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Integrally positive-definite functions on groups*

by

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1. Introduction. A complex-valued function φ on a group G is called *positive-definite*, according to a classical definition, if

$$(1)\qquad \sum_{j=1}^{m}\sum_{k=1}^{m}\bar{a}_j a_k \varphi(x_j^{-1}x_k) \geqslant 0$$

for all finite subsets $\{x_1, \ldots, x_m\}$ of G and all sequences $\{a_1, \ldots, a_m\}$ of complex numbers. Positive-definite functions play a vital rôle in the theory of unitary representations of locally compact groups. See, for example, [6], § 30, or the detailed and interesting treatment in [1], §§ 13-15. For a topological group G, let $\mathfrak{P}(G)$ denote the set of all continuous positive-definite functions on G.

Besides the definition (1), there is a second notion of positive-definiteness meaningful for locally compact groups G. Let λ be a left Haar measure on G (normalized by $\lambda(G) = 1$ if G is compact). A Borel measurable function φ is said to be *integrally positive-definite* if the function

$$(2)\qquad (x, y) \to \varphi(y^{-1}x)\overline{f(y)}f(x) \quad \text{is in } \mathfrak{L}_1(G\times G) \text{ for all } f\in\mathfrak{L}_1(G)$$

and

$$(3)\qquad \int_{G\times G} \varphi(y^{-1}x)\overline{f(y)}f(x)\,d\lambda\times\lambda(x, y) \geqslant 0 \quad \text{for all } f\in\mathfrak{L}_1(G).$$

It is well known that a function in $\mathfrak{P}(G)$ belongs to $\mathfrak{L}_\infty(G)$ and is integrally positive-definite. It is also well known that if φ is in $\mathfrak{L}_\infty(G)$ and (3) holds for all $f\in\mathfrak{L}_1(G)$, then φ is locally λ-almost everywhere equal to a continuous positive-definite function. See for example [6], § 30, Theorems III and IV. Actually all λ-measurable φ's satisfying conditions (2) and (3) are in $\mathfrak{L}_\infty(G)$ (see [3], § 32).

In this note, we study Borel measurable functions φ on G for which (2) and (3) hold not for all $f\in\mathfrak{L}_1(G)$ but for all $f\in\mathfrak{L}_p(G)\cap\mathfrak{L}_{p'}(G)$, where p

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is a fixed number in $]1, \infty[$. It turns out that such functions φ need not be locally λ-almost everywhere equal to a positive-definite function. Thus integral positive-definiteness implies ordinary positive-definiteness only in very restricted cases. A precise statement of our result follows.

THEOREM. *Let G be a non-discrete group that either contains a compact open subgroup or is locally compact Abelian. There is a Borel measurable function φ on G with the following properties. If $2 \leqslant p < \infty$ and $f \in \mathfrak{L}_p(G) \cap \mathfrak{L}_{p'}(G)$ $(1/p+1/p' = 1)$, then*

$$(x, y) \to \varphi(y^{-1}x)\overline{f(y)}f(x)$$

is in $\mathfrak{L}_1(G\times G)$ and

$$\int_G \int_G \varphi(y^{-1}x)\overline{f(y)}f(x)\,dy\,dx \geqslant 0.$$

However, φ is not in $\mathfrak{L}_\infty(G)$. In particular, φ differs from every continuous positive-definite function on G on a set of positive measure.

For all facts and terminology from harmonic analysis used here without explanation or reference, see the monograph [2].

2. The general technique.

LEMMA. *Let G be a locally compact group and let p be a real number > 1. Let φ be a function such that*

(i) $\varphi \in \mathfrak{L}_p(G)$;

(ii) $\lim_{n\to\infty} \|\varphi - \varphi_n\|_p = 0$ *for some sequence* $\{\varphi_n\}$ *in* $\mathfrak{P}(G) \cap \mathfrak{L}_p(G)$;

(iii) $\Delta^{-1/p'}\varphi \in \mathfrak{L}_1(G)$ *(Δ is the modular function for G).*

Then (2) *and* (3) *hold for all* $f \in \mathfrak{L}_{p'}(G) \cap \mathfrak{L}_p(G)$.

If φ only satisfies conditions (i) *and* (ii), *then* (2) *and* (3) *hold for all* $f \in \mathfrak{L}_p(G) \cap \mathfrak{L}_{p'}(G) \cap \mathfrak{L}_1(G)$.

Proof. We first prove the second assertion. Thus we consider φ as in (i) and (ii) and f in $\mathfrak{L}_p(G) \cap \mathfrak{L}_{p'}(G) \cap \mathfrak{L}_1(G)$. It is known that the function

$$\tilde{f} * \varphi(x) = \int_G \overline{f(y)}\varphi(y^{-1}x)\,dy$$

exists and is finite for λ-almost all $x \in G$ and is a function in $\mathfrak{L}_p(G)$, for which $\|\tilde{f}*\varphi\|_p \leqslant \|f\|_1\|\varphi\|_p$ (see [2], Corollary (20.14)). By Hölder's inequality, the integral

$$\int_G \int_G |\overline{f(y)}|\,|\varphi(y^{-1}x)|\,dy\,|f(x)|\,dx$$

is finite. Fubini's theorem implies that the integral in (3) exists and is finite for our current φ and f. Since

$$\|\tilde{f}*\varphi_n - \tilde{f}*\varphi\|_p \leqslant \|f\|_1\|\varphi_n - \varphi\|_p,$$

we can use Hölder's inequality to write

$$(4)\quad \Big|\int_G \tilde{f}*\varphi_n(x)f(x)\,dx - \int_G \tilde{f}*\varphi(x)f(x)\,dx\Big| \leqslant \|\tilde{f}*\varphi_n - \tilde{f}*\varphi\|_p\|f\|_{p'} \leqslant \|f\|_1\|f\|_{p'}\|\varphi_n-\varphi\|_p.$$

Since

$$\int_G \tilde{f}*\varphi_n(x)f(x)\,dx = \int_G\int_G \overline{f(y)}\varphi_n(y^{-1}x)f(x)\,dy\,dx,$$

and since (3) holds for the functions φ_n (which are in $\mathfrak{P}(G)$), (4) proves (3) for our current φ and f.

Now suppose that (iii) also holds, and consider $f \in \mathfrak{L}_p(G) \cap \mathfrak{L}_{p'}(G)$. Let $\{f_n\}$ be a sequence of functions in $\mathfrak{L}_p(G) \cap \mathfrak{L}_{p'}(G) \cap \mathfrak{L}_1(G)$ such that

$$\lim_{n\to\infty}\|f-f_n\|_p = \lim_{n\to\infty}\|f-f_n\|_{p'} = 0.$$

(It is an elementary exercise to construct the f_n.) By [2], Corollary (20.14.iv), and Hölder's inequality, we have

$$(5)\quad \Big|\int_G (\tilde{f}_n*\varphi)f_n\,d\lambda - \int_G (\tilde{f}*\varphi)f\,d\lambda\Big|$$
$$\leqslant \int_G |\tilde{f}_n*\varphi|\cdot|f_n-f|\,d\lambda + \int_G |\tilde{f}_n*\varphi - \tilde{f}*\varphi|\cdot|f|\,d\lambda$$
$$\leqslant \|f_n\|_p\|\Delta^{-1/p'}\varphi\|_1\|f_n-f\|_{p'} + \|(f_n-f)*\varphi\|_p\|f\|_{p'}$$
$$\leqslant (\|f_n\|_p\|f_n-f\|_{p'} + \|f_n-f\|_p\|f\|_{p'})\|\Delta^{-1/p'}\varphi\|_1.$$

The limit of the last line of (5) is zero. Applying the preceding paragraph to each f_n, we infer that

$$\int_G (\tilde{f}*\varphi)f\,d\lambda \geqslant 0.$$

That is, (3) holds for our current φ and f's, and the proof is complete.

Let G be discrete. For all $x \in G$, we have

$$|\varphi_n(x) - \varphi(x)| \leqslant \|\varphi_n - \varphi\|_p.$$

Thus every φ as in the lemma is the pointwise limit of a sequence in $\mathfrak{P}(G)$ and so φ is in $\mathfrak{P}(G)$. The lemma provides us with no new information in this case.

The remaining sections are devoted to the proof of the theorem.

3. The compact case. Let G be a compact infinite group. There is a function φ in $\bigcap_{2\leqslant p<\infty} \mathfrak{L}_p(G)$ satisfying all the properties listed in the lemma, and such that φ is not in $\mathfrak{L}_\infty(G)$.

We construct φ by looking at representations of G. Let Σ be the set of all equivalence classes of continuous, irreducible, unitary representations of G. Since G is infinite, so is Σ (see [3], Theorem (28.1)). For each $\sigma \in \Sigma$, let $U^{(\sigma)}$ be a fixed member of σ. Let H_σ be the (finite-dimensional!) Hilbert space on which $U^{(\sigma)}$ acts, and let d_σ be the dimension of H_σ. Let $\mathfrak{B}(H_\sigma)$ be the algebra of all linear operators on H_σ, and let $\mathfrak{E}(\Sigma)$ be the product algebra $\mathsf{P}_{\sigma\in\Sigma} \mathfrak{B}(H_\sigma)$. For $f \in \mathfrak{L}_1(G)$, the Fourier transform $\hat{f}$ of f is the element of $\mathfrak{E}(\Sigma)$ such that

$$\langle \hat{f}(\sigma)\xi, \eta\rangle = \int_G \overline{\langle U_x^{(\sigma)}\xi, \eta\rangle} f(x)\,dx$$

for all $\sigma \in \Sigma$ and all $\xi, \eta \in H_\sigma$.

We are concerned with certain subalgebras of $\mathfrak{E}(\Sigma)$. Following von Neumann [7], we first introduce a family of norms for operators A on a finite-dimensional Hilbert space H. Let $|A|$ be the positive-definite square root of $AA^\sim$ ("$\sim$" denotes adjoint). Let $a_1, a_2, \ldots, a_n$ be the eigenvalues of $|A|$. For $1 \leqslant p < \infty$, let

$$\|A\|_{\varphi_p} = \Big[\sum_{j=1}^{n} a_j^p\Big]^{1/p},$$

and let $\|A\|_{\varphi_\infty} = \max\{a_1, a_2, \ldots, a_n\}$. All functions $A \to \|A\|_{\varphi_p}$ are norms on $\mathfrak{B}(H)$, and $\|A\|_{\varphi_\infty}$ is the operator norm of A. Following Kunze [5], we use the φ_p-norms to pick out some useful subspaces of $\mathfrak{E}(\Sigma)$. For $1 \leqslant p < \infty$, let $\mathfrak{E}_p(\Sigma)$ be the set of all $E = (E_\sigma) \in \mathfrak{E}(\Sigma)$ such that

$$\|E\|_p = \Big[\sum_{\sigma\in\Sigma} d_\sigma \|E_\sigma\|_{\varphi_p}^p\Big]^{1/p} < \infty,$$

and let $\mathfrak{E}_\infty(\Sigma)$ be the set of all $(E_\sigma) \in \mathfrak{E}(\Sigma)$ such that

$$\|E\|_\infty = \sup\{\|E_\sigma\|_{\varphi_\infty} : \sigma \in \Sigma\} < \infty.$$

We will need the following two facts.

I. A function $f \in \mathfrak{L}_\infty(G)$ is λ-almost everywhere equal to a function in $\mathfrak{P}(G)$ if and only if each operator $\hat{f}(\sigma)$ is positive-definite. In this case, we have $\hat{f} \in \mathfrak{E}_1(\Sigma)$.

II. Suppose that $1 \leqslant p \leqslant 2$ and that $E \in \mathfrak{E}_p(\Sigma)$. There is a unique function $\check{E} \in \mathfrak{L}_{p'}(G)$ $(1/p + 1/p' = 1)$ such that $(\check{E})^\wedge = E$. The mapping $E \to \check{E}$ is linear, and the inequality $\|\check{E}\|_{p'} \leqslant \|E\|_p$ obtains.

For continuous f, Theorem I is due to Kreĭn [4], § 7. The extension to $\mathfrak{L}_\infty$-functions is due to the writers and will appear in the monograph [3], § 34. Theorem II is due to Kunze [5], Corollary 8.1.

Recall that Σ is infinite. Choose any $E \in \mathfrak{E}(\Sigma)$ such that $E \in \mathfrak{E}_p(\Sigma)$ for all $p \in]1, \infty]$, all E_σ are positive-definite, and $E \notin \mathfrak{E}_1(\Sigma)$. For example, choose a sequence $\{\sigma_k\}$ of distinct elements of Σ, and let $E_{\sigma_k} = (kd_{\sigma_k}^2)^{-1} I_{\sigma_k}$, where I_{σ_k} is the identity operator on H_{σ_k}. Let $E_\sigma = 0$ for all other values of σ. In any case, given such an E, let us order the set $\{\sigma \in \Sigma : E_\sigma \neq 0\}$ as an infinite sequence $\{\sigma_k\}_{k=1}^\infty$. For every positive integer n, let $E^{(n)}$ be the element of $\mathfrak{E}(\Sigma)$ such that $E_{\sigma_k}^{(n)} = E_{\sigma_k}$ for $k \in \{1, 2, \ldots, n\}$ and $E_\sigma^{(n)} = 0$ for all other σ's. Also let φ_n be the trigonometric polynomial on G for which $\hat{\varphi}_n = E^{(n)}$. Each φ_n belongs to $\mathfrak{P}(G)$ by I.

For each $p \in]1, 2]$, there is a unique $\varphi^{(p)} \in \mathfrak{L}_{p'}(G)$ of the form $\check{E}$, as in II. Since $(\varphi^{(p)})^\wedge = E$, the uniqueness theorem for Fourier transforms shows that all the functions $\varphi^{(p)}$ are almost everywhere the same; we denote this function by φ. For $p \in]1, 2]$, we have

$$(7)\qquad \|\varphi_n - \varphi\|_{p'} \leqslant \|E^{(n)} - E\|_p = \Big[\sum_{k=n+1}^{\infty} d_{\sigma_k}\|E_{\sigma_k}\|_{\varphi_p}^p\Big]^{1/p}.$$

Since E is in $\mathfrak{E}_p(\Sigma)$, the right side of (7) has limit 0 as $n \to \infty$, and so φ_n converges to φ in the $\mathfrak{L}_{p'}$-metric for $p' \in [2, \infty[$. This shows that φ has properties (i)-(iii) of the lemma.

Now assume that φ is in $\mathfrak{L}_\infty(G)$. Theorem I shows that $\hat{\varphi}$ is in $\mathfrak{E}_1(\Sigma)$. This contradicts our choice of E, and completes the proof of the theorem in the case that G is compact.

4. The case of a compact open subgroup. Now let G be a non-discrete locally compact group containing a compact open subgroup J. According to § 3, there is a function ψ in $\bigcap_{2\leqslant p<\infty} \mathfrak{L}_p(J)$ enjoying properties (i)-(iii) of the lemma and not in $\mathfrak{L}_\infty(J)$. Define $\varphi(x) = \psi(x)$ for $x \in J$ and $\varphi(x) = 0$ for $x \in G \cap J'$. Plainly (i) and (iii) of the lemma hold for φ and for $p \in [2, \infty[$. We have

$$\lim_{n\to\infty} \int_J |\psi - \psi_n|^p \, d\lambda = 0$$

for some sequence $\{\psi_n\}$ in $\mathfrak{P}(J) \cap \mathfrak{L}_p(J)$. Define $\varphi_n(x) = \psi_n(x)$ for $x \in J$ and $\varphi_n(x) = 0$ for $x \in G \cap J'$. Clearly

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_p = 0$$

and each φ_n belongs to $\mathfrak{L}_p(G)$. It is a routine matter using (1) to check that each φ_n belongs to $\mathfrak{P}(G)$. Thus φ enjoys properties (i)-(iii) of the lemma.

Assume that φ belongs to $\mathfrak{L}_\infty(G)$. Then ψ would belong to $\mathfrak{L}_\infty(J)$, contrary to our selection of ψ.

5. Anzai's example. The construction of this section was suggested to the first-named author by the late Hirotada Anzai in a conversation in 1953. It is a pleasure to record here our debt to him. Consider the additive group R of real numbers, and as usual regard R as its own character group. Take Haar measure on R to be $(2\pi)^{-1/2}$ times Lebesgue measure. On the character group R, define the function g by $g(y) = \min\{1, 1/|y|\}$ and g_n as $g\xi_{[-n,n]}$ for $n \in \{1, 2, \ldots\}$ (ξ_A denotes the characteristic function of the set A). For every $p \in]1, \infty[$, g is in $\mathfrak{L}_p(R)$, and g is not in $\mathfrak{L}_1(R)$. Define φ_n as $\check{g}_n$ and φ as $\check{g}$; the latter is the inverse $\mathfrak{L}_p$ transform for any $p \in]1, 2]$. Thus φ belongs to $\mathfrak{L}_{p'}(R)$ for all $p' \in [2, \infty[$. Applying a classical Fourier inversion theorem to each g_n and some elementary calculus, we see that

$$\lim_{n\to\infty} \varphi_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \int_x^\infty \frac{\sin(u)}{u^2}\, du \tag{8}$$

for $x > 0$ and that the φ_n's are even functions. It is well known that

$$\lim_{n\to\infty} \|\varphi_n - \varphi\|_{p'} = 0.$$

A subsequence $\{\varphi_{n_k}\}_{k=1}^\infty$ of $\{\varphi_n\}_{n=1}^\infty$ converges almost everywhere. Thus φ is the even function that is equal to the integral in (8) for $x > 0$. It is obvious that all of the functions φ_n are in $\mathfrak{P}(R)$. It is elementary, albeit a little tedious, to show that $\varphi \in \mathfrak{L}_1(R)$. Since

$$\lim_{x\to 0} \varphi(x) = \infty,$$

it is impossible for φ to be in $\mathfrak{L}_\infty(R)$. Another citation of the lemma now completes the proof of the theorem for the group R.

6. The Abelian case. Let G be an arbitrary non-discrete locally compact Abelian group. A classical structure theorem ([2], (24. 30)) asserts that G has the form $G_0 \times R^a$, where G_0 is a locally compact Abelian group containing a compact open subgroup and a is a non-negative integer. Sections 4 and 5 show that the theorem holds for G_0 and each factor R. The theorem for all locally compact Abelian groups therefore follows from the following assertion and a simple induction.

Let G and H be locally compact groups. If φ and ψ are functions on G and H, respectively, as in the theorem and the lemma, then $(x, y) \to \varphi(x)\psi(y)$ defines a function on $G \times H$ with all the properties listed in the theorem and the lemma.

Let $\Phi(x, y) = \varphi(x)\psi(y)$ for $(x, y) \in G \times H$. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences for φ and ψ as in (ii) of the lemma, and let $\Phi_n(x, y) = \varphi_n(x)\psi_n(y)$ for $(x, y) \in G \times H$. Clearly the functions $(x, y) \to \varphi_n(x)$ and $(x, y) \to \psi_n(y)$ are

positive-definite and continuous on $G \times H$. Since the pointwise product of positive-definite functions is positive-definite ([8], p. 14), it follows that each Φ_n belongs to $\mathfrak{P}(G \times H)$. For functions on $G \times H$ having the form $f(x, y) \to g(x)h(y)$, it is obvious that $\|f\|_p = \|g\|_p\|h\|_p$, since

$$\int_{G\times H} f(x, y)\,dx\,dy = \int_G f(x)\,dx \int_H g(y)\,dy.$$

Hence (i) and (iii) of the lemma obviously hold ($2 \leqslant p < \infty$). Also, we have

$$\|\Phi - \Phi_n\|_p \leqslant \|\varphi\|_p \|\psi - \psi_n\|_p + \|\varphi - \varphi_n\|_p \|\psi_n\|_p,$$

so that (ii) of the lemma holds for $2 \leqslant p < \infty$. Finally, we show that Φ is not in $\mathfrak{L}_\infty(G \times H)$. Otherwise, we have $|\varphi(x)\psi(y)| \leqslant M$ for almost all (x, y) in $G \times H$.

There is a set $D \subset H$ of finite positive measure contained in $\{y \in H : \psi(y) \neq 0\}$. Then $|\varphi(x)\psi(y)| \leqslant M$ for almost all (x, y) in $G \times D$. Hence for some $y \in D$, we have $|\varphi(x)\psi(y)| \leqslant M$ for almost all $x \in G$. That is, $|\varphi(x)| \leqslant M/|\psi(y)|$ for almost all $x \in G$. This contradicts the fact that $\varphi \notin \mathfrak{L}_\infty(G)$.

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