COROLLARY 3. *Let $\{x_n\}$ be a basis of a Banach space E with the associated sequence of coefficient functionals $\{f_n\}$, and let φ be the canonical mapping of E into $[f_n]^*$. Then*

$$1 \leqslant r([f_n])\, C(\{x_n\}) \leqslant C(\{\varphi(x_n)\}) \leqslant C(\{x_n\}). \tag{33}$$

If $\{x_n\}$ is an unconditional basis of E, we also have

$$1 \leqslant r([f_n])\, C_u(\{x_n\}) \leqslant C_u(\{\varphi(x_n)\}) \leqslant C_u(\{x_n\}). \tag{34}$$

Remark 3. Let us also mention that by (10) and (12′) we have the following formula for the computation of $C(\{\varphi(x_n)\})$:

$$C(\{\varphi(x_n)\}) = \sup_n \sup_{\substack{f\in[f_j]\\ \|f\|\leqslant 1}} \|s_n^*(f)\|_{[\varphi(x_j)]} = \sup_n \sup_{\substack{x\in E\\ \|x\|_{[f_j]}\leqslant 1}} \|s_n(x)\|_{[f_j]}. \tag{35}$$

With the aid of (35) it is easy to obtain again, directly, formula (30). In fact, by (35), $\|x\|_{[f_j]} \leqslant \|x\|$ $(x \in E)$ and (12′) we have

$$C(\{\varphi(x_n)\}) \geqslant \sup_n \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} \|s_n(x)\|_{[f_j]} = C(\{f_n\}),$$

whence, since by theorem 1 we have $C(\{\varphi(x_n)\}) \leqslant C(\{f_n\})$, we obtain (30).

References

[1] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17 (1958), p. 151-164.

[2] J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. 15 (1958), p. 1057-1071.

[3] В. Ф. Гапошкин и М. И. Кадец, *Операторные базисы в пространствах Банаха*, Матем. сб. 71 (103) (1963), p. 3-12.

[4] Л. А. Гуревич, *О базисе в сопряженном пространстве*, Труды семин. по функц. анализу 6 (1958), p. 42-43.

[5] I. Singer, *Об одной теореме И. М. Гельфанда*, Успехи матем. наук 17, 1 (103), (1962), p. 169-176.

[6] — *Weak* bases in conjugate Banach spaces II*, Rev. Math. Pures Appl. 8 (1963), p. 575-584.

INSTITUTE OF MATHEMATICS, ROUMANIAN ACADEMY OF SCIENCES

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On compact metric spaces and the group of Baire equivalences*

by

E. R. LORCH (New York)

Dedicated to Professors
S. Mazur and W. Orlicz,
distinguished leaders of the distinguished
Polish school in Functional Analysis

If (E, τ) is a topological space, the class of Baire sets generated by τ plays a fundamental role in innumerable problems. The present paper investigates questions which arise from the circumstance that many topologies τ on E lead to the same class of Baire sets. The group of Baire equivalences seems to play a fundamental role in this investigation. Clearly, the group is unique for all these topologies. The problems considered lead quickly to deep questions concerning Baire sets and projective sets. For this reason, we limit ourselves to topologies τ which are metric and compact and where classic topology provides some answers to these deep questions. No particular gain would be obtained by considering complete separable metric spaces instead of compact ones and the present procedure has the advantage of setting the stage for the non-metric case. It may also be pointed out that the preponderance of the objects favored in many branches of mathematics (algebraic topology, for instance) have a metric structure. By virtue of classical theorems on generalized homeomorphisms, the present paper presents a background for the comparative study of all compact metric spaces.

The proper structure to be placed on the collection of topologies τ is, paradoxically enough, a topological structure! In fact, at least three such topologies can be introduced, of which one in particular is dominant. As for the group of Baire equivalences, there is a uniform topology assigned to it for each τ. A principal result of this paper is to show that

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the group is complete in this topology. It should be pointed out here that weaker topologies than this one can also play a natural role and will be considered on another occasion.

Problems of a nature parallel to those under investigation here are considered in a recent joint paper with Hing Tong [3]. It is shown there that by varying the topology τ on E, bounded Baire functions may be approximated uniformly by continuous functions and that a non-trivial Baire set may have any given ordinal order α, once more by varying τ.

1. Preliminaries. Let E be a set with points $x, y, \ldots$ Topologies on E will be designated by $\sigma, \tau, \tau', \ldots$ The symbol (E, τ) represents the topological structure on E given by τ. Henceforth, topologies on E will be compact. We shall use "compact" to imply "separated". We note at this stage that if E is finite, the conclusions we draw below are trivial. If E is denumerable, each τ is metrizable. If τ is metrizable, then the set E is finite, denumerable, or has the cardinality $\mathfrak{c}$ of the continuum. Our results are principally concerned with this case. However, in sections 1 and 2 we shall not restrict τ to be metrizable. Since we shall deal simultaneously with many topologies on E, we shall appropriately qualify topological terms; thus we shall say that a set is *τ-open* or *σ-closed* and that a function is *τ-continuous*. The field of real numbers will be denoted by R.

Let C_τ denote the algebra of all τ-continuous real-valued functions on (E, τ). These functions will be denoted by $f, g, h, \ldots$ A *τ-zero set* is a set M such that $M = \{x : f(x) = 0\}$ where f is some element of C_τ. A *τ-cozero set* is the complement in E of a τ-zero set. Zero sets are τ-closed. In a metric space each closed set is a zero set. The family of τ-Baire sets is the family of sets generated by the family of τ-zero sets and τ-cozero sets by the operations of denumerable intersection, denumerable union, and complementation. The family of τ-Borel sets is that obtained by the same operations but starting with the family of τ-closed sets and τ-open sets. In a metric space the family of τ-Baire sets coincides with that of τ-Borel sets. A function $f : E \to R$ is called a *τ-Baire function* if the set $f^{-1}(M)$ is a τ-Baire set for any Baire (equals Borel) set M in R. The totality of bounded real τ-Baire functions will be denoted by I_τ. Note that for any two topologies (compact!) τ and τ', $C_\tau = C_{\tau'}$ if and only if $\tau = \tau'$. Similarly, $I_\tau = I_{\tau'}$ if and only if the families of τ and τ' Baire sets are identical.

Consider now the family $\mathscr{F}$ of all compact topologies on E. Introduce in $\mathscr{F}$ an equivalence relation as follows: $\tau \sim \tau'$ if and only if $I_\tau = I_{\tau'}$. This equivalence relation partitions $\mathscr{F}$ into mutually disjoint equivalence classes. If $\tau \sim \tau'$ we shall call (τ, τ') a *coherent pair*. Consider one of the equivalence classes in $\mathscr{F}$. Let us denote it by $\mathscr{T}$. Thus if $\tau_0 \in \mathscr{T}$, then $\mathscr{T} = \{\tau : \tau \sim \tau_0\}$. We shall refer to topologies τ, τ', τ_1, etc. in $\mathscr{T}$ as coherent topologies. In view of the fact that I_τ is fixed for all $\tau \in \mathscr{T}$, we shall simply write I from now on.

2. The topological structure of $\mathscr{T}$. There can be associated to $\mathscr{T}$ a topological structure in a rather natural way as follows. Suppose $\tau_0 \in \mathscr{T}$. Let $f_1, \ldots, f_n$ be any functions in I which are τ_0-continuous. Let $\mathscr{U}(\tau_0) = \mathscr{U}(\tau_0; f_1, \ldots, f_n)$ represent the totality of topologies τ such that f_i is τ-continuous, $i = 1, \ldots, n$. That is,

$$\mathscr{U}(\tau_0; f_1, \ldots, f_n) = \{\tau : \tau \in \mathscr{T} \text{ and } f_i \in C_\tau,\ i = 1, \ldots, n\}.$$

The sets $\mathscr{U}$ so obtained by varying τ_0 and the f_i appropriately in all possible ways form the base of a topology. This topology will be called the *metatopology* on $\mathscr{T}$. That these sets form a base can be seen from the fact that if $\mathscr{U}(\tau; f_1, \ldots, f_n)$ and $\mathscr{U}(\sigma; g_1, \ldots, g_m)$ have the topology ϱ in common, then the intersection of these two sets is $\mathscr{U}(\varrho; f_1, \ldots, f_n; g_1, \ldots, g_m)$.

If τ_0 is fixed, then the sets $\mathscr{U}(\tau_0; f_1, \ldots, f_n)$ where n and $f_1, \ldots, f_n$ are allowed to vary constitute a neighborhood base of the metatopology at τ_0.

Lemma 2.1. *If A is a set which is τ-closed but not τ'-closed, there exists a τ'-continuous function f such that if $\sigma \in \mathscr{U}(\tau'; f)$, then the set A is not σ-closed.*

Proof. Let x_0 belong to the τ'-closure of A, $x_0 \notin A$. Then since $\{x_0\}$ and A are τ-closed sets, there exists a τ-zero set B such that $x_0 \in B$, $A \cap B = \emptyset$. Now the class of τ-Baire sets is identical with the class of τ'-Baire sets. Thus B is a τ'-Baire set. Furthermore, the zero sets of a compact topology form a base for the ι-topology and the Baire sets are clopen in the ι-topology (see [2], p. 211, for the definition of the ι-topology). Thus there exists a τ'-zero set Z, $Z = Z(f)$, where $f \in C_{\tau'}$, such that $x_0 \in Z$ and $Z \subset B$. Thus $Z \cap A = \emptyset$.

Now let σ be any coherent topology in which f is σ-continuous; that is, let $\sigma \in \mathscr{U}(\tau'; f)$. Then A is not σ-closed. For if it were, the σ-compact set, $\{y : y = f(x), x \in A\}$, would be bounded away from 0 and hence, since f is τ'-continuous, there would exist a τ'-closed set C for which $E - Z \supset C \supset A$ contradicting the fact that x_0 is in the τ'-closure of A.

Proposition 2.2. *The sets $\mathscr{U}(\tau; f_1, \ldots, f_n)$ of the metatopology are both open and closed. The metatopology is separated.*

By definition of the metatopology, $\mathscr{U}$ is open. We show that $\mathscr{T} - \mathscr{U}$ is open. Let $\tau' \in \mathscr{T} - \mathscr{U}$. Then there exists an index i such that f_i is not τ'-continuous. Thus there exists a closed set of real numbers, A_0, such that the set $A = f_i^{-1}(A_0)$ is not τ'-closed (obviously A is τ-closed). According to the preceding lemma, there exists a neighborhood of τ', $\mathscr{U}(\tau'; f)$ such that A is not a closed set of any σ in $\mathscr{U}(\tau'; f)$. This implies that $\mathscr{U}(\tau; f_i) \cap \mathscr{U}(\tau'; f) = \emptyset$ and hence $\mathscr{U}(\tau; f_1, \ldots, f_n) \cap \mathscr{U}(\tau'; f) = \emptyset$.

By the above result, to show that the metatopology is separated, it suffices to show that if τ and τ' are distinct, there exists a neighborhood of τ, $\mathscr{U}(\tau; f_1, \ldots, f_n)$ which does not contain τ'. But this is obvious since $\tau \neq \tau'$ implies that there exists a function f such that $f \in C_\tau$ while $f \notin C_{\tau'}$. Thus $\tau' \notin \mathscr{U}(\tau; f)$.

Now, let τ be a topology and let there exist τ-continuous functions $f_1, \ldots, f_n$ which separate the points of E. This means that $f_i(x) = f_i(y)$ for $i = 1, \ldots, n$ imply $x = y$. Then $\mathscr{U}(\tau; f_1, \ldots, f_n)$ consists of one topology only, namely τ. Thus the metatopology is discrete at the point τ.

Suppose that the functions $f_1, \ldots, f_n$ separate points of E. Consider the map $\Phi: E \to R^n$ defined by $\Phi(x) = (f_1(x), \ldots, f_n(x))$. It is easy to see that $\Phi(E)$ is a closed bounded set E' in R^n hence E is homeomorphic to E'. That is, E is essentially a compact set in R^n. Conversely, for any compact set E' in R^n, the projection functions f_i given by $f_i(\xi_1, \ldots, \xi_n) = \xi_i$, $i = 1, \ldots, n$, are continuous and separate points. Note that since E' is either finite, denumerable, or has the power c, the same is true of E. Also E is metrizable.

It is a fact that the metatopology is not discrete in general. It is not compact in general, for example, when E is a denumerable set.

It may be pointed out here, that another topology which may be imposed on E is obtained by selecting a denumerable collection $\{f_n\}$ of functions which are τ_0-continuous and defining a neighborhood $\mathscr{U}(\tau_0; f_1, f_2, \ldots)$ to consist of all τ such that f_i is τ-continuous, $i = 1, 2, \ldots$ In the case which will be of special interest to us, the case in which $\mathscr{T}$ consists of *metrizable* coherent topologies, this topology on $\mathscr{T}$ is discrete since sequences $\{f_n\}$ exist which separate points of E. Thus for metrizable topologies this "denumerable" topology is too fine to be of interest.

3. The group of Baire equivalences. If τ is a compact topology on E, a Baire equivalence is a bijective map which along with the inverse map transforms τ-Baire sets into τ-Baire sets. Since the topologies in $\mathscr{T}$ all have the same Baire sets, the property of being a Baire equivalence is independent of $\tau \in \mathscr{T}$ and therefore we shall henceforth not refer to an initial topology. The Baire equivalences obviously form a group which will be denoted by $\mathfrak{G}$. Elements of $\mathfrak{G}$ will be denoted by $\mathfrak{g}, \mathfrak{g}_1, \mathfrak{h}, \ldots$ The neutral element of $\mathfrak{G}$ will be denoted by $\mathfrak{e}$. If $x \in E$, and $\mathfrak{g} \in \mathfrak{G}$, then $\mathfrak{g}: E \to E$ and $x \to \mathfrak{g}x$.

Consider the map $\mathfrak{g}: (E, \tau) \to (E, \sigma)$ where $\tau \in \mathscr{T}$ and σ is arbitrary. There is a unique topology σ which makes the map $\mathfrak{g}$ a homeomorphism. This is the topology whose open sets are the images under $\mathfrak{g}$ of the open

sets of τ. This topology will be designated by $\tau_\mathfrak{g}$. Thus $\mathfrak{g}: (E, \tau) \to (E, \tau_\mathfrak{g})$ is a homeomorphism. Since τ is compact, so is $\tau_\mathfrak{g}$. Note also that if τ is metrizable, so is $\tau_\mathfrak{g}$.

From now on, we shall consider only compact metrizable topologies. It is clear that the propositions of the preceding sections are valid in this context.

PROPOSITION 3.1. *If $\tau \in \mathscr{T}$ and $\mathfrak{g} \in \mathfrak{G}$, the compact metrizable topology $\tau_\mathfrak{g}$ defined above is in $\mathscr{T}$.*

Consider the identity map $I: (E, \tau) \to (E, \tau_\mathfrak{g})$. If M is a $\tau_\mathfrak{g}$-open set, then since M is the image under $\mathfrak{g}$ of a τ-open set and since $\mathfrak{g}$ is a Baire equivalence (for τ), M is a τ-Baire set. Conversely, if N is a τ-Baire set, its image under I is a $\tau_\mathfrak{g}$-Baire set. This follows from results of Suslin and Lusin for complete separable metric spaces, hence for compact metrizable spaces (see [2] and [4]; also [1], p. 397).

Let $\tau \in \mathscr{T}$ and let f be τ-continuous. Let $\mathfrak{g} \in \mathfrak{G}$ and define $f_\mathfrak{g}$ by $f_\mathfrak{g}(x) = f(\mathfrak{g}^{-1}x)$. Then $f_\mathfrak{g}$ is $\tau_\mathfrak{g}$ continuous. If $\mathfrak{g}$ and $\mathfrak{g}' \in \mathfrak{G}$, then $(f_\mathfrak{g})_{\mathfrak{g}'} = f_{\mathfrak{g}'\mathfrak{g}}$. Also $(\tau_\mathfrak{g})_{\mathfrak{g}'} = \tau_{\mathfrak{g}'\mathfrak{g}}$.

For a given $\mathfrak{g} \in \mathfrak{G}$, consider the map $\bar{\mathfrak{g}}: \mathscr{T} \to \mathscr{T}$ defined by $\bar{\mathfrak{g}}(\tau) = \tau_\mathfrak{g}$. Let $\bar{\mathfrak{G}}$ denote the totality of maps $\bar{\mathfrak{g}}$ so obtained as $\mathfrak{g}$ varies in $\mathfrak{G}$. Since $(\bar{\mathfrak{g}}' \cdot \bar{\mathfrak{g}})(\tau) = \bar{\mathfrak{g}}'(\bar{\mathfrak{g}}(\tau)) = \bar{\mathfrak{g}}'(\tau_\mathfrak{g}) = (\tau_\mathfrak{g})_{\mathfrak{g}'} = \tau_{\mathfrak{g}'\mathfrak{g}}$ we see that $\overline{\mathfrak{g}' \cdot \mathfrak{g}} = \bar{\mathfrak{g}}' \cdot \bar{\mathfrak{g}}$. Thus the map $\Psi: \mathfrak{G} \to \bar{\mathfrak{G}}$ defined by $\Psi(\mathfrak{g}) = \bar{\mathfrak{g}}$ is a homomorphism of $\mathfrak{G}$ onto $\bar{\mathfrak{G}}$.

PROPOSITION 3.2. *If E is not a finite set, the map $\Psi: \mathfrak{G} \to \bar{\mathfrak{G}}$ defined above is an isomorphism. If E is finite both $\mathscr{T}$ and $\bar{\mathfrak{G}}$ are sets with one element.*

Let $\mathfrak{g} \in \mathfrak{G}$, $\mathfrak{g} \neq \mathfrak{e}$. Then there are two points x, y with $x \neq y$ and $\mathfrak{g}x = y$. If $\tau \in \mathscr{T}$ is such that one of $\{x\}$ and $\{y\}$ is τ-open while the other is not, then the first statement of the proposition is valid since $\tau \neq \tau_\mathfrak{g}$, and hence $\bar{\mathfrak{g}} \neq \bar{\mathfrak{e}}$.

Suppose $\{x\}$ and $\{y\}$ are both τ-open. If E is not finite let x' be a τ-limit point. If we "interchange" x and x', we obtain a coherent topology τ' for which x is a limit point. (More precisely, the open sets of τ' are obtained from those of τ by substituting x for x' and x' for x wherever either occurs. Clearly, τ' is coherent.) Thus we now have $\tau' \neq \tau'_\mathfrak{g}$. Now suppose that $\{x\}$ and $\{y\}$ are both τ-limit points. Then we obtain a coherent topology as follows. We "pinch together" τ bringing to coincidence x and y. Then we "remove" the point y and make $\{y\}$ an open set. The resulting topology is τ'. Clearly, τ' is coherent and $\tau' \neq \tau_\mathfrak{g}$. (The τ'-open neighborhoods of x are unions $A \cup (B - \{y\})$ where A and B are τ-open sets containing x and y resp.)

If E is finite, $\mathscr{T}$ is a set with one element and hence $\bar{\mathfrak{G}}$ also has only one element.

PROPOSITION 3.3. *For each $\bar{\mathfrak{g}} \in \bar{\mathfrak{G}}$, the map $\bar{\mathfrak{g}}: \mathscr{T} \to \mathscr{T}$ defined by $\bar{\mathfrak{g}}(\tau) = \tau_\mathfrak{g}$ is a homeomorphism in the metatopology.*

In the first place, $\bar{\mathfrak{g}}$ is injective. For suppose $\tau_{\mathfrak{g}} = \tau_{\mathfrak{g}'}$. If $\boldsymbol{M}$ is τ-open, then its image under $\mathfrak{g}$ belongs to both $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{g}'}$ hence, $\boldsymbol{M}$ is τ'-open. This means that $\tau = \tau'$. Next $\bar{\mathfrak{g}}$ is surjective. If $\tau' \in \mathscr{T}$, let $\tau = \tau'_{\mathfrak{g}^{-1}}$. Then $\tau_{\mathfrak{g}} = \tau'$.

Now, suppose that $\tau \to \tau_{\mathfrak{g}} = \tau'$ and let $\mathscr{U}(\tau'; f'_1, \ldots, f'_n)$ be any basic neighborhood of τ' in the metatopology. Then there exist τ-continuous functions $f_1, \ldots, f_n$ such that $f_{1\mathfrak{g}} = f'_1, \ldots, f_{n\mathfrak{g}} = f'_n$. It is now easy to verify that $\bar{\mathfrak{g}}$ maps $\mathscr{U}(\tau; f_1, \ldots, f_n)$ onto $\mathscr{U}(\tau'; f'_1, \ldots, f'_n)$. Hence $\bar{\mathfrak{g}}$ is a homeomorphism.

If $\tau \in \mathscr{T}$, the set $\{\tau_{\mathfrak{g}} : \mathfrak{g} \in \mathfrak{G}\} = \{\bar{\mathfrak{g}}(\tau) : \bar{\mathfrak{g}} \in \bar{\mathfrak{G}}\}$ is the *orbit* under $\bar{\mathfrak{G}}$ determined by $\tau (= \tau_{\mathfrak{e}})$. Two orbits either coincide or have no topologies in common. The space of orbits is very important. We shall return to it on another occasion.

4. Topological structures of the group of Baire equivalences. We shall show how for each $\tau \in \mathscr{T}$ there exists a topological structure on the group $\mathfrak{G}$ of Baire equivalences. We shall denote the group with this topology by ${}_\tau\mathfrak{G}$. In fact, we shall introduce a left and a right topology for each τ. These will be represented by ${}_\tau\mathfrak{G}$ and $\mathfrak{G}_\tau$ respectively. The neighborhoods of $\mathfrak{e}$ will first be defined and then the neighborhoods of an arbitrary element $\mathfrak{g}$ will be obtained by group "translation", that is to say, by group multiplication. If the multiplication is on the left (right), we obtain the left (right) topology ${}_\tau\mathfrak{G}$ ($\mathfrak{G}_\tau$).

Let $f_1, \ldots, f_n$ be τ-continuous functions. These functions may or may not distinguish points of $\boldsymbol{E}$. The points of $\boldsymbol{E}$ are partitioned in disjoint τ-zero sets of the type

$$\boldsymbol{M} = \bigcap_{i=1}^{n} \{x : f_i(x) = a_i\}$$

where $(a_1, \ldots, a_n)$ is any element in the range of the map from $\boldsymbol{E}$ to $\boldsymbol{R}^n$ given by $x \to (f_1(x), \ldots, f_n(x))$. These sets will be called *sets of indeterminacy associated with* $f_1, \ldots, f_n$. Every τ-zero set may be obtained in this way. In fact, if f is τ-continuous, then $\boldsymbol{Z}(f) = \{x : f(x) = 0\}$.

Let $\mathfrak{h}$ be an element of $\mathfrak{G}$ which transforms each $\boldsymbol{M} = \boldsymbol{M}(a_1, \ldots, a_n)$ into itself. The totality of these Baire equivalences $\mathfrak{h}$ is a subgroup of $\mathfrak{G}$ which we shall denote by $\mathfrak{H} = \mathfrak{H}(\tau; f_1, \ldots, f_n)$. The totality of these subgroups $\mathfrak{H}$ obtained by varying $f_1, \ldots, f_n$ in all possible ways will constitute the base of neighborhoods of $\mathfrak{e}$ for the topology ${}_\tau\mathfrak{G}$ (and also for $\mathfrak{G}_\tau$). Note that

$$\mathfrak{H}(\tau; f_1, \ldots, f_n; g_1, \ldots, g_m) = \mathfrak{H}(\tau; f_1, \ldots, f_n) \cap \mathfrak{H}(\tau; g_1, \ldots, g_m).$$

If $\mathfrak{g}$ is arbitrary in $\mathfrak{G}$, the base of neighborhoods of $\mathfrak{g}$ in ${}_\tau\mathfrak{G}$ ($\mathfrak{G}_\tau$) is the family of all the left (right) cosets of the form $\mathfrak{g}\cdot\mathfrak{H}$ ($\mathfrak{H}\cdot\mathfrak{g}$), where $\mathfrak{H}$ varies over all neighborhoods of $\mathfrak{e}$. From now on, we shall essentially deal with the left topology only and shall not state results for the two topologies simultaneously.

The topology ${}_\tau\mathfrak{G}$ is separated. It is sufficient to show that if $\mathfrak{g} \neq \mathfrak{e}$, then $\mathfrak{g}$ and $\mathfrak{e}$ are separated from each other. Suppose that for some $x \neq y$, $\mathfrak{g}x = y$. Let f be τ-continuous and such that $f(x) \neq f(y)$. Then $\mathfrak{H}(\tau; f)$ and $\mathfrak{g}\cdot\mathfrak{H}(\tau; f)$ are neighborhoods of $\mathfrak{e}$ and $\mathfrak{g}$ respectively. If these neighborhoods have a point in common, then $\mathfrak{g} \in \mathfrak{H}(\tau; f)$. However, $\mathfrak{g}$ obviously is not in $\mathfrak{H}(\tau; f)$ because of the relations involving x, y, f, and $\mathfrak{g}$.

The topology ${}_\tau\mathfrak{G}$ does not in general make $\mathfrak{G}$ a topological group. For in a topological group the map $\mathfrak{g} \to \mathfrak{g}^{-1}$ is a homeomorphism. However, under this map, the neighborhood $\mathfrak{g}\cdot\mathfrak{H}$ of $\mathfrak{g}$ is transformed into $\mathfrak{H}\cdot\mathfrak{g}^{-1}$. It is clear that the topology ${}_\tau\mathfrak{G}$ is a uniform topology. In fact, the sets $\mathfrak{V}(\mathfrak{H})$ in $\mathfrak{G} \times \mathfrak{G}$ defined by

$$\mathfrak{V}(\mathfrak{H}) = \{(\mathfrak{g}, \mathfrak{g}') : \mathfrak{g}'^{-1}\cdot\mathfrak{g} \in \mathfrak{H}\}, \quad \text{where} \quad \mathfrak{H} = \mathfrak{H}(\tau; f_1, \ldots, f_n),$$

are a base for a uniform structure on $\mathfrak{G}$ and the induced topology is the topology ${}_\tau\mathfrak{G}$. Note that $\mathfrak{V}(\mathfrak{H})$ contains the diagonal in $\mathfrak{G} \times \mathfrak{G}$; $\mathfrak{V}^{-1} = \mathfrak{V}$ since $\mathfrak{g}^{-1}\mathfrak{g}' = (\mathfrak{g}'^{-1}\mathfrak{g})^{-1}$ and since $\mathfrak{H}$ is a group; also $\mathfrak{V}^2 = \mathfrak{V}$, once more because $\mathfrak{H}$ is a group.

PROPOSITION 4.1. *The topology of* ${}_\tau\mathfrak{G}$ *is discrete if and only if* τ *has the property: there exist* τ*-continuous functions* $f_1, \ldots, f_n$ *which separate the points of* $\boldsymbol{E}$.

If there exist τ-continuous functions $f_1, \ldots, f_n$ which separate the points of $\boldsymbol{E}$, then $\mathfrak{H}(\tau; f_1, \ldots, f_n) = \{\mathfrak{e}\}$. Thus the topology of $\mathfrak{G}$ is discrete. If the τ-continuous functions $f_1, \ldots, f_n$ do not separate points, there exists a $\mathfrak{g} \in \mathfrak{H}$, $\mathfrak{g} \neq \mathfrak{e}$, hence every neighborhood of $\mathfrak{e}$ contains elements distinct from $\mathfrak{e}$.

5. The completeness of ${}_\tau\mathfrak{G}$. Since the topology of ${}_\tau\mathfrak{G}$ is derived from a uniformity, it is pertinent to raise the question as to whether the space ${}_\tau\mathfrak{G}$ is complete. We shall prove that this is indeed the case. The proof uses significantly the fact that τ is a compact metric space. We choose to prove completeness using the apparatus of generalized Cauchy sequences defined on directed sets.

Let M be a directed set. Then $\{\mathfrak{g}_m : m \in M\}$ is a generalized Cauchy sequence, if given any neighborhood $\mathfrak{H} = \mathfrak{H}(\tau; f_1, \ldots, f_n)$ of $\mathfrak{e}$ there exists $m_0 \in M$, such that $p, m > m_0$ imply $\mathfrak{g}_p^{-1}\mathfrak{g}_m \in \mathfrak{H}$.

Let $\sigma = \sigma(f_1, \ldots, f_n)$ be the weak topology induced on $\boldsymbol{E}$ by the functions $f_1, \ldots, f_n$. Then $\sigma \subset \tau$, that is, σ is weaker than τ. Note that σ is compact but does not in general separate points. Let $a_1, \ldots, a_n$ be fixed and consider the τ-zero set

$$Z = \bigcap_{i=1}^{n} \{x : f_i(x) = a_i\}.$$

Let $S \in \sigma$, that is, let S be an open set in σ. Then either $S \supset Z$ or $S \cap Z = \emptyset$.

Consider $\bigcup \sigma$ where the union is formed over all sets $\{f_1, \ldots, f_n\}$ of τ-continuous functions. Clearly $\tau \supset \bigcup \sigma$. However, $\bigcup \sigma$ is a base for a compact topology which separates points of E. For if $x, y \in E$, $x \neq y$, there exists $f \in C_\tau$ such that $f(x) \neq f(y)$. Thus there exist two $\sigma(f)$ open sets, non-intersecting, containing respectively x and y. This implies that the topology generated by $\bigcup \sigma$ is τ. (In fact, we may delete the words "generated by".)

We shall now construct a new topology τ' on E. Consider the topology $\sigma = \sigma(f_1, \ldots, f_n)$ introduced above. Since $\{\mathfrak{g}_m\}$ is a generalized Cauchy sequence, given $\mathfrak{H} = \mathfrak{H}(\tau; f_1, \ldots, f_n)$ there exists m_0 such that $m > m_0$ and $p > m_0$ imply that $\mathfrak{g}_p^{-1}\mathfrak{g}_m \in \mathfrak{H}$. We shall show that if $S \in \sigma$, then $\mathfrak{g}_p(S) = \mathfrak{g}_m(S)$. We have seen above that S is a union of zero sets Z. Now, since $\mathfrak{h} = \mathfrak{g}_p^{-1}\mathfrak{g}_m \in \mathfrak{H}$, then $\mathfrak{h}(Z) = Z$. Thus if $x \in Z$, there exists $y \in Z$ such that $\mathfrak{g}_p x = \mathfrak{g}_m y$. This shows that $\mathfrak{g}_p(Z) = \mathfrak{g}_m(Z)$ and hence $\mathfrak{g}_p(S) = \mathfrak{g}_m(S)$.

Let $\sigma' = \sigma'(f_1, \ldots, f_n)$ be the family of all sets of the form $\mathfrak{g}_m(S)$, $S \in \sigma$, $m > m_0$. Since σ is a topology and $\mathfrak{g}_m$ is a bijection, σ' is also a topology. Let τ' be the topology generated by $\bigcup \sigma'$, where the union is formed over all n-tuples of τ-continuous functions $f_1, \ldots, f_n$, n variable. We shall study the properties of the topology τ'.

We now introduce a fundamental map from E to E. If $x \in E$, then $\{x\}$ is a τ-zero set, say, $\{x\} = Z(f)$. Thus for each x, for sufficiently large m, that is, for m greater than some m_0, $\mathfrak{g}_m x$ is constant, $\mathfrak{g}_m x = y$. In anticipation of future results we define the map $x \to y$ by $\mathfrak{g} : E \to E$, $\mathfrak{g}x = y$.

Proposition 5.1. *The map* $\mathfrak{g} : (E, \tau) \to (E, \tau')$ *is a homeomorphism. Thus* τ' *is a compact, metrizable (in particular, separated) topology.*

If $x_1 \neq x_2$, then $\mathfrak{g}x_1 = \mathfrak{g}_m x_1 \neq \mathfrak{g}_m x_2 = \mathfrak{g}x_2$ for sufficiently large m. Hence $\mathfrak{g}$ is injective.

If $\sigma = \sigma(f_1, \ldots, f_n)$ is any topology as defined earlier, then as we have seen there is an index m_0 such that for $m > m_0$, $\mathfrak{g}_m$ is constant on the sets of indeterminacy of σ. If $y \in E$, there is precisely one set of indeterminacy $Z = Z_\sigma$ such that $\mathfrak{g}_m(Z) \ni y$. Also, $\mathfrak{g}_m(Z) \supset \mathfrak{g}(Z)$. The sets $Z = Z_\sigma$ so obtained are τ-closed and it is easy to see that they have the finite intersection property. Since τ is compact, there is a point x common to all Z. We shall see that $\mathfrak{g}x = y$. Suppose for a moment that $\mathfrak{g}x = y_1 \neq y$. Let f be chosen in C_τ such that $\{x\} = Z(f)$. Consider $\sigma = \sigma(f)$ and let Z be the set of indeterminacy of σ such that $\mathfrak{g}_m(Z)$ is constant for large m and $y \in \mathfrak{g}_m(Z)$. Then $x \notin Z$. This contradiction shows that $\mathfrak{g}x = y$. We have thus shown that $\mathfrak{g}$ is a surjective map. This argument shows that for any τ-zero set Z, $\mathfrak{g}(Z) = \mathfrak{g}_m(Z)$ for large m.

To show that $\mathfrak{g} : (E, \tau) \to (E, \tau')$ is a homeomorphism, we shall prove that $\mathfrak{g}$ maps τ-closed sets into τ'-closed sets and that $\mathfrak{g}^{-1}$ does the reverse. Consider a τ-closed set; it is a τ-zero set Z. Now if $Z = Z(f)$, then $Z' = \mathfrak{g}(Z)$ is a closed set in the topology $\sigma' = \sigma'(f)$ and since $\sigma' \subset \tau'$, Z' is τ'-closed. Conversely, consider any τ'-closed set. Note first that the family of sets $\bigcup \sigma'$ is a base for τ' (not merely a subbase). Hence any τ'-closed set is the intersection of sets Z' where $Z' = \mathfrak{g}(Z)$ and Z is τ-closed. Since $\mathfrak{g}^{-1}[\bigcap Z'] = \bigcap \mathfrak{g}^{-1}(Z') = \bigcap Z$, we see that $\mathfrak{g}^{-1}$ maps τ'-closed sets into τ-closed sets.

It is now obvious that τ' is a compact metrizable (hence separated) topology.

Proposition 5.2. *The topologies* τ *and* τ' *have the same Baire sets. The transformation* $\mathfrak{g} : (E, \tau) \to (E, \tau)$ *is a Baire equivalence. Thus* $\tau' \in \mathcal{T}$ *and* $\mathfrak{g} \in \mathfrak{G}$; *also* $\tau' = \tau_{\mathfrak{g}}$.

Let Z' be any τ'-closed set. Then there exists a τ-closed set Z such that $Z' = \mathfrak{g}(Z)$. We have also, for sufficiently large m, $Z' = \mathfrak{g}_m(Z)$. Now $\mathfrak{g}_m$ is a τ-Baire equivalence hence Z' is a τ-Baire set. This shows that any τ'-Baire set is a τ-Baire set.

Consider the identity transformation $I : (E, \tau) \to (E, \tau')$. Note that both spaces are metric, separable, and complete. The transformation I is Baire measurable (the pre-image of a τ'-Baire set is a τ-Baire set) and it is bijective. It results from theorems due to Suslin and Lusin that I maps τ-Baire sets into τ'-Baire sets (see [5], [4]; also [1], p. 397). Thus τ and τ' have the same Baire sets. Hence $\tau' \in \mathcal{T}$.

If B is a τ-Baire set and since $\mathfrak{g} : (E, \tau) \to (E, \tau')$ is a homeomorphism, $\mathfrak{g}(B)$ is a τ'-Baire set. By the preceding paragraph, $\mathfrak{g}(B)$ is a τ-Baire set. Similarly the pre-image under $\mathfrak{g}$ of a τ-Baire set is a τ-Baire set. Hence $\mathfrak{g} : (E, \tau) \to (E, \tau)$ is a Baire equivalence. This means that $\mathfrak{g} \in \mathfrak{G}$. That $\tau' = \tau_{\mathfrak{g}}$ is clear from the fact that $Z' = \mathfrak{g}(Z)$ is τ'-closed if and only if Z is τ-closed.

We are now in a position to prove the completeness theorem.

Theorem 5.3. *The uniform space* ${}_\tau\mathfrak{G}$ *is complete. If* $\{\mathfrak{g}_m\}$ *is a generalized Cauchy sequence, it converges to the limit* $\mathfrak{g}$ *defined above.*

Let $\mathfrak{H} = \mathfrak{H}(\tau; f_1, \ldots, f_n)$ be any neighborhood of the identity $\mathfrak{e}$ in ${}_\tau\mathfrak{G}$. To show that $\{\mathfrak{g}_m\} \to \mathfrak{g}$, it is sufficient to show that for m sufficiently large, $\mathfrak{g}^{-1}\cdot\mathfrak{g}_m \in \mathfrak{H}$. We have seen in the proofs of this section that there exists an m_0 such that $m > m_0$ implies that for any τ-zero set of indeterminacy Z associated with $f_1, \ldots, f_n$, $\mathfrak{g}_m(Z) = \mathfrak{g}(Z)$. This implies that $\mathfrak{g}^{-1}\cdot\mathfrak{g}_m(Z) = Z$. Thus $\mathfrak{g}^{-1}\cdot\mathfrak{g}_m \in \mathfrak{H}(\tau; f_1, \ldots, f_n)$.

References

[1] C. Kuratowski, *Topologie*, Vol. I, 4-ème éd., Warszawa 1956.

[2] E. R. Lorch, *Compactification, Baire functions and Daniell integration*, Acta Scient. Math. (Szeged) 24 (1963), p. 204-218.

[3] — and Hing Tong, *Continuity of Baire functions and order of Baire sets*, J. Math. Mech. 16 (1967), p. 991-996.

[4] N. Lusin, *Sur la classification de M. Baire*, C. R. Acad. Sc. Paris 164 (1917), p. 91-94.

[5] M. Suslin, *Sur une définition des ensembles mesurables Borel*, ibidem 164 (1917), p. 88-91.

COLUMBIA UNIVERSITY, NEW YORK, N. Y.

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Integrally positive-definite functions on groups*

by

E. HEWITT (Seattle, Wash.) and K. A. ROSS (Eugene, Oregon)

1. Introduction. A complex-valued function φ on a group G is called *positive-definite*, according to a classical definition, if

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{a}_j a_k \varphi(x_j^{-1}x_k) \geqslant 0 \tag{1}$$

for all finite subsets $\{x_1, \ldots, x_m\}$ of G and all sequences $\{\alpha_1, \ldots, \alpha_m\}$ of complex numbers. Positive-definite functions play a vital rôle in the theory of unitary representations of locally compact groups. See, for example, [6], § 30, or the detailed and interesting treatment in [1], §§ 13-15. For a topological group G, let $\mathfrak{P}(G)$ denote the set of all continuous positive-definite functions on G.

Besides the definition (1), there is a second notion of positive-definiteness meaningful for locally compact groups G. Let λ be a left Haar measure on G (normalized by $\lambda(G) = 1$ if G is compact). A Borel measurable function φ is said to be *integrally positive-definite* if the function

$$(x, y) \to \varphi(y^{-1}x)\overline{f(y)}f(x) \quad \text{is in } \mathfrak{L}_1(G\times G) \text{ for all } f \in \mathfrak{L}_1(G) \tag{2}$$

and

$$\int_{G\times G} \varphi(y^{-1}x)\overline{f(y)}f(x)\,d\lambda\times\lambda(x, y) \geqslant 0 \quad \text{for all } f\in\mathfrak{L}_1(G). \tag{3}$$

It is well known that a function in $\mathfrak{P}(G)$ belongs to $\mathfrak{L}_\infty(G)$ and is integrally positive-definite. It is also well known that if φ is in $\mathfrak{L}_\infty(G)$ and (3) holds for all $f\in\mathfrak{L}_1(G)$, then φ is locally λ-almost everywhere equal to a continuous positive-definite function. See for example [6], § 30, Theorems III and IV. Actually all λ-measurable φ's satisfying conditions (2) and (3) are in $\mathfrak{L}_\infty(G)$ (see [3], § 32).

In this note, we study Borel measurable functions φ on G for which (2) and (3) hold not for all $f\in\mathfrak{L}_1(G)$ but for all $f\in\mathfrak{L}_p(G)\cap\mathfrak{L}_{p'}(G)$, where p

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