STUDIA MATHEMATICA, T. XXXI. (1968)

An interpolation theorem on Banach function spaces

by

TETSUYA SHIMOGAKI (Sapporo)

Dedicated to Professors
S. Mazur and W. Orlicz

1. Introduction. An operator U, that maps a Banach space X into itself, is called a *Lipschitz operator*, if $U0 = 0$ and if $\|Uf - Ug\| \leqslant K\|f-g\|$ $(f, g \in X)$ for some $K > 0$. The smallest K in this inequality is called the *bound* of U. By $\mathrm{Lip}(X; K)$ we denote the class of all Lipschitz operators U from X into itself with the bound not exceeding K. Let L^1 and L^∞ be the space of Lebesgue measurable functions on a (finite or infinite) interval $(0, l)$. In his papers [9, 10] Orlicz proved the following theorem (in the case when $l < +\infty$):

THEOREM A. *If* $U \in \mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty)$, U *is a Lipschitz operator on an arbitrary Orlicz space* L^M, *that is,*

$$\|Uf - Ug\|_M \leqslant K_M \|f-g\|_M, \quad f, g \in L^M,$$

where $\|\cdot\|_M$ *is the norm of the space* L^M.

This theorem was generalized by several authors to the case where the Orlicz spaces L^M are replaced by rearrangement invariant Banach function spaces X. For operators U given by an integral transformation it was proved by Lorentz [2]; for the case when L^∞ is dense in X and U is linear by Mitjagin [6]; for general case, but for quasi-linear operators by Calderón [1], and for Lipschitz operators by Lorentz and Shimogaki [4, 5]. The present paper is concerned with interpolation theorems for operators U of weak type (1,1) (see [11]). By ω-$\mathrm{Lip}(L^1; K)$ we denote the class of operators U satisfying $U0 = 0$ and

$$\|Uf - Ug\|_1^* \leqslant K\|f-g\|_1, \quad f, g \in L^1, \tag{1.1}$$

where $\|\cdot\|_1^*$ is defined for a measurable h by

$$\|h\|_1^* = \sup_{y>0}\{y d_h(y)\}, \quad d_h(y) = \mathrm{mes}\{t : |h(t)| > y\}. \tag{1.2}$$

icm©

Since $\|\cdot\|_1^* \leqslant \|\cdot\|_1$ holds, $\mathrm{Lip}(L^1; K) \subset \omega\text{-}\mathrm{Lip}(L^1; K)$ stands, but the converse does not hold. Now we call a Banach space X to have the *interpolation property for the class* $\omega\text{-}\mathrm{Lip}(L^1; K_1) \frown \mathrm{Lip}(L^\infty; K_\infty)$, if, for each $U \in \omega\text{-}\mathrm{Lip}(L^1; K_1) \frown \mathrm{Lip}(L^\infty; K_\infty)$, U can be considered as a Lipschitz operator from X into itself. The aim of this paper is to obtain a necessary and sufficient condition in order that a Banach function space X has the interpolation property for the class $\omega\text{-}\mathrm{Lip}(L^1; K_1) \frown \mathrm{Lip}(L^\infty; K_\infty)$. In what follows, a Banach space $(X, \|\cdot\|)$, whose elements are complex-valued integrable functions over the interval $(0, l)$, will be called a *Banach function space*, if it satisfies the following conditions:

(1.3) $|g| \leqslant |f|$ (1), $f \in X$ *implies* $g \in X$ *and* $\|g\| \leqslant \|f\|$;

(1.4) $0 \leqslant f_n \uparrow$, $\|f_n\| \leqslant M$ $(n = 1, 2, \ldots)$ *implies* $f_0 = \bigcup_{n\geqslant 1} f_n \in X$ *and* $\|f_0\| = \sup_n \|f_n\|$.

It follows that the norm of X is always *semi-continuous*, that is, $0 \leqslant f_n \uparrow f$ implies $\|f\| = \sup_n \|f_n\|$. The space $(X, \|\cdot\|)$ is called *rearrangement invariant*, if $0 \leqslant f \in X$ implies $g \in X$ and $\|f\| = \|g\|$ for each g equi-measurable with f. Orlicz spaces and Lorentz spaces $\Lambda(\Phi)$ and $M(\Phi)$ (see [3]) are rearrangement invariant spaces. For non-negative number $a > 0$ the function f_a is given by $f_a(x) = f(ax)$, if $ax \leqslant l$, $f_a(x) = 0$ if $ax > l$. We write also

$$\sigma_a f = f_a. \tag{1.5}$$

It is easy to see that σ_a is a bounded linear operator on X, if X is rearrangement invariant. The main theorem is the following:

THEOREM 1. *Let $(X, \|\cdot\|)$ be a rearrangement invariant Banach function space. In order that X has the interpolation property for the class $\omega\text{-}\mathrm{Lip}(L^1; K_1) \frown \mathrm{Lip}(L^\infty; K_\infty)$, it is necessary and sufficient that there exist positive numbers K and p $(0 \leqslant p < 1)$ such that* (2)

$$\|\sigma_a\|_X \leqslant K a^{-p}, \qquad 0 < a < 1. \tag{1.6}$$

We call a Banach function space X to have the *Hardy-Littlewood Property* and write $X \in \mathrm{HLP}$ (see [3]), if $f \in X$ implies $\theta_f \in X$, where

$$\theta_f(x) = \sup_{0\leqslant y\leqslant l} \int_x^y \frac{f(t)}{y-x}\, dt, \qquad 0 < x < l. \tag{1.7}$$

(1) $g \leqslant f$ means that $g(t) \leqslant f(t)$ holds almost everywhere.

(2) $\|\sigma_a\|_X$ denotes the norm of the linear operator σ_a on X. (1.6) is equivalent to the condition $a\|\sigma_a\|_X < 1$ for an $a > 1$ [8].

In course of the proof of Theorem 1 it is shown that the interpolation property considered here is actually equivalent with that of HLP. In Section 2 we use the following theorem:

THEOREM B. *$X \in HLP$ if and only if* (1.6) *holds. Furthermore, in this case,*

$$\|\theta_f\| \leqslant \Big(\int_0^1 \|\sigma_a\|_X da\Big) \|f\| \leqslant \frac{K}{1-p} \|f\| \tag{1.8}$$

for all non-negative decreasing $f \in X$.

This theorem was proved in [3, 7] in case of $l < +\infty$. It is, however, easy to see that the result remains valid for an arbitrary rearrangement invariant Banach function space X of integrable functions (or, more generally, X of locally integrable functions) over $(0, l)$, $0 < l \leqslant +\infty$ (cf. [3], (9), and [7], Theorem 1). In Section 2 the proof of Theorem 1 is given. In Section 3 supplementary results are stated and a theorem analogous to Theorem 1 for quasi-linear operators is proved.

2. Proof of Theorem 1. For $a > 0$, $f^{(a)}$ denotes the a-truncation of f, that is, $f^{(a)}(x) = f(x)$ if $|f(x)| \leqslant a$, $f^{(a)}(x) = a\,\mathrm{sign}\, f(x)$ if $|f(x)| > a$. For a measurable function f, f^* is the *decreasing rearrangement of* $|f|$. We write $g \dot{\prec} f$, if $\|g^* \chi_{(0,a)}\|_1^* \leqslant \|f^* \chi_{(0,a)}\|_1$ for every $0 < a < l$, where $\chi_{(0,a)}$ is the characteristic function of the interval $(0, a)$. By D we denote the set of all non-negative decreasing functions belonging to L^1.

LEMMA 1. *There exists a positive number γ such that $\|g\| \leqslant \gamma \|f\|$ holds for each g, f with $g \dot{\prec} f$, $f \in X$, if and only if* (1.6) *holds.*

Proof. Suppose first that (1.6) holds for X. By $\tilde{f}$ we denote the function defined by

$$\tilde{f}(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad 0 < x < l. \tag{2.1}$$

If $f \in D$, $\tilde{f}$ coincides with θ_f. By Theorem B, $X \in \mathrm{HLP}$, hence $\tilde{f} \in X$. Furthermore, on account of (1.8) we have

$$\|\tilde{f}\| = \|\theta_f\| \leqslant \frac{K}{1-p} \|f\|.$$

We have also

$$\|\tilde{f}\chi_{(0,a)}\|_1^* = \int_0^a f(x)\, dx = \|f\chi_{(0,a)}\|_1. \tag{2.2}$$

Now assume that $g \dot{\prec} f$, $f \in X$. For every a $(0 < a < l)$ one has

$$\|g^* \chi_{(0,a)}\|_1^* \leqslant \|f^* \chi_{(0,a)}\|_1 = \|\tilde{f^*} \chi_{(0,a)}\|_1^*.$$

From this and (2.2) it follows that

$$ag^*(a) \leqslant \int_0^a f^*(x)\,dx,$$

hence $g^* \leqslant \tilde{f}^*$. Since $\tilde{f}^* \in X$ and $\|\tilde{f}^*\| \leqslant K\|f\|/(1-p)$, we see that $g \in X$ and $\|g\| \leqslant \|\tilde{f}^*\| \leqslant K\|f\|/(1-p)$.

Conversely, if such a γ exists, then $\tilde{f}$ must belong to X for each $f \in D$, because $\tilde{f} \dot{\succ} f$, as is shown above. Therefore, $X \in \mathrm{HLP}$ ([3], Lemma).

The next lemma is analogous to that of [5].

LEMMA 2. *For $U \in \omega\text{-}\mathrm{Lip}(L^1; 1) \cap \mathrm{Lip}(L^\infty; 1)$ and for $f \in L^1$ we have $Uf \dot{\prec} f$.*

Proof. For each $0 < a < l$ we put $\beta = (Uf)^*(a)$. For arbitrary, but fixed y with $y > \beta$, we can find a measurable set e_1 such that $\mathrm{mes}(e_1) = d_{Uf}(y) \leqslant a$ and $y\chi_{e_1} \leqslant |Uf|$. We can also find a measurable set e_2 such that $\mathrm{mes}(e_2) = a$ and

$$\int_0^a f^*(t)\,dt = \int_{e_2} |f(t)|\,dt.$$

If $y \leqslant f^*(a)$, then

$$y d_{Uf}(y) \leqslant f^*(a)\,a \leqslant \int_0^a f^*(t)\,dt.$$

On the other hand, if $\alpha = f^*(a) < y$, then

$$(y-\alpha)\,d_{Uf}(y) \leqslant (y-\alpha)\,d_{|Uf|-\alpha\chi_{e_1}}(y-\alpha) \leqslant \|Uf-(Uf)^{(\alpha)}\|_1^*.$$

Since $U \in \omega\text{-}\mathrm{Lip}(L^1; 1) \cap \mathrm{Lip}(L^\infty; 1)$,

$$\|U(f^{(\alpha)})\|_\infty \leqslant \|U(f^{(\alpha)}) - U0\|_\infty \leqslant \|f^{(\alpha)}\|_\infty \leqslant \alpha,$$

which implies

$$|Uf-(Uf)^{(\alpha)}| \leqslant |Uf - U(f^{(\alpha)})|.$$

Therefore we obtain

$$(y-\alpha)\,d_{Uf}(y) \leqslant \|Uf-(Uf)^{(\alpha)}\|_1^* \leqslant \|Uf-U(f^{(\alpha)})\|_1^* \leqslant \|f-f^{(\alpha)}\|_1.$$

We have also $\alpha d_{Uf}(y) = \alpha\,\mathrm{mes}(e_1) \leqslant \alpha a$. Consequently, we get

$$y d_{Uf}(y) \leqslant \|f-f^{(\alpha)}\|_1 + \alpha a = \int_{e_2} |f-f^{(\alpha)}|(t)\,dt + \alpha\,\mathrm{mes}(e_2) = \int_0^a f^*(t)\,dt.$$

Since $y > \beta$ is arbitrary, it follows that

$$\|(Uf)^*\chi_{(0,a)}\|_1^* \leqslant \|f^*\chi_{(0,a)}\|_1.$$

Again, a being arbitrary, we see that $Uf \dot{\prec} f$ holds.

Proof of Theorem 1. Sufficiency. Suppose that (1.6) holds, and let

$$U \in \omega\text{-}\mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty).$$

We put $K_0 = \mathrm{Max}(K_1, K_\infty)$ and $U_0 = U/K_0$. We fix $h \in X \cap L^\infty$ and define the operator V by

$$Vf = U_0(f+h) - U_0 h, \quad f \in L^1 \cup L^\infty.$$

V belongs to $\omega\text{-}\mathrm{Lip}(L^1; 1) \cap \mathrm{Lip}(L^\infty; 1)$. Hence for each $f \in X$, $Vf \dot{\prec} f$. From Lemma 1 it follows that $Vf \in X$ and $\|Vf\| \leqslant K\|f\|(1-p)$, which means

$$\|U_0(f+h) - U_0 h\| \leqslant \frac{K}{1-p}\|f\|,$$

hence

$$\|U_0 f - U_0 h\| = \|V(f-h)\| \leqslant \frac{K}{1-p}\|f-h\|, \quad f \in L^1.$$

For arbitrary $f, g \in X$, we consider the truncations $f^{(n)}, g^{(n)}$. Then $U_0(f^{(n)})$ and $U_0(g^{(n)})$ converge to $U_0 f$ and $U_0 g$, respectively, in measure, since $U_0 \in \omega\text{-}\mathrm{Lip}(L^1; 1)$. Thus for a properly chosen sequence n_i, $U_0(f^{(n_i)})$ and $U_0(g^{(n_i)})$ converge almost everywhere to $U_0 f$ and $U_0 g$ simultaneously. Since

$$\|U_0(f^{(n_i)}) - U_0(g^{(n_i)})\| \leqslant \frac{K}{1-p}\|f^{(n_i)} - g^{(n_i)}\| \leqslant \frac{K}{1-p}\|f-g\|,$$

we have

$$\|U_0 f - U_0 g\| \leqslant \frac{K}{1-p}\|f-g\|$$

by virtue of Fatou's Property, which is implied by the semi-continuity of $\|\cdot\|$. Consequently, we have

$$\|Uf - Ug\| \leqslant \frac{KK_0}{1-p}\|f-g\|, \quad f, g \in X.$$

Necessity. If (1.6) does not hold, then $X \notin \mathrm{HLP}$, and furthermore, as is shown in [7], Theorem 1, there exists an $f \in X \cap D$ for which $\tilde{f}_0 \notin X$. Now we define a linear operator T, which maps $f \in L^1 \cup L^\infty$ into $\tilde{f}$, that is,

$$(Tf)(x) = \tilde{f}(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad 0 < x < l.$$

This operator T obviously belongs to $\omega\text{-}\mathrm{Lip}(L^1; 1)$ and to $\mathrm{Lip}(L^\infty; 1)$, but can not be a Lipschitz operator from X into itself. Therefore, X has not the interpolation property.

3. Supplementary results. We denote by $|||U|||_X$ the bound of a Lipschitz operator U from X into itself. In [5] it is shown that if

$$U \in \mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty),$$

then, for an arbitrary rearrangement invariant X, $|||U|||_X \leqslant K_X = K_\infty \|\sigma_c\|_X$ where $c = K_\infty/K_1$. Moreover, K_X is the best possible. For the class ω-$\mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty)$ we similarly have

THEOREM 2. *If X satisfies (1.6) with K and p $(0 \leqslant p < 1)$, we have*

$$|||U|||_X \leqslant \frac{K \cdot K_\infty}{1-p} \|\sigma_c\|_X, \quad c = K_\infty/K_1. \tag{3.1}$$

More precisely, if we put $\gamma = \sup_{0 \neq f \in D} \|\tilde{f}\|/\|f\|$,

$$\sup\{|||U|||_X: U \in \omega\text{-}\mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty)\} = \gamma K_\infty \|\sigma_c\|_X, \quad c = K_\infty/K_1. \tag{3.2}$$

The proof is quite the same to that of [5]. So we omit it. Determination of $\|\sigma_c\|_X$ for some fundamental rearrangement invariant spaces is also found in [5].

When X is one of the Orlicz spaces, $\Lambda(\varphi)$-spaces, and $M(\varphi)$-spaces, the conditions equivalent to (1.6) (that is, equivalent to $X \in \mathrm{HLP}$) are given in [3], [5]. Thus these conditions are also equivalent to the interpolation property considered here. For example, as for Orlicz spaces L^M, L^M has the interpolation property for the class ω-$\mathrm{Lip}(L^1; K_1) \cap \mathrm{Lip}(L^\infty; K_\infty)$ if and only if N, the complementary function of M, satisfies the Δ_2-condition.

When $\|\cdot\|$ of X is *continuous*, that is, $f_n \downarrow 0$ implies $\|f_n\| \downarrow 0$, we can extend Theorem 1 to spaces X of locally integrable functions. In fact, we have

THEOREM 3. *Let X be a rearrangement invariant Banach function space over $(0, \infty)$ consisting of locally integrable functions. The statement of Theorem 1 remains true if U is replaced by the unique extension of U onto X.*

The proof is derived from the fact that we can prove that U is a Lipschitz operator from $Y = X \cap L^1 \cap L^\infty$ (equipped with the norm of X) into itself. Since $\|\cdot\|$ of X is continuous, Y is dense in X, hence there is a unique extension of U onto X, which is also a Lipschitz operator from X into itself.

Finally we give an interpolation theorem for quasi-linear operators which is analogous to Theorem 1. An operator T is called *quasi-linear*, if $|T(\lambda f)| = |\lambda|\,|Tf|$ and

$$|T(f+g)| \leqslant c\{|Tf| + |Tg|\}, \quad f, g \in \text{Domain of } T.$$

The following theorem is a special case of Theorem 10 of [1]. But, for the convenience of readers, we show here a direct proof of it by making use of the notion $\precsim$.

THEOREM 4. *Let T be a quasi-linear operator of weak type (1,1) with the bound K_1 and of type (∞, ∞) with the bound K_∞. If a rearrangement invariant Banach function space X satisfies (1.6), T is an operator from X into itself and*

$$\|Tf\| \leqslant \frac{2cK}{1-p} \mathrm{Max}(K_1, K_\infty) \cdot \|f\|, \quad f \in X, \tag{3.3}$$

holds, where K and p are the numbers appeared in (1.6).

Proof. For $f \in X$ we prove that $Tf \precsim 2cK_0 f$, where $K_0 = \mathrm{Max}(K_1, K_\infty)$, which in turn implies (3.3) by Lemma 1. Putting $T_0 = T/K_0$, we obtain $\|T_0 f\|_1^* \leqslant \|f\|_1$ and $\|T_0 g\|_\infty \leqslant \|g\|_\infty$ for $f \in L^1$ and $g \in L^\infty$, respectively. Since $\|h_1 + h_2\|_1^* \leqslant 2\{\|h\|_1^* + \|h_2\|_1^*\}$ holds, we have for $f \in L_1$

$$\|T_0 f\|_1^* \leqslant 2c\{\|T(f - f^{(a)})\|_1^* + \|Tf^{(a)}\|_1^*\}.$$

For each a, $0 < a < l$, put $\alpha = f^*(a)$ and take a measurable set e such that $\mathrm{mes}(e) = a$ and $\|(T_0 f)^* \chi_{(0,a)}\|_1^* = \|(T_0 f)\chi_e\|_1^*$. Then we get

$$\begin{aligned}\|(T_0 f)^* \chi_{(0,a)}\|_1^* = \|(T_0 f)\chi_e\|_1^* &\leqslant 2c\{\|T_0(f - f^{(\alpha)})\chi_e\|_1^* + \|(T_0 f^{(\alpha)})\chi_e\|_1^*\} \\ &\leqslant 2c\{\|f - f^{(\alpha)}\|_1 + \alpha\,\mathrm{mes}(e)\},\end{aligned}$$

because $\|T_0 f^{(\alpha)}\|_\infty \leqslant \|f^{(\alpha)}\|_\infty$. The last term is equal to

$$2c\int_0^a f^*(t)\,dt = 2c\|f^* \chi_{(0,a)}\|_1.$$

Since a is arbitrary, we have $T_0 f \precsim 2cf$, hence $Tf \precsim 2cK_0 f$.

We note that condition (1.6) is also related to an interpolation property concerning with the complete continuity of linear operators. This is studied in [8].

References

[1] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), p. 273-299.

[2] G. G. Lorentz, *Bernstein polynomials*, Toronto 1953.

[3] — *Majorants in spaces of integrable functions*, Amer. Jour. Math. 77 (1955), p. 484-492.

[4] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for operators in function spaces*, Jour. Functional Analysis 1 (1967).
[5] — *Majorants for interpolation theorems*, Memorial Volume to Dr. Ananda Raur (to appear).
[6] B. S. Mitjagin, *An interpolation theorem for modular spaces*, Mat. sbornik 66 (1965), p. 373-482.
[7] T. Shimogaki, *Hardy-Littlewood majorants in function spaces*, Jour. Math. Soc. Japan 17 (1965), p. 365-373.
[8] — *On the complete continuity of operators in an interpolation theorem*, J. Fac. Sci. Hokkaido Univ. Ser. 1.20 (1968), p. 109-114.
[9] W. Orlicz, *Ein Satz über die Erweiterung von linearen Operationen*, Studia Math. 5 (1934), p. 127-140.
[10] — *On a class of operations over the space of integrable functions*, ibidem 14 (1954), p. 302-309.
[11] A. Zygmund, *Trigonometric series II*, Cambridge 1959.

DEPARTMENT OF MATHEMATICS
HOKKAIDO UNIVERSITY

Reçu par la Rédaction le 28. 2. 1968

STUDIA MATHEMATICA, T. XXXI. (1968)

A note on a Szegö type properties of semi-spectral measures

by

W. MLAK (Kraków)

The present paper deals with some properties of semi-spectral measures of representations of subalgebras of $C(X)$, provided such measures exist. The properties in question were first proved for representations associated with completely non-unitary contractions in [7]. We try in the present paper to bring to light some prediction-free essential points of the reasonings involved in the proofs of similar properties within the frames of the general theory of representations of function algebras. We also present some examples of operator-theoretic interpretation of some simple features concerning function algebras in connection with the above-mentioned properties of semi-spectral measures.

Let H be a complex Hilbert space. The inner product of $f, g \in H$ is denoted by (f, g). $\|f\|$ stands for the norm of f induced by this product. We write $L(H)$ for the algebra of all linear bounded operators in H. $\|T\|$ stands for the norm of $T \in L(H)$, T^* for the adjoint of T. I is the identity operator in H.

Suppose we are given a compact Hausdorff space X. $C(X)$ ($C_R(X)$) is the Banach algebra of all complex (real) continuous functions on X with the norm

$$(*) \qquad \|u\| = \sup_X |u(x)|.$$

We say that $A \subset C(X)$ is the *algebra* (strictly: *subalgebra of* $C(X)$) if the following conditions are satisfied:

(1) A is a closed subspace of $C(X)$ which is closed under multiplication, i.e. $u, v \in A$ implies $uv \in A$.

(2) The function $u_0(x) \equiv 1$ belongs to A.

(3) The functions of A separate the points of X.

The mapping $\varphi : A \to L(H)$ is called the *representation* of A if the following holds true:

(4) φ is a homomorphism of A into $L(H)$ such that $\varphi(u_0) = I$.

(5) $\|\varphi(u)\| \leqslant \|u\|$ for every $u \in A$ ($\|u\|$ is defined by $(*)$).