geht. Es gibt jedoch noch viele andere Varianten, bei denen die Methode der unendlichen Gleichungen (auch kombiniert mit funktionentheoretischen Methoden) sich bewährt.

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Stability of order convergence and regularity in Riesz spaces

by

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Dedicated to Professors Stanislaw Mazur and Władysław Orlicz on the occasion of the 60th anniversary of their scientific research

1. Introduction. The years from 1928 until 1936, a period of rapid growth for Banach and Hilbert space theory, were also the time that the foundations were laid for the functional analytic theory of linear vector lattices. This was done, independently, by Riesz [10, 11], Kantorovitch [5, 6] and Freudenthal [1], and it is interesting to observe now, more than thirty years later, the different methods of approach. Riesz was interested primarily in what is now called the order dual space of a given partially ordered vector space, and he presented an extended version of his short 1928 Congress note in a 1940 Annals of Mathematics paper, a translation of a 1937 Hungarian paper. Freudenthal, in 1936, proved a "spectral theorem" for vector lattices, the significance of which is illustrated by the fact that the Radon-Nikodym theorem in integration theory as well as the spectral theorem for Hermitian operators in Hilbert space are corollaries, although it was not until early in the fifties that a direct method was indicated for deriving the spectral theorem for Hermitian operators from the abstract spectral theorem. Finally, around 1935, Kantorovitch began his extensive investigation of the algebraic and convergence properties of vector lattices, with applications to linear operator theory. A few years later, curiously enough between 1940 and 1944, important contributions to the subject were published by Nakano [6, 7, 8], Ogasa-pur [9], Yosida [13, 14] in Japan and Kakutani [2, 3] in the United States. Of the more recent progress we only mention the work by Kantorovitch, Nakano and their schools. In contrast with Banach and Hilbert space theory, however, where in recent books the main body of the theory has been welded into a unified and elegant whole, there is only a very small number of textbooks on partially or-
odered vector spaces (and in the existing ones the treatment is mainly restricted to the research of one school). This explains why the present paper is written in a somewhat expository style. There is one chapter on linear vector lattices in the Bourbaki work on integration, and we will follow Bourbaki’s example in calling any linear vector lattice a Riesz space. In the present paper we discuss several special properties which a Riesz space may possess (stability of order convergence, regularity, existence of a strong unit) and we prove that certain combinations of these properties are possible only if the space is of finite dimension. We believe that our results in this direction are a little more general than previously known results (cf. e.g. Theorem VI. 4.2 in [12]); our main purpose, however, is to present straightforward proofs.

2. Convergence in Riesz spaces. We recall the definitions of an ordered vector space and of a Riesz space. The real linear space $L$, with elements $f, g, \ldots$ and with null element 0, is called an ordered vector space if $L$ is partially ordered in such a manner that the partial ordering (denoted by $\preceq$) is compatible with the algebraic structure of $L$, i.e., $f \preceq g$ implies $f + h \preceq g + h$ for every $h \in L$, and $f \preceq g$ implies $af \preceq gf$ for every real $a \geq 0$. The ordered vector space $L$ is called a Riesz space if $L$ is a lattice with respect to the partial ordering, i.e., if for every pair $f, g \in L$ the supremum $\sup(f, g)$ and the infimum $\inf(f, g)$ with respect to the partial ordering exist in $L$. It will be assumed throughout the present paper that $L$ is a Riesz space. The notation $f \preceq g$ means that $f \preceq g, f \neq g$, and the subset $L^+ = \{f : f \in L, f \geq 0\}$ is called the positive cone of $L$. The positive cone has the property that if $f, g \in L^+$ and $a, b$ are non-negative real numbers, then $af + bg \in L^+$. Furthermore, if $f$ and $-f$ are simultaneously members of $L^+$, then $f = 0$.

The following abbreviations are widely used and well-known especially in the case that the elements of $L$ are real functions. Given any $f \in L$, we set

$$f^+ = \sup\{f, 0\}, \quad f^- = \sup\{-f, 0\}, \quad |f| = \sup\{|f|, -f|\}.$$  

It is not difficult to prove that $f^+, f^-$ and $|f|$ are members of $L^+$, $f = f^+ - f^-$, and $|f| = f^+ + f^-$. The formula $f = f^+ - f^-$ shows, in particular, that every element of $L$ is the difference of two elements from the positive cone. As might be expected, the triangle inequality

$$|f| - |g| \leq |f + g| \leq |f| + |g|$$

holds; the proof is derived by showing first that

$$(f + g)^+ \leq f^+ + g^+$$

and

$$(f + g)^- \leq f^- + g^-.$$
We finally recall that every Riesz space \( L \) has the dominated decomposition property, i.e., if \( 0 \leq u \leq v_1 + v_2 \) with \( v_1, v_2 \in L^+ \), then there exist \( u_1, u_2 \in L^+ \) such that \( u = u_1 + u_2 \) and \( u_i \leq v_i \) for \( i = 1, 2 \).

3. Projection properties and completeness properties. The subset \( D \) of the Riesz space \( L \) is called solid if \( f \leq g \) implies \( f \leq \|f\| \). The subset \( A \) of \( L \) is called an ideal (or order ideal) if \( A \) is a solid linear subspace of \( L \). The ideal \( A \) is called a band if \( A \) is closed under the operation of taking suprema, i.e., if for every subset of \( A \) possessing a supremum in \( L \) this supremum is a member of \( A \). It easily follows from the remarks on disjointness in the preceding section that for any arbitrary subset \( D \) of \( L \) the disjunct complement \( D^c \) of \( D \) (i.e., the set of all \( f \) such that \( f \perp g \) for all \( g \in D \)) is a band.

It is evident from the definitions that an arbitrary intersection of ideals (or bands) is an ideal (or band). Given the arbitrary subset \( D \) of \( L \), the intersection of all ideals including \( D \) is called the ideal generated by \( D \). The band generated by \( D \) is defined similarly. If \( D \) consists of one element \( f \), it is customary to speak about the principal ideal (principal band) generated by \( f \). Evidently, the principal ideal generated by \( f \) consists of all \( g \in L \) such that \( |g| \leq |f| \) for some positive number \( e \) (depending on \( g \), therefore). It is a little more difficult to prove that, for every subset \( D \) of \( L \), the band generated by \( D \) consists of all \( g \) such that \( |g| \) is the supremum of some subset of the ideal generated by \( D \).

We will prove two simple theorems, particular cases of which are well-known.

Theorem 3.1. Let \( A \) and \( B \) be ideals in the Riesz space \( L \).

(i) The algebraic sum \( A + B \) is an ideal. Given \( f \geq 0 \) in \( A + B \), there exists at least one decomposition \( f = f_1 + f_2 \) such that \( f_1 \in A, f_2 \in B \) and \( f_1, f_2 \geq 0 \).

(ii) We have \( A \perp B \) if and only if \( A \cap B = \{0\} \), where \( \{0\} \) is the ideal consisting only of the null element, i.e., if and only if \( A + B \) is a direct sum of \( A \cap B \). In this case, therefore, the decomposition \( f = f_1 + f_2 \) with \( f_1 \in A, f_2 \in B \) is unique, and \( f \geq 0 \) implies \( f_1, f_2 \geq 0 \).

Proof. (i) In order to show that \( A + B \) is an ideal, it is sufficient to prove that \( A + B \) is solid, i.e., we have to prove that \( f_1 + f_2 \) and \( |g| \leq |f| \) imply \( g \in A + B \). Hence, let \( f = f_1 + f_2 \) with \( f_1 \in A, f_2 \in B \), and let \( |g| \leq |f| \). Then

\[
g = g - g' + g'' \leq |f| \]
fially, we have $g_{\cdot A} + B$ and so $g = g_{\cdot A} + B$. This is the desired result.

Assume now that $f \geq 0$ and $f = f'' + f''$ with $f' + A$ and $f'' + B$. Then $0 \leq f < 1$, so by the dominated decomposition property there exists a decomposition $f = f_1 + f_2$ such that $0 < f_1 < f'$ and $0 < f_2 < f''$. Since $f_1 + A$ and $f'' + B$, it follows that $f_1 + A$ and $f'' + B$.

(ii) Assume first that $A \cap B$. Given $f \in A \cap B$, we have now that $f \leq f_1$, so $f = 0$. This shows that $A \cap B = \{0\}$. Conversely, assuming that $A \cap B = \{0\}$, it has been proved that $\inf\{f_1, f_2\} = 0$ for all $f_1, f_2 \in A$. This is evident because $\inf\{f_1, f_2\}$ is a member of $A \cap B$, and so must be the null element.

Theorem 3.2. If $A$ and $B$ are ideals in the Riesz space $I$ such that $A \oplus B = I$, then $B = A^d$ and $A = B^d$. In other words, $A$ and $B$ are new bands, the disjoint complement of the other one.

Proof. It follows from $A \oplus B = I$ that $A \cap B = \{0\}$, so $B \subseteq A$, and hence $B \subseteq A^d$. In order to prove the converse inclusion, assume that $0 \leq a \in A$. By hypothesis there exists a decomposition $a = u_1 + u_2$ with $u_1 \in A$ and $u_2 \in B$, and it follows from the preceding theorem that $u_1, u_2 \geq 0$. Hence $0 \leq u_1 \leq u_2 A^d$, which implies that $u_1 \in A^d$. But $u_2 \in B$ holds just as well, so $u_2 = 0$. It follows that $u = u_1 \in B$, and it has thus been proved that $A^d \subseteq B$. The final result is that $B = A^d$.

In view of the last theorem it is now appropriate to call any band $A$ such that $A \oplus A^d = I$ holds a projection band. If $A$ is simultaneously a principal band and a projection band, then $A$ is called a principal projection band.

Theorem 3.3. (i) The band $A$ in the Riesz space $I$ is a projection band if and only if, for any $u \in I^+$, the element

$$u_1 = \sup\{v : v \in A, 0 \leq v \leq u\}$$

exists, and in this case $u_1$ is the component of $u$ in $A$. Similarly,

$$u_2 = \sup\{w : w \in A^d, 0 \leq w \leq u\}$$

is then the component of $u$ in $A^d$. In other words, if $u_1$ and $u_2$ are these suprema, then $u = u_1 + u_2$ with $u_1 \in A$ and $u_2 \in A^d$.

(ii) The principal band $A_0$, generated by the element $v \in I^+$, is a projection band if and only if, for any $u \in I^+$, the element

$$u_1 = \sup\{w \cap (w \cap (u_0, u)) \mid u = 1, 2, \ldots, \}$$

exists, and in this case $u_1$ is the component of $u$ in $A$.

Proof. (i) Let $A$ be a projection band, so $I = A \oplus A^d$, and let $u \in I^+$ have the decomposition $u = u_1 + u_2$. Define the subset $V$ of $I$ by $V = \{v : v \in A, 0 \leq v \leq u\}$.

We have to prove that $u_1 = \sup V$ for any $v \in V$ we have $u \leq v \geq 0$, and $u \in I^+$ does have the composition $u = v = u_1 + u_2$, so $u_1 \leq v \geq 0$ by Theorem 3.1 (ii), i.e., $v \in u_1$. This shows that $u_1$ is an upper bound of $V$. On the other hand, $u_1$ is a member of $V$, and so $u_1 = \sup V$.

Assume now, conversely, that $A$ is a band with the property that

$$u_1 = \sup\{v : v \in A, 0 \leq v \leq u\}$$

exists for any given $u \in I^+$. Since $u_1$ is then a member of $A$, it will be sufficient for the proof of $A \oplus A^d = I$ to show that $u_1 = u_2 = v$ is a member of $A^d$. If not, we have $v = \sup\{u_1, u_2\} > 0$ for some $v \in A$. Then $0 < u < A$, and also $p \leq u_1$. Hence $u_1 + p \in A$ as well as $u_1 + p \leq u_1 + u_2 = u$, so $u_1 + p$ is a member of the set $\{v : v \in I^+, 0 \leq v \leq u\}$. But then $u_1 + p$ is less than or equal to the supremum $u_1$ of this set, so $u_1 + p \leq u_1$, i.e., $p < 0$, which contradicts $p > 0$.

(ii) Given the principal band $A_0$ generated by $v \in I^+$ and given $u \in I^+$, we set

$$W = \{w : w \in A_0, 0 \leq w \leq u\}$$

and

$$W' = \{\inf\{w, w\} : u = 1, 2, \ldots, \}$$

It is evident that $W' \subseteq W$, and so any upper bound of $W$ is an upper bound of $W'$. In order to prove the converse, note that for any given $w \in W$ there exists a subset $D$ of the ideal generated by $v$ such that $w = \sup D$. Every element of $D$ is majorized by a positive multiple of $v$, and is also majorized by $w$ and hence by $u$. It follows that every element of $D$ is majorized by an element of $W$. This shows that any upper bound of $W'$ is an upper bound of $D$, and so of $w$, and hence of $W$. It has been proved that $W$ and $W'$ have the same upper bounds. In particular, the supremum of $W'$ exists if and only if the supremum of $W$ exists, and these suprema are then the same. By part (i), $A$ is a projection band if and only if $u_1 = \sup W$ exists, i.e., $v$ if and only if

$$w_1 = \sup W \leq \sup\{w_0 : w_0 \in I^+\}$$

exists, and in this case $w_1$ is the component of $w$ in $A$.

The Riesz space $L$ is said to have the projection property if every band is a projection band, and $L$ is said to have the principal projection property if every principal band is a projection band. Furthermore, as well-known, $L$ is called Dedekind complete (a $K$-space in the Soviet terminology) if every subset which is bounded from above has a supremum, and $L$ is called Dedekind-complete if every finite or countable subset which is bounded from above has a supremum.

It is obvious that Dedekind completeness implies Dedekind completeness, and it is well-known that Dedekind completeness implies the
projection property. It is obvious again that the projection property implies the principal projection property, and it is easy to prove that Dedekind $\sigma$-completeness also implies the principal projection property. Indeed, let $L$ be Dedekind $\sigma$-complete, and let $A$ be the principal band generated by $a \in L^+$. According to the last theorem, we have to prove that

$$\sup\{\inf(u, u_n) : n = 1, 2, \ldots\}$$

exists for any given $u \in L^+$. Writing $u_n = \inf(u, u_n)$ for $n = 1, 2, \ldots$, we have $0 \leq u_n \leq u$ for all $n$, so $\sup u_n$ exists on account of the Dedekind $\sigma$-completeness of $L$. Finally, we observe that if $L$ has the principal projection property, then $L$ is Archimedean. For the proof we have to show that if $0 \leq u \leq u$ holds for $n = 1, 2, \ldots$, then $v = 0$. It follows from $0 \leq u \leq u$ that $\inf(u, u_n) = u_n$ for all $n$, and it follows from the principal projection property that $\sup\{\inf(u, u_n)\} = u_0$, i.e., $u_0 = \sup u_n$ exists. But then

$$2u = \sup 2u_n = \sup u_n = u_0,$$

so $u_0 = 0$. This implies that $u = 0$ for all $n$, so $v = 0$.

4. Atoms. The element $f \neq 0$ in the Riesz space $L$ is called an atom whenever it follows from $0 \leq u \leq |f|$, $0 \leq u \leq |f|$ and $u \perp v$ that $u = 0$ or $v = 0$. If $f$ is an atom, then $\inf(a, \{f\}) = 0$ for every non-zero $a \neq 0$. If $f$ is an atom and $0 \leq |u| \leq |f|$, then either $g = 0$ or $g$ is an atom. If $f$ is an atom, then either $f > 0$ or $f < 0$. Indeed, since $0 < f^+ \leq |f|$, $0 < f^- \leq |f|$ and $f^+ \perp f^-$, we must have either $f^+ = 0$ or $f^- = 0$. In an Archimedean Riesz space we can say more.

**Theorem 4.1.** In an Archimedean Riesz space the following holds:

(i) If $f$ is an atom in $L$ and $0 \leq u \leq |f|$, then $u = a f$ for some real $a$.

(ii) If $f$ and $g$ are atoms in $L$, then either $f \perp g$ or $f = a g$ for some real $a \neq 0$.

(iii) If $A$ is the principal band generated by the atom $f$ in $L$, then $A$ consists of all real multiples of $f$, and $A$ is a projection band.

**Proof.** (i) This part is well-known; we briefly recall the proof. It may be assumed that $f > 0$ and $0 < u \leq |f|$ (the case $u = 0$ is trivial). The set of numbers $\beta : \beta u \leq f$ is non-empty and bounded from above since $L$ is Archimedean. Let $a = \sup\{\beta : \beta u \leq f\}$, so $1 \leq a < \infty$ and $au \leq f$. We will prove that $f = au$. If not, we have $v = f - au > 0$ and so, on account of $(v - su^+)^+ \perp v$ for $s \downarrow 0$, there exists a number $s$ such that $0 < s < a$ and $(v - su)^+ > 0$. It follows that

$$0 < (v - su)^+ = (f - (a + e)v)^+ = f^+ = f,$$

and so, since $a > 1$, we have that

$$0 < (f - (a + e)v)^+ \leq 2af.$$

Next, note that $0 < (f - (a + e)v)^+$ since otherwise $(a + e)v < f$, against the definition of $u$. Hence

$$0 < (f - (a + e)v)^+ = ((a + e)v - f)^+ \leq (a + e)v < 2af.$$

But (1) and (2) are contradictory since $2af$ is an atom. It follows that $f = au$, where $1 \leq a < \infty$ as observed above. Hence $u = a f$.

(ii) Assuming that $f$ and $g$ are atoms, we set $u = \inf\{|f|, |g|\}$. If $u = 0$, then $f \perp g$. If $u > 0$, then part (i) shows that $f = a u$ and $g = a u$, with $a \neq 0$, $a \neq 0$, and so $f = a u$. If $a = 0$, then $f = g$. If $a \neq 0$, then $f = g$.

(iii) By part (i) every element in the ideal generated by the atom $f$ is a real multiple of $f$. Now assume (as we may) that the atom $f$ is positive, and that $\epsilon L^+$ is an element in the band $A$ generated by $f$. Then $v$ is the supremum of a set of elements each of which is a non-negative multiple of $f$. It follows (since $L$ is Archimedean) that $v$ is also a non-negative multiple of $f$. Hence, the band $A$ consists of all real multiples of $f$. For the proof that $A$ is a projection band, it is sufficient to show that

$$\sup\{\inf(u, u_n) : n = 1, 2, \ldots\}$$

exists for every $u \in L^+$ (cf. Theorem 3.3 (ii)). For $u = 1, 2, \ldots$, we have $\inf(u, u_n) = a u_n$ for an appropriate increasing sequence $(a_n)$ of non-negative numbers. Since $a_{n+1} \leq u$ holds for all $n$, it is impossible that $a_n \uparrow \infty$. Hence $a_n \uparrow a < \infty$, and so

$$\inf(u, u_n) = a u_n \rightarrow a f,$$

where it has been used again that $L$ is Archimedean. This shows that the desired supremum exists, and so $A$ is a projection band.

**Theorem 4.2.** If $L$ is Archimedean and $(e_1, \ldots, e_n)$ is a set of mutually disjoint atoms in $L$ with the property that there exists no non-zero element in $L$ disjoint to $e_1, \ldots, e_n$, then $L$ is $n$-dimensional and $(e_1, \ldots, e_n)$ is a basis of $L$ in the algebraic sense. The algebraic decomposition of any $f \in L$ as a sum of real multiples of the basis elements is exactly the decomposition of $f$ as a sum of components of $f$ in the bands generated by the basis elements.

**Proof.** Since $(e_1, \ldots, e_n)$ is a linearly independent system, the dimension of $L$ is at least $n$. The bands $B_1, \ldots, B_n$ generated by $e_1, \ldots, e_n$ are projection bands by the preceding theorem; given $u \in L^+$, let $a_1 e_1, \ldots, a_n e_n$ be the corresponding components of $u$. Then $0 \leq a_i e_i \leq u$ for $i = 1, \ldots, n$, and so

$$0 \leq a_1 e_1 + \cdots + a_n e_n = \sup(a_1 e_1, \ldots, a_n e_n) \leq u,$$
where we have used that the sum and the supremum of a finite number of disjoint non-negative elements are identical. Thus

\[ w = u - (a_1 e_1 + \ldots + a_n e_n) \geq 0, \]

and so \( 0 \leq w \leq u - a_i e_i \) for \( i = 1, \ldots, n \). But \( u - a_i e_i \) is the component of \( u \) in the band \( B_i^u \); it follows that \( w \in B_i^u \), i.e., \( w \perp e_i \) for \( i = 1, \ldots, n \). Hence \( w = 0 \) by hypothesis, so \( u = a_1 e_1 + \ldots + a_n e_n \). The desired results follow immediately.

**Theorem 4.3.** If the Archimedean Riesz space \( L \) has the property that any system of mutually disjoint non-zero elements is finite, then \( L \) is of finite dimension, say of dimension \( n \), and there exists a basis \( \{ e_1, \ldots, e_n \} \) of mutually disjoint atoms.

**Proof.** Assuming that \( L \) does not consist exclusively of the null element, it will be proved first that \( L \) contains an atom. Indeed, \( L \) contains an element \( u > 0 \), and if \( u \) is an atom we are ready. If not, there exist \( u_1, u_2 \in L \) such that \( 0 < u_1 < u, 0 < u_2 < u \) and \( u_1 \perp u_2 \). If one of \( u_1, u_2 \) is an atom, we are ready; if not, we proceed and obtain non-zero and mutually disjoint elements \( u_1, u_2, u_3, u_4 \). The procedure breaks off after a finite number of steps since by hypothesis there is no infinite disjoint system of non-zero elements. Hence, \( L \) contains an atom \( e_1 \); let \( B_1 \) be the corresponding principal projection band. If \( B_1^u \neq (0) \), it is proved similarly that \( B_1^u \) contains an atom \( e_i \); this procedure again breaks off after a finite number of steps, and the desired result follows then from the preceding theorem.

Every Archimedean space \( L \) of finite dimension \( n \) is of the kind described in the last theorem, and hence there exists a basis \( \{ e_1, \ldots, e_n \} \) of mutually disjoint atoms. The partial ordering in \( L \) is such that \( f = a_1 e_1 + \ldots + a_n e_n > 0 \) if and only if all coefficients other than the last \( a_n \) are \( > 0 \). We obtain, therefore, as a corollary the known result that \( L \) is isomorphic to \( n \)-dimensional number space \( \mathbb{R}^n \) with coordinatewise ordering.

5. Spaces with a strong unit and with stable order convergence. The element \( e > 0 \) in the Riesz space \( L \) is called a strong unit if the ideal generated by \( e \) is already the whole space \( L \). Evidently, \( e > 0 \) is a strong unit if and only if for any given \( f \in L \) there exists a positive number \( a \) such that \( |f| < ae \). In a space with a strong unit \( e \) relatively uniform convergence of a sequence \( f_n \) to \( f \) is equivalent to \( e \)-uniform convergence of \( f_n \) to \( f \).

We will prove now that in an Archimedean Riesz space with a strong unit stability of the order convergence is a severe restriction upon the space. Precisely, the following theorem holds:

**Theorem 5.1.** Let the Archimedean Riesz space \( L \) have a strong unit, and let \( L \) be either Dedekind \( \sigma \)-complete or have the projection property. Then order convergence in \( L \) is stable if and only if \( L \) is of finite dimension.

**Proof.** If \( L \) is Archimedean and of finite dimension, then (according to the remarks in the final paragraph of the preceding section) \( L \) is isomorphic to \( \mathbb{R}^n \) for some \( n \), and hence it is evident that \( L \) has a strong unit and order convergence is stable.

Conversely, assume that the Archimedean space \( L \) has a strong unit \( e \), and is either Dedekind \( \sigma \)-complete or has the projection property. Assume also that order convergence in \( L \) is stable and that in \( L \) there exists an infinite system of mutually disjoint non-zero elements. Let \( \{ f_n : n = 1, 2, \ldots \} \) be a countable subsystem, and denote by \( B_n \) the band generated by \( f_n \). The space \( L \) has the principal projection property (we recall that Dedekind \( \sigma \)-completeness implies the principal projection property, and so of course does the projection property); for \( n = 1, 2, \ldots \), let \( p_n \) be the component of \( e \) in \( B_n \), and let \( s_n = f_1 + \ldots + f_{n+1} \). Evidently the sequence \( \{ s_n \} \) is increasing, and \( p = \sup_{n \in \mathbb{N}} s_n \) exists. If \( L \) is Dedekind \( \sigma \)-complete, this is evident; if \( L \) has the projection property, and if \( B \) is the band generated by the system \( \{ p_1, p_2, \ldots \} \), then the component \( p \) of \( e \) in \( B \) satisfies \( p = \sup_{n \in \mathbb{N}} s_n \), i.e., \( p = s_n \downarrow 0 \). Indeed, assume that \( 0 \leq v \leq p - s_n \) holds for all \( n \). Then, since \( p - s_n \) has the component 0 in the bands \( B_1, \ldots, B_n \), the same holds for \( v \). It follows that \( v \leq p_n \) for all \( n \), and so \( v \leq B \). On the other hand, we have \( v \leq B \) since \( 0 \leq v \leq p \). Hence \( v = 0 \), i.e., \( p = s_n \downarrow 0 \). In any case, therefore, \( s_n \) converges in order to \( p \). Observing now that order convergence and relatively uniform convergence are equivalent on account of the stability of order convergence, we obtain that \( s_n \rightarrow p \) (n.a.), which implies (as observed above) that \( s_n \) converges \( \epsilon \)-uniformly to \( p \). Hence, given \( \epsilon > 0 \) such that \( 0 < \epsilon < 1 \), there exists a natural number \( N \) such that \( p - s_n < \epsilon \). Taking components in \( B_N \) for any \( n > N \), we obtain \( p_n < s_n \), which is impossible on account of \( s_n \neq 0 \). We have derived, therefore, a contradiction. Hence, every system of mutually disjoint non-zero elements in \( L \) is finite. It follows then from the preceding theorem that \( L \) is of finite dimension.

In order to illustrate the fact that it is really the existence of a strong unit which forces an Archimedean Riesz space with stable order convergence to be of finite dimension, we present the following example. Let \( L \) be the Riesz space of all real sequences \( f = (f(1), f(2), \ldots) \) with only finitely many non-zero coordinates, and with pointwise ordering. This space is Dedekind complete and order convergence is stable. There is no strong unit in \( L \), in agreement with the fact that \( L \) is not of finite dimension. Note that every principal band in \( L \), considered as an Riesz space on its own, has a strong unit and, in agreement with the last theorem, every principal band is of finite dimension.
The theorem in the present section, under the extra hypothesis that \( L \) is Dedekind complete, is known (cf. for example Theorem VI. 4.2 in [12]). The proof is then based on the representation of \( L \) as the Riesz space of all real continuous functions on a certain compact topological space. The simple proof presented here avoids this representation theorem.

6. Regular Riesz spaces. The Riesz space \( L \) is called regular if the following conditions are satisfied:

(i) \( L \) is Archimedean.

(ii) Order convergence in \( L \) is stable.

(iii) For any sequence \( \{u_n\} \) in \( L^\infty \) there exists a sequence \( \{\lambda_n\} \) of positive real numbers such that the sequence \( \{\lambda_n u_n\} \) is bounded.

All Archimedean spaces of finite dimension are regular. The space presented as an example in the preceding section satisfies (i) and (ii), but not (iii). The (real) sequence space \( L_\infty \) satisfies (i) and (iii), but not (ii). The spaces \( L_p(1 \leq p < \infty) \) are regular, and the same holds more generally for spaces \( L_p(1 \leq p < \infty) \) of \( p \)-th power summable functions with respect to a countable additive measure.

The notion of a regular Riesz space is due to Kantorovitch [4]. In the original definition the space was assumed to be also Dedekind complete; in the present discussion we will not need this extra assumption. We first recall a simple lemma.

**Lemma 6.1.** (i) If \( L \) is regular and if \( \{\lambda_n\} \) is an arbitrary sequence in \( L \), then there exists a sequence \( \{\lambda_n u_n\} \) of positive real numbers such that \( \lambda_n u_n \rightarrow 0 \).

(ii) If \( L \) is regular and the double sequence \( \{f_{n,k}\} \) in \( L \) has the property that \( f_{n,k} \rightarrow f_k \) for every \( n \), then there exists an element \( u > 0 \) such that for every \( n \) the sequence \( \{f_{n,k}; k = 1, 2, \ldots\} \) converges \( u \)-uniformly to \( f_k \).

**Proof.** (i) Let \( \{\mu_n\} \) be a sequence of positive real numbers such that \( \{\mu_n f_n\} \) is bounded, so \( 0 \leq \mu_n f_n \leq v \) for some \( v \in L^\infty \) and all \( n \). Let \( \lambda_n = n^{-1} \mu_n \) for every \( n \). Then \( 0 \leq \lambda_n f_n \leq n^{-1} v \) holds for every \( n \), and so \( \lambda_n f_n \rightarrow 0 \).

(ii) Since order convergence and relatively uniform convergence are equivalent, there exists for every \( n \) an element \( u_n \in L^\infty \) such that \( f_{n,k}; k = 1, 2, \ldots\) converges \( u_n \)-uniformly to \( f_k \). Let \( \lambda_n > 0 \) such that the sequence \( \{\lambda_n u_n\} \) is bounded, say \( \lambda_n u_n \leq u \) for all \( n \). It follows that for every \( n \) the sequence \( \{f_{n,k}; k = 1, 2, \ldots\} \) converges \( u \)-uniformly to \( f_k \).

The Riesz space \( L \) is said to have the diagonal property if it follows from \( f_{n,k} \rightarrow f_n \) and \( f_n \rightarrow f \) that there exists a diagonal sequence \( \{f_{n,k_n}\}; n = 1, 2, \ldots\) with \( k(1) < k(2) < \ldots \) such that \( f_{n,k_n} \rightarrow f \).

**Theorem 6.2.** The Archimedean Riesz space \( L \) is regular if and only if \( L \) has the diagonal property.

**Proof.** Assume first that \( L \) is regular. Let \( f_{n,k} \rightarrow f_n \) and \( f_n \rightarrow f \). By the preceding lemma there exists a \( c \in L^\infty \) such that every sequence \( \{f_{n,k}; k = 1, 2, \ldots\} \) converges \( u \)-uniformly to \( f_n \), and so there is for every \( n \) a natural number \( k(n) \) such that \( |f_{n,k(n)} - f_n| \leq n^{-1} u \).

We may assume here that \( k(1) < k(2) < \ldots \) and that \( f_{n,k(n)} \rightarrow f_n \) for every \( n \).

For the converse, assume that \( L \) is Archimedean and possesses the diagonal property. In order to prove that order convergence in \( L \) is stable, assume that \( f_n \rightarrow 0 \), i.e., \( f_n \leq w_k \downarrow 0 \) for an appropriate sequence \( \{w_k\} \) in \( L^\infty \). Let \( f_n = w_k \) for \( n = k, k + 1, \ldots \). Then \( f_n \rightarrow 0 \) for \( n = 1, 2, \ldots \), and so by the diagonal property there exists a diagonal sequence \( \lambda_n u_n \rightarrow 0 \) with \( k(1) < k(2) < \ldots \). In other words, we have \( \lambda_n u_n \rightarrow 0 \). Now, for any natural number \( n \) satisfying \( k(n) < k(n+1) \), let \( \lambda_n = n \). It is not difficult to see that \( \lambda_n \rightarrow \infty \) and \( \lambda_n u_n \rightarrow 0 \), so \( \lambda_n f_n \rightarrow 0 \).

It remains to prove that for any sequence \( \{u_n\} \) in \( L^\infty \) there exists a sequence \( \{\lambda_n u_n\} \) such that \( \lambda_n u_n \rightarrow 0 \) such that \( \{\lambda_n u_n\} \) is bounded. Set \( f_n = k^{-1} u_n \) for \( n = k, k + 1, \ldots \). Then \( f_n \rightarrow 0 \) for \( n = 1, 2, \ldots \) and so \( f_{n,k} \rightarrow 0 \) for appropriate \( k(n) \). In other words, setting \( \lambda_n = (k(n)^{-1}) \), we have \( \lambda_n u_n \rightarrow 0 \), which implies that \( \lambda_n u_n \rightarrow 0 \).

Just as in Theorem 5.1 we will assume now that \( L \) is either Dedekind complete or \( L \) has the projection property. In Theorem 5.1 it was proved that if, in addition, \( L \) has a strong unit, then stability of the order convergence implies that \( L \) is of finite dimension. We will weaken now the condition that \( L \) has a strong unit, and assume only that every principal band in \( L \) has a strong unit; we will prove that regularity of \( L \) implies now that \( L \) is of finite dimension.

**Theorem 6.3.** Let the Riesz space \( L \) be either Dedekind complete or have the projection property, and let every principal band in \( L \) have a strong unit. Then \( L \) is regular if and only if \( L \) is of finite dimension.

**Proof.** We need only prove that regularity implies finite dimensionality. Observe first that, in view of Theorem 5.1, every principal band in \( L \) is of finite dimension. Furthermore, by Theorem 4.3, it will be sufficient to prove that any system of mutually disjoint non-zero elements is finite. Assume, therefore, that there exists a countably infinite system \( \{f_n; n = 1, 2, \ldots\} \) of mutually disjoint non-zero elements. By the regularity of \( L \) there exists a corresponding sequence \( \{\lambda_n; \lambda_n > 0\} \) such that \( \{\lambda_n f_n\} \) is bounded; say \( \lambda_n f_n \leq u \in L^\infty \) for all \( n \). It follows that all elements \( f_n \) are included in the principal band of finite dimension generated
by $u$. On the other hand, $(f_n; n = 1, 2, \ldots)$ is a linearly independent system by one of the remarks in section 2. Contradiction. Hence, $L$ must be of finite dimension.

References


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**Invariante Masse positiver Kontraktionen in $C(X)$**

**von**

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**vom 60. Jahrestag ihrer wissenschaftlichen Forschung in Verbindung gewidmet**

1. **Einzleitung.** Wir betrachten einen kompakten Hausdorffraum $X$, der quasi-stohech ist, d.h. der Banachverband $C(X)$ ist bedingt $\sigma$-ordnungsvollständig, und einen positiven Operator $T$ in $C(X)$, der konstante Funktionen invariant lässt (einen Markov-operator). Bekanntlich erscheinen solche Voraussetzungen oft in der Theorie der messbaren Abbildungen und in der Theorie der Markov-­prozesse, wenn man den Körper aller messbaren Mengen (modulo Nullmengen) mit dem Körper aller offen-­geschlossenen Teilmengen eines kompakten Hausdorffraumes identifiziert.


In bezug auf die $\sigma$-Ordnungsvollständigkeit von $C(X)$ zeichnen sich ordnungsgemäße Masse und Operatoren aus, die wir $\sigma$-additiv nennen. Wir stellen dann die Frage: Wann sind alle invariante Masse eines $\sigma$-additiven Operators $\sigma$-additiv? Theorem 2 antwortet darauf mit dem Mittelergodensatz und der endlichen Dimension der Menge aller invarianten Funktionen.

Umgekehrt behandeln wir auch die Frage: Wann kann kein $\sigma$-additives Mass invariant sein? Eine Antwort darauf ergibt sich aus der Charakterisierung (Theorem 3) des von allen $\sigma$-additiven, invarianten Massen annullierten Bandes. Aus Theorem 3 folgt auch die von Ito [2] bewiesene

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