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Continuity for linear maps on Banach algebras *

by

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Dedicated to Professors
Stanisław Mazur and Władysław Orlicz
on the occasion of the 40th anniversary
of their scientific research

1. Introduction. Let B be a complex Banach algebra with an involution $x \to x^*$ and an identity. There are important situations where one must deal with linear maps on B which are not homomorphisms. We mention the theory of positive linear functionals ([5], Chapt. IV). More generally, we consider positive linear maps of B into a B^*-algebra A (linear maps T on B such that $T(x^*x) \geqslant 0$ in A for all x in B). In the special case where B is also a B^*-algebra these maps have been extensively studied; see [6] where further references can also be found.

In this note we investigate the continuity of linear maps T on B into a complex normed linear space. It is a very special case of our Theorem 2.5 that any positive linear map of B into a B^*-algebra is continuous.

2. A continuity theorem. Throughout this paper let B denote a complex Banach algebra with an involution $x \to x^*$ and an identity e. It is not assumed that the involution is continuous on B. Let H be the set of self-adjoint elements of B and let R denote the radical of B. Clearly $B = H + iH$ and $R = R^*$. As usual, we call an element x *normal* if $xx^* = x^*x$.

2.1. LEMMA. *Let x be a normal element of B. Suppose that x^{-1} exists and that x can be expressed in the form $x = a + w$, $a \in H$, $w \in R$. If $x^n \in H$, for some positive integer n, then $x \in H$.*

Proof. We can write $w = u + iv$, where u and v lie in $H \cap R$. This gives $x = h + iv$, where $h = a + u \in H$. Since x is normal, $hv = vh$. Moreover, $x^{-1}h = e - ix^{-1}v$. Consequently, as $v \in R$, we see that $x^{-1}h$ has a two-sided inverse in B. Therefore h^{-1} exists in B.

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By assumption, $x^n \in H$ for some $n > 1$. By the binomial theorem and $hv = vh$ we can write $x^n = (h+iv)^n = f+ig$, where

$$f = h^n - \binom{n}{2} h^{n-2} v^2 + \ldots,$$

$$g = nh^{n-1}v - \binom{n}{3} h^{n-3} v^3 + \ldots$$

Thus f and g lie in H. Since $x^n \in H$, we have $g = 0$. Therefore $vz = 0$, where

$$z = nh^{n-1} - \binom{n}{3} h^{n-3} v^2 + \ldots$$

But z is the sum of an invertible element and an element in R. Hence z^{-1} exists. It follows that $v = 0$, so that $x = u \in H$.

2.2. LEMMA. *Let z lie in the closure of H. Then $z = a+w$, where $a \in H$, $w \in R$.*

Proof. Let π be the natural homomorphism of B onto B/R. The definition $[\pi(x)]^* = \pi(x^*)$ defines an involution on the semi-simple Banach algebra B/R. Now by [4], p. 1068, $\|\pi(x)\|_1 = \|\pi(x^*)\|$ is a Banach algebra norm for B/R. Consequently, the uniqueness of norm property [3] for B/R implies that the involution on B/R is continuous.

Next let $z = \lim h_n$, $h_n \in H$. Then $\pi(z) = \lim \pi(h_n)$ is selfadjoint in B/R.

The case $n = 2$ of the following result was established by Ford [2]. As in [5] we use the notation $\nu(x)$ to denote the spectral radius of x.

2.3. LEMMA. *Let $h \in H$. Suppose that $\nu(e-h) < 1$. Then, for each $n = 2, 3, \ldots$, there exists $x \in H$ such that $x^n = h$.*

Proof. Take any $n = 2, 3, \ldots$ Consider the binomial expansion

$$(1+z)^{-1/n} = \sum_{r=0}^{\infty} c_r z^r.$$

Then the series

$$x = \sum_{r=0}^{\infty} c_r (-1)^r (e-h)^r \tag{2.1}$$

converges in B and $x^n = h$. Our aim is to show that x is self-adjoint.

First observe that h^{-1} exists ([5], p. 12). Consequently x^{-1} exists. Next we show that x is normal. To this end let A denote a maximal commutative $*$-subalgebra of B containing h. Whether or not the involution is continuous, it is known that A is closed ([5], p. 182). Therefore $x \in A$, so that x is normal.

Lemma 2.2 and formula (2.1) show that $x = a+w$, where $a \in H$, $w \in R$. All the hypotheses of Lemma 2.1 are now seen to be fulfilled, so we get $x \in H$.

The following variant is sometimes useful:

2.4. LEMMA. *Let $h \in H$, where $\mathrm{sp}(h) \subset (0\ \infty)$. Then, for each $n = 2, 3, \ldots$, there exists $x \in H$, $x^n = h$, where $\mathrm{sp}(x) \subset (0, \infty)$.*

Proof. Consider $v = h/\|h\|$. Then $\mathrm{sp}(v) \subset (0, 1]$ so that $\nu(e-v) < 1$. Lemma 2.3 shows that v, and hence h, has a self-adjoint n^{th} root, $x^n = h$. Let A be a maximal commutative $*$-subalgebra of B containing h and let $\mathfrak{M}$ be its space of maximal ideals. Formula (2.1) shows that $x(M) > 0$ for each $M \in \mathfrak{M}$, so that ([5], Theorem 4.1.3) $\mathrm{sp}(x) \subset (0, \infty)$.

2.5. THEOREM. *Let T be a linear mapping of B into a complex normed linear space E. Suppose that there exists an integer $n \geqslant 2$ and a real number $c > 0$ such that*

$$\|T(u^n + v^n)\| \geqslant c\|T(v^n)\|$$

for all u, v which are self-adjoint in B. Then T is continuous.

Proof. Let $h \in H$, where $\nu(h) < 1$. Lemma 2.3 shows that there exist elements $u, v \in H$ with $u^n = e+h$, $v^n = e-h$. Then our norm hypothesis yields

$$2\|T(e)\| \geqslant c\max\big(\|T(e+h)\|, \|T(e-h)\|\big).$$

Thus $4\|T(e)\| \geqslant c\|T(e+h) - T(e-h)\| = 2c\|Th\|$. Let k be any self-adjoint element and $\varepsilon > 0$. The last inequality applied to $h = k/\big(\nu(k)+\varepsilon\big)$ shows that

$$\|T(k)\| \leqslant 2c^{-1}\|T(e)\|\nu(k) \tag{2.2}$$

for all $k \in H$.

Formula (2.2) shows that T vanishes on the radical R of B inasmuch as $R = R^*$. Thus T defines a linear mapping $T^{\#}$ of B/R into E by the rule $T^{\#}(x+R) = T(x)$.

It is easy to see that $\mathrm{sp}(y+z) = \mathrm{sp}(y)$ for all $y \in B$, $z \in R$. Then, for $k \in H$, using (2.2) we get

$$\|T^{\#}(k+R)\| \leqslant 2c^{-1}\|T(e)\|\nu(k+z) \leqslant 2c^{-1}\|T(e)\|\|k+z\|$$

for all $z \in R$. Therefore

$$\|T^{\#}(k+R)\| \leqslant 2c^{-1}\|T(e)\|\|k+R\|, \qquad k \in H. \tag{2.3}$$

Let $x_n \to x$ in B, where $x_n = u_n + iv_n$, $x = u+iv$ and u_n, v_n; u, $v \in H$. We wish to show that $T(x_n) \to T(x)$, and accomplish this by showing that $T(u_n) \to T(u)$ and $T(v_n) \to T(v)$ (even though $u_n \to u$, $v_n \to v$ may not hold).

Let π be the natural homomorphism of B onto B/R. The proof of Lemma 2.2 gives the continuity of the involution in B/R. Therefore $\pi(u_n)\pm i\pi(v_n)\to\pi(u)\pm i\pi(v)$. Hence $\pi(u-u_n)\to 0$, where $\pi(u-u_n)$ is self-adjoint in B/R. Formula (2.3) then shows that $T^{\#}[\pi(u-u_n)]\to 0$. Consequently $T(u_n)\to T(u)$. Likewise $T(v_n)\to T(v)$.

2.6. COROLLARY. *Let E be a B^*-algebra. Suppose that, for some integer n, $T(h^n)\geqslant 0$ in E for all $h\in H$. Then T is continuous.*

Proof. Let $u, v\in H$. Then ([1], p. 15)

$$\|T(u^n+v^n)\|\geqslant\|T(v^n)\|$$

and we may apply Theorem 2.5. In particular, if f is any linear functional on B with $f(h^2)\geqslant 0$ for all $h\in H$, we have f continuous.

Theorem 2.5 shows the continuity of T in other situations as well. Let S denote the cone in B consisting of all finite sums of the form $\sum h_j^2$, where each $h_j\in H$. Let T be a linear map of B into $L([0,1])$, say. Then T is continuous if $T(g)\geqslant 0$ almost everywhere for each $g\in S$.

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Über nukleare Folgenräume

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T. und Y. Kōmura haben in [1] bewiesen, daß jeder nukleare Raum eingebettet werden kann in einen Raum $(s)^A$, s der Raum der schnell fallenden Folgen, A eine genügend große Indexmenge. Ihr Verfahren läßt sich für nukleare Folgenräume so vereinfachen, daß gleichzeitig eine zusätzliche Einsicht in die Struktur der nuklearen Folgenräume gewonnen werden kann.

Es sei P eine Menge von Folgen $a=(a_n)\geqslant 0$ (d.h. alle $a_n\geqslant 0$) mit den Eigenschaften: a) zu jedem n gibt es ein $a\in P$ mit $a_n\neq 0$, b) P ist gerichtet, d.h. zu $a^{(1)},\ldots,a^{(k)}$ in P gibt es stets ein $a\in P$ und ein $M>0$ mit $a^{(i)}\leqslant Ma$, d.h. $a_n^{(i)}\leqslant Ma_n$ für alle i und n.

Für gegebenes P ist $\lambda(P)$ der Raum aller Folgen $x=(x_n)$ mit $p_a(x)=\sum_{n=1}^{\infty}a_n|x_n|<\infty$ für alle $a\in P$. Die $p_a(x)$ sind die Halbnormen der auf $\lambda(P)$ definierten Topologie T.

Ist M eine Menge von Folgen $b=(b_n)$, so besteht die normale Hülle von M aus allen Folgen $y=(y_n)$ mit $|y_n|\leqslant|b_n|$, für alle n für ein geeignetes $b\in M$. Eine Menge heißt *normal*, wenn sie mit ihrer normalen Hülle zusammenfällt.

Bezeichnet λ' die normale Hülle der Menge, die aus allen positiven Vielfachen der Folgen aus P besteht, so ist λ' ein linearer Folgenraum. Aus der Definition von $\lambda(P)$ folgt, daß $\lambda(P)$ gleich $(\lambda')^x$ ist, dem a-dualen Raum zu λ' (vgl. [2], § 30). $\lambda(P)$ ist also ein vollkommener Folgenraum. Da für ein $u=(u_n)$ aus λ' stets $\sum_{n=1}^{\infty}|u_n|\,|x_n|<\infty$ für jedes x aus $\lambda(P)$ gilt, ist λ' ein normaler Teilraum von λ^x, braucht jedoch nicht mit λ^x zusammenzufallen. Die Topologie T von $\lambda(P)$, die auch durch alle $p_b(x)$, $b\geqslant 0$, aus λ' erzeugt werden kann, ist also im allgemeinen schwächer als die durch die positiven Elemente aus λ^x definierte normale Topologie.

Wie in [2], § 30, 4. (5) ist $\lambda(P)$ als projektiver Limes der durch die einzelnen a aus P definierten vollständigen Räume λ_a darstellbar, also vollständig in der Topologie T (vgl. [2], § 19, 10. (2)). Schließlich ergibt