sum_{-\infty}^{+\infty} a_n e^{int}, \quad |t| \leqslant \pi.$$

It is well-known that the mapping $a \to A$ of l^1 onto the algebra of continuous 2π-periodic functions with absolutely convergent Fourier series is an *algebraic isomorphism* and is *norm decreasing*.

Let (Φ, Ψ) be a pair of complimentary Young functions. By $[l^{M\Phi}]$ ($[l^\Phi]$) we shall denote the complex Banach algebra of all *bounded* linear transformations of $l^{M\Phi}(l^\Phi)$ into $l^{M\Phi}(l^\Phi)$.

We begin with the following simple lemma:

LEMMA 3.1. *For every $a \in l^1$, the Laurent operator $L_a \in [l^\Phi]$, and so also $L_a \in [l^{M\Phi}]$.*

Proof. From $l^1 \subset l(\Phi) \subset l^\infty$ it follows that for all $a \in l^1$ and for all $x \in l(\Phi)$ the infinite series

$$\sum_{m=-\infty}^{+\infty} a(n-m)x(m)$$

converges absolutely for all $x \in l(\Phi)$. In order to prove that $L_a \in [l^\Phi]$ we have to show that

$$\sup(\|a*x\|_\Phi : \|x\|_\Phi \leqslant 1) < \infty.$$

To this end observe that for all $x \in l(\Phi)$ and $y \in l(\Psi)$ we have

$$\sum |(a*x)(m)||y(m)| \leqslant \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |a(n)||x(n-m)||y(m)|$$
$$= \sum_{n=-\infty}^{+\infty} |a(n)| \Big(\sum_{m=-\infty}^{+\infty} |x(n-m)||y(m)| \Big) \leqslant \|a\|_1 \|x\|_\Phi \|y\|_{M\Psi}$$

by Hölder's inequality and the fact that the Orlicz norms are translation invariant. Hence,

$$\sup(\|a*x\|_\Phi : \|x\|_\Phi \leqslant 1) \leqslant \|a\|_1.$$

The proof for the space $l^{M\Phi}$ is of course the same.

For all $a \in l^1$, we set

$$\|L_a\|_\Phi = \sup(\|a*x\|_\Phi : \|x\|_\Phi \leqslant 1) = |||a|||_\Phi$$

and

$$\|L_a\|_{M\Phi} = \sup(\|a*x\|_{M\Phi} : \|x\|_{M\Phi} \leqslant 1) = |||a|||_{M\Phi}.$$

Then by (3.1) we have $|||a|||_\Phi \leqslant \|a\|_1$ and $|||a|||_{M\Phi} < \|a\|_1$ for all $a \in l^1$. Further, $a = a*e$ implies that $\|a\|_\Phi \leqslant \|e\|_\Phi \cdot |||a|||_\Phi$ and $\|a\|_{M\Phi} \leqslant \|e\|_{M\Phi} |||a|||_{M\Phi}$ for all $a \in l^1$. Observe also that

$$2^{-1}|||a|||_{M\Phi} \leqslant |||a|||_\Phi \leqslant 2|||a|||_{M\Phi}$$

for all $a \in l^1$.

Lemma 3.1 can now be strengthened to the following statement:

THEOREM 3.2. *The mapping $a \to L_a$ of l^1 into $[l^\Phi]$ $([l^{M\Phi}])$ is an algebraic isomorphism and norm decreasing.*

Proof. It is easy to see that $a*b \to L_a(L_b)$. That the mapping $a \to L_a$ is one-to-one follows from the above inequalities $\|a\|_\Phi \leqslant \|e\|_\Phi \|L_a\|_\Phi$ and $\|a\|_{M\Phi} \leqslant \|e\|_{M\Phi} \|L_a\|_{M\Phi}$.

Under the norms $|||\cdot|||_\Phi$, $|||\cdot|||_{M\Phi}$ the space l^1 is also a *commutative normed algebra*. Their closures in $[l^\Phi]$ and $[l^{M\Phi}]$ will be denoted by l^1_Φ and $l^1_{M\Phi}$ respectively. Furthermore, we shall frequently identify l^1 with the commutative subalgebras of $[l^\Phi]$ $([l^{M\Phi}])$ of the Laurent operators of the form L_a.

Concerning $|||\cdot|||_\Phi$ and $|||\cdot|||_{M\Phi}$ we have the following result:

THEOREM 3.3. *For all $a \in l^1$ we have $|||a|||_\Phi = |||a|||_{M\Psi}$.*

Proof. Observe that

$$|||a|||_\Phi = \sup\Big(\Big|\sum_{n,k} a_k x_{n-k} y_n\Big| : \|x\|_\Phi \leqslant 1, \|y\|_{M\Phi} \leqslant 1\Big)$$
$$= \sup\Big(\Big|\sum_{n,k} a_k y'_{m-k} x'_n\Big| : \|x'\|_\Phi \leqslant 1, \|y'\|_{M\Psi} \leqslant 1\Big)$$
$$= \sup(\|a*y'\|_{M\Psi} : \|y'\|_{M\Psi} \leqslant 1) = |||a|||_{M\Psi},$$

and the proof is finished.

Remark. Of course $|||a|||_{M\Psi}$ is the norm of the transposed matrix and so is equal to its norm.

4. The spectra of the elements of the algebras l^1_Φ and $l^1_{M\Phi}$. Let X be a commutative Banach algebra with unit e. The set of all non-zero homomorphisms of X into the algebra of complex numbers is usually referred to as the *Gelfand representation space of X* or the *spectrum of X* and will be denoted by $\mathrm{sp}(X)$. Since X has a unit element, $\mathrm{sp}(X)$ is a closed subset of the unit ball of the Banach dual X^* of X, and so $\mathrm{sp}(X)$ is a compact Hausdorff space in the weak*-topology of X^*. If $a \in X$, then we recall that its *spectrum* $\mathrm{sp}(a, X)$ consists of all the complex numbers λ such that $\lambda e - a$ is *singular*. Since X has a unit element, it follows that $\mathrm{sp}(a, X) = \{h(a) : h \in \mathrm{sp}(X)\}$ (see [11], (3.1.6)).

It is well-known that the non-zero complex homomorphisms of the convolution algebra l^1 are of the type

$$h_t(a) = \sum_{-\infty}^{+\infty} a_n e^{int}, \qquad -\pi < t \leqslant \pi$$

(see [6], p. 71-72), and $\mathrm{sp}(l^1)$ is homeomorphic to the interval $(-\pi, \pi]$, where $-\pi$ and π are identified.

Since the commutative Banach algebras l^1_Φ and $l^1_{M\Phi}$ have equivalent norms, their spectra are identical. We shall now determine the spectrum of l^1_Φ. To this end, observe that $|||\cdot|||_\Phi \leqslant \|\cdot\|_1$ implies that $\mathrm{sp}(l^1_\Phi) \subset \mathrm{sp}(l^1)$. Since the Banach algebra l^1 is *commutative, completely regular* and *semi-simple* ($h(a) = 0$ for all $h \in \mathrm{sp}(l^1)$ implies $a = 0$), it follows from a general theorem of C. E. Rickart (see [11], Corollary (3.7.7)) that indeed for every Young function Φ we have that $\mathrm{sp}(l^1_\Phi) = \mathrm{sp}(l^1)$ and that their topologies are the same. For the sake of completeness we shall present a proof of this result here for this special case.

THEOREM 4.1. *For every Young function Φ we have $\mathrm{sp}(l^1_\Phi) = \mathrm{sp}(l^1)$.*

Proof. Observe that $\operatorname{sp}(l^1_\Phi)$ is a closed subset of $\operatorname{sp}(l^1)$. If there exists a number $-\pi < t_0 \leqslant \pi$ such that $t_0 \notin \operatorname{sp}(l^1_\Phi)$, then for some $\delta > 0$ we infer that $|t-t_0| \leqslant \delta$ implies $t \notin \operatorname{sp}(l^1_\Phi)$. Let $a \in l^1$ be such that

$$A(t) = \sum_{-\infty}^{+\infty} a(n)e^{int} \qquad (|t| \leqslant \pi)$$

satisfies $A(t) = 1$ for all $t \in U_\delta = \{t : |t-t_0| < \delta\}$, $A(t) = 0$ for $t \in \operatorname{sp}(l^1_\Phi)$ and $0 < A(t) \leqslant 1$ for all $|t| \leqslant \pi$. Since $A(t) = 0$ for all $t \in \operatorname{sp}(l^1_\Phi)$, it follows that a is in the radical of l^1_Φ, and so the operator L_{e-a} has an inverse T in l^1_Φ. It is obvious that there is an element $b \in l^1$ such that $0 \leqslant B(t) \leqslant 1$ for all $|t| \leqslant \pi$, $B(t_0) = 1$ and $B(t) = 0$ for all $t \notin U_\delta$. Then, since l^1 is semi-simple, it follows that $b(e-a) = 0$. Hence, $0 = (L_b L_{e-a})T = L_b(L_{e-a}T) = L_b L_e = L_b$ and a contradiction is obtained since by Theorem 3.2 the mapping $a \to L_a$ is an isomorphism.

From Theorem 4.1 we can now at once draw the following conclusion:

THEOREM 4.2. *For every Young function Φ, the spectrum*

$$\operatorname{sp}(a, l^1_\Phi) = \operatorname{sp}(a, l^1_{M\Phi}) = \Big\{\sum_{-\infty}^{+\infty} a(n)e^{int},\ |t| \leqslant \pi\Big\} = \operatorname{sp}(a, l^1)$$

for all $a \in l^1$. Consequently, for every $a \in l^1$ the spectral radius $\nu(a, l^1)$ of a satisfies

$$\begin{aligned}\nu(a, l^1) &= \nu(a, l^1_\Phi) = \nu(a, l^1_{M\Phi}) \\ &= \max\Big(\Big|\sum_{-\infty}^{+\infty} a(n)e^{int}\Big| : |t| \leqslant \pi\Big) \leqslant |||a|||_\Phi \leqslant \|a\|_1,\end{aligned}$$

and similarly for $|||\cdot|||_{M\Phi}$.

We shall now determine the spectral radius of a Laurent operator L_a, $a \in l^1$.

THEOREM 4.3. *For every $a \in l^1$, we have*

$$\max\Big(\Big|\sum_{-\infty}^{+\infty} a(n)e^{int}\Big| : |t| \leqslant \pi\Big) = |||a|||_2 .$$

In particular, $|||a|||_2 \leqslant |||a|||_\Phi$ and $|||a|||_2 \leqslant |||a|||_{M\Phi}$ for all $a \in l^1$, and if $a(n) \geqslant 0$ for all $n \in Z$, then $|||a|||_2 = |||a|||_{M\Phi} = |||a|||_\Phi = \|a\|_1$.

Proof. For every $a \in l^1$ we have by definition

$$|||a|||_2 = \sup\Big(\Big|\sum_{n=\infty}^{+\infty} a(n)\Big(\sum_{k=\infty}^{+\infty} x(n-k)y(k)\Big)\Big| : \|x\|_2 \leqslant 1 \text{ and } \|y\|_2 \leqslant 1\Big).$$

Hence,

$$\Big|\sum a(n)e^{int}\Big| \leqslant |||a|||_2$$

will follow if we can determine two sequences $\{x_l\}$, $\{y_l\}$ $(l = 1, 2, \ldots)$ of elements of $l^2(Z)$ such that $\|x_l\|_2 \leqslant 1$, $\|y_l\|_2 \leqslant 1$ $(l = 1, 2, \ldots)$ and $\sum_{-\infty}^{+\infty} x_l(n-k)y_l(k)$ tends to e^{int} as $l \to \infty$ boundedly for $n \in Z$. To this end, let $|t_0| \leqslant \pi$ and let $\delta_l(t) = l/2$ whenever $-t_0 - 1/l \leqslant t \leqslant -t_0 + 1/l$, $t \neq \pm\pi$ and zero otherwise and if $t_0 = -\pi$ we let $\delta_l(t) = l$ for $+\pi \leqslant t \leqslant \pi - 1/l$ and zero otherwise and similarly for $t_0 = \pi$. Then we set $f_l = g_l = \sqrt{2\pi\delta_l}$, and hence $\|f_l\|_2^2 = \|g_l\|_2^2 = 2\pi$. If $\{x_l\}$, $\{y_l\}$ are the sequences of Fourier coefficients of f_l and g_l respectively $(l = 1, 2, \ldots)$, then $\|x_l\|_2^2 = \|y_l\|_2^2 = 1$ and

$$\Big|\sum_{+\infty}^{+\infty} x_l(n-k)y_l(k)\Big| \leqslant 1$$

for all $n \in Z$. Now

$$\sum_{k=-\infty}^{+\infty} x_l(n-k)y_l(k) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \delta_l(t)e^{int}\,dt$$

and it tends to e^{int_0} boundedly as $l \to \infty$. Thus we obtain that

$$\Big|\sum a(n)e^{int_0}\Big| \leqslant |||a|||_2 \quad \text{for all } |t_0| \leqslant \pi .$$

In order to prove the converse inequality observe that if $a \in l^1$, then

$$\begin{aligned}|||a|||_2^2 &= \sup(\|a * x\|_2^2 : \|x\|_2 \leqslant 1) \\ &= \sup\Big(\frac{1}{2\pi}\Big|\int_{-\pi}^{+\pi}\Big(\sum a(n)e^{int}\Big)f(t)\,dt\Big|^2 : \|f\|_2 \leqslant 1\Big),\end{aligned}$$

and so

$$|||a|||_2 \leqslant \max\Big(\Big|\sum a(n)e^{int}\Big| : |t| \leqslant \pi\Big).$$

Since the spectral radius is always less than or equal to the norm, the stated inequalities follow. If $a(n) \geqslant 0$ for all $n \in Z$, then setting $t = 0$ we obtain that

$$\|a\|_1 = \sum a(n) \leqslant |||a|||_2 \leqslant |||a|||_\Phi \leqslant \|a\|_1,$$

and so all the inequalities are equalities. This completes the proof of the theorem.

Remark. There exists a converse to the last statement of the previous theorem. Namely, *if $a(n) \geqslant 0$ for all $n \in Z$ and $a * x \in l(\Phi)$, for all $x \in l(\Phi)$, then $a \in l^1$, provided $\Phi(u) > 0$ for all $u > 0$.* The proof is as follows. If $a \notin l^1$, then there exist natural numbers p_n such that

$$\sum_{k=-p_n}^{p_n} a(k) > n^2 \qquad (n = 1, 2, \ldots).$$

Let $\{I_n\}$ be a sequence of disjoint intervals of Z of length $4p_n$ and let J_n be an interval with the same midpoint as I_n and of length $2p_n$. Let γ_n be defined by $\Phi(\gamma_n) = (n^2 p_n)^{-1}$. Define x by $x(k) = \gamma_n$ if $k \in I_n$ $(n = 1, 2, \ldots)$ and $x(k) = 0$ otherwise. Then $x \in l(\Phi)$ and from $y(k) = (a * x)(k) > n^2 \gamma_n$ if $k \in J_n$ $(n = 1, 2, \ldots)$ it follows that

$$\sum_{-\infty}^{+\infty} \Phi\big((\lambda y(k)\big) \geqslant \sum_{n=1}^{\infty} \sum_{J_n}^{\infty} \Phi\big((\lambda y(k)\big) \geqslant \sum_{1}^{\infty} 2p_n \Phi(\lambda n^2 \gamma_n)$$

$$\geqslant \sum_{\lambda n \geqslant 1} 2p_n \Phi(\lambda n^2 \gamma_n) \geqslant \sum_{\lambda n \geqslant 1} 2np_n \Phi(\gamma_n) = \sum_{\lambda n \geqslant 1} 2n^{-1} = \infty,$$

where $\lambda > 0$ is arbitrary. Hence, $y \notin l(\Phi)$ and a contradiction is obtained.

For a similar result due to A. C. Zaanen for the spaces $L^p(-\infty, +\infty)$, $1 \leqslant p < \infty$, we refer the reader to [16].

5. The spectrum of a Laurent operator $L_a \in [l^\Phi]$, $a \in l^1$. Let X_1 be a closed subalgebra of a Banach algebra X with unit. If $a \in X_1$, then in general $\mathrm{sp}(a, X) \subset \mathrm{sp}(a, X_1)$. In our case, l^1_Φ $(l^1_{M\Phi})$ is a closed subalgebra of $[l^\Phi]$ $([l^{M\Phi}])$, and so for all $a \in l^1$ we have

$$\mathrm{sp}(a, [l^\Phi]) = \mathrm{sp}(a, [l^{M\Phi}]) \subset \mathrm{sp}(a, l^1_\Phi) = \mathrm{sp}(a, l^1_{M\Phi}).$$

We shall show, however, that these spectra are equal.

We shall need the following two simple lemmas.

LEMMA 5.1. *If $T \in l^1_\Phi$ and $T^{-1} \in [l^\Phi]$, then T^{-1} commutes with the elements of l^1_Φ, and similarly for $l^1_{M\Phi}$.*

Proof. Since l^1_Φ is a commutative subalgebra of $[l^1_\Phi]$, it follows that for all $S \in l^1_\Phi$ we have $S = STT^{-1} = TST^{-1}$, and so $T^{-1}S = ST^{-1}$ for all $S \in l^1_\Phi$ and the proof of the lemma is finished.

From the spectral radius theorem for commutative algebras (see [11], (3.1.7)) the following result follows immediately:

LEMMA 5.2. *Let $T \in l^1_\Phi$ and let X be a closed commutative subalgebra of $[l_\Phi]$ containing l^1_Φ. Then*

$$\sup\big(|h(T)| : h \in \mathrm{sp}(l^1_\Phi)\big) = \sup\big(|h(T)| : h \in \mathrm{sp}(X)\big),$$

and similarly for $l^1_{M\Phi}$.

We are now in a position to prove the main result.

THEOREM 5.3. *For all $a \in l_1$ and for all Young functions Φ we have*

$$\mathrm{sp}(a, l^1) = \mathrm{sp}(L_a, l^1_\Phi) = \mathrm{sp}(L_a, l^1_{M\Phi}) = \mathrm{sp}(L_a, [l^\Phi]) = \mathrm{sp}(L_a, [l^{M\Phi}])$$

and the spectral radius of L_a is equal to $|||a|||_2$.

Proof. We have only to show that if $a \in l^1$, then $\mathrm{sp}(L_a, l^1_\Phi) = \mathrm{sp}(L_a, [l^\Phi])$. To this end, we have to show that if $a \in l^1$ and L_a^{-1} exists and is an element of $[l^\Phi]$, then $L_a^{-1} \in l^1_\Phi$. In order to prove that $L_a^{-1} \in l^1_\Phi$ we have to show that $h(L_a) \neq 0$ for all $h \in \mathrm{sp}(l^1_\Phi)$. Assume this is not so, then by Theorem 4.1, there is an element t_0 such that $|t_0| \leqslant \pi$ and $\sum a(n) e^{int_0} = 0$. Let $\beta = \|L_a^{-1}\|$ and set again

$$A(t) = \sum a(n) e^{int} \quad (|t| \leqslant \pi).$$

Then $\beta > 0$ implies that there is an open interval I in $(-\pi, \pi)$ such that $t_0 \in I$ and $I \subset \{t : |A(t)| < 1/2\beta\}$. It is easy to see that there is an element $b \in l^1$ such that $0 \leqslant B(t) \leqslant 1$ $(|t| \leqslant \pi)$, $B(t_0) = 1$ and $B(t) = 0$ for $t \notin I$. From Lemma 4.1 it follows that there is a commutative closed subalgebra X of $[l^\Phi]$ such that $L_a^{-1} \in X$ and $l^1_\Phi \subset X$. Then $L_b = L_b L_a L_a^{-1}$ and Lemma 4.2 imply that

$$\sup\big(|h(L_b)| : h \in \mathrm{sp}(l^\Phi_1)\big) = \sup\big(|h(L_b)| : h \in \mathrm{sp}(X)\big)$$

$$= \sup\big(|h(L_b L_a L_a^{-1})| : h \in \mathrm{sp}(X)\big) \leqslant \beta \sup\big(|h(L_b L_a)| : h \in \mathrm{sp}(X)\big)$$

$$= \beta \sup\big(|h(L_b L_a)| : h \in \mathrm{sp}(l^1_\Phi)\big) = \beta \max\big(|A(t) B(t)| : |t| \leqslant \pi\big) \leqslant \beta (1/2\beta) = 1/2.$$

But

$$\sup\big(|h(L_b)| : h \in \mathrm{sp}(l^1_\Phi)\big) = \max\big(|B(t)| : |t| \leqslant \pi\big) = 1,$$

and a contradiction is obtained. This completes the proof of the theorem.

Remark. If $1 \leqslant p \leqslant \infty$, then Theorem 5.3 states that

$$\mathrm{sp}(L_a, [l^p]) = \{\sum a(n) e^{int} : |t| \leqslant \pi\} \quad \text{for all } a \in l^1.$$

This result is due to Krabbe [5]. We shall present here another proof of Krabbe's result.

If one can show directly that $|||a|||_2 \leqslant |||a|||_\Phi$ $(|||a|||_2 \leqslant |||a|||_{M\Phi})$ for all $a \in l^1$; then it follows from Theorem 4.3 that $\mathrm{sp}(l^1) = \mathrm{sp}(l^1_\Phi)$ and another proof of Theorem 4.1 was obtained. The final result would follow then in the form of Theorem 5.3. For general Young functions I have not been able to prove the above inequalities directly, i.e., without using Theorem 4.1. If, on the other hand, we deal with l^p-spaces $(1 \leqslant p \leqslant \infty)$, then these inequalities follow immediately from the celebrated M. Riesz convexity theorem (see [15], p. 95). Indeed, from the M. Riesz convexity theorem it follows that there exists a constant $\theta = \theta(p)$ such that $0 < \theta < 1$ and $|||a|||_2 \leqslant |||a|||_p^\theta |||a|||_q^{1-\theta}$, where $1/p + 1/q = 1$, and so by Theorem 3.3 we obtain $|||a|||_2 \leqslant |||a|||_p$ for all $1 \leqslant p \leqslant \infty$. This proof bases Krabbe's result on the basic properties of the convolution algebra l^1, the spectral radius theorem, the classical Riesz-Fischer theorem and the M. Riesz convexity theorem.

6. On certain Laurent operators on reflexive Orlicz spaces l^Φ. In [3], P. Hartman determines the spectrum of the following kind of Laurent operators.

Let $F(t)$ $(|t| \leqslant \pi)$ be a complex-valued function of *bounded variation* and let F have the Fourier expansion

$$F(t) \sim \sum_{-\infty}^{+\infty} f(n) e^{int} \quad (|t| \leqslant \pi).$$

We shall assume that F is normalized as to be right continuous for $-\pi \leqslant t < \pi$ and left continuous at $t = \pi$. We can assign to every such F a *Laurent operator* L_f as follows:

$$y(n) = \sum_{-\infty}^{+\infty} f(n-k) x(k), \quad n \in Z.$$

According to a result of Stečkin [12], *the Laurent operator* L_f *is a bounded operator on* l^p *whenever* $1 < p < \infty$, which is a consequence of the fact that *the Hilbert transform is a bounded operator on all* l^p*-spaces with* $1 < p < \infty$. For such Laurent operators on the l^p-spaces $(1 < p < \infty)$ Hartman showed that $\mathrm{sp}(L_f, [l^p]) =$ *closure of the range of* F.

A Young function Φ is said *to have the property* δ_2 whenever $\Phi(u) > 0$ for all $u > 0$, and there exists two constants $u_0 > 0$ and $m > 0$ such that $\Phi(2u) \leqslant m\Phi(u)$ for all $0 \leqslant u \leqslant u_0$. *An Orlicz space* l^Φ $(l^{M\Phi})$ *is reflexive if and only if* Φ *and* Ψ *have the property* δ_2 (see [7], p. 60).

If l^Φ $(l^{M\Phi})$ *is reflexive, then the Laurent operator* L_f *as defined above is a bounded operator on* l^Φ. This immediately follows from the result of Stečkin for the l^p-spaces $(1 < p < \infty)$ and Theorem 6.1 in [1]. Then essentially the same argument as given in Hartman's paper [3] shows that *for every normalized* F *and for every reflexive Orlicz space* l^Φ, $\mathrm{sp}(L_f, [l^\Phi]) =$ *closure of the range of* F.

Finally, we would like to point out that a slight modification of the proof of Theorem 4.3 shows that also in this case the spectral radius of L_f is equal to

$$|||f|||_2 = \sup(\|f * x\|_2 : \|x\|_2 \leqslant 1).$$

As a consequence we obtain the well-known result that if H denotes the *Hilbert transform*

$$y(n) = \sum_{-\infty}^{+\infty}{}' x(k)/(n-k) \quad (n \in Z),$$

where $\sum'$ means that the terms with $k = n$ are omitted, then the norm $\|H\|_p$ $(1 < p < \infty)$ of the Hilbert transform satisfies $\|H\|_p = \|H\|_q \geqslant \|H\|_2 = \pi$, where $1 < p < \infty$ and $1/p + 1/q = 1$. Of course, for every reflexive l^Φ, we have also $\|H\|_\Phi \geqslant \pi$ and $\|H\|_{M\Phi} \geqslant \pi$.

Remark. In an oral communication D. Boyd pointed out to me the following converse: *If the Hilbert transform* H *is bounded on* l^Φ, *then* l^Φ *is reflexive.*

References

[1] D. W. Boyd, *The Hilbert transform on rearrangement invariant spaces*, Can. J. Math. 19 (1967), p. 599-616.

[2] M. Eisen and H. Gindler, *On the spectrum of a Laurent form*, Amer. Math. Monthly 73 (1966), p. 79-90.

[3] P. Hartman, *On Laurent operators on* l_p, Proc. Amer. Math. Soc. 8 (1957), p. 45-48.

[4] G. L. Krabbe, *Abelian rings and spectra of operators on* l_p, ibidem 7 (1956), p. 783-790.

[5] — *On the spectra of certain Laurent matrices*, ibidem 8 (1957), p. 894-897.

[6] L. Loomis, *An introduction to abstract harmonic analysis*, Princeton 1953.

[7] W. A. J. Luxemburg, *Banach function spaces*, Thesis, Assen 1955.

[8] — *Rearrangement-invariant Banach function spaces*, Proc. Symp. in Analysis, Queen's papers in Pure and Applied Mathematics 10 (1967), p. 83-144.

[9] — and A. C. Zaanen, *Notes on Banach function spaces*, Proc. Acad. Sci. Amsterdam A 66 (1963); Note I, p. 135-147; Note II, p. 148-153; Note III, p. 239-250; Note IV, p. 251-263; Note V, p. 496-504.

[10] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Int. Acad. Pol. Sci. Lettres, Cl. Math. Natur., Série A (1932), p. 207-220; and *Über Räume* (L^M), ibidem, Série A (1936), p. 93-107.

[11] C. E. Rickart, *General theory of Banach algebras*, Princeton 1960.

[12] S. B. Stečkin, *On bilinear forms*, Dokl. Ak. Nauk SSSR, N. S. 71 (1950), p. 237-240 (in Russian).

[13] O. Toeplitz, *Theorie der quadratischen und bilinearen Formen von unendlichvielen veränderlichen. I Teil: Theorie der L-Formen*, Math. Annalen 70 (1911), p. 351-376.

[14] A. C. Zaanen, *Linear analysis*, Amsterdam 1953.

[15] A. Zygmund, *Trigonometric series*, vol. II, Cambridge 1959.

[16] Wiskundige Opgaven met de Oplossingen 20 (1955), p. 41-43, Problem 40.

Reçu par la Rédaction le 4. 3. 1968