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On the constants of basic sequences in Banach spaces

by

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Dedicated to Professors
S. Mazur and W. Orlicz
in honour of the fortieth anniversary
of their scientific activity

1. A sequence $\{x_n\}$ in a Banach space E is called a *basic sequence* (respectively, an *unconditional basic sequence*) if it is a basis (respectively, an unconditional basis) of its closed linear span $[x_n]$ in E (see [1]). It is well known that $\{x_n\}$ is a basic sequence (respectively, an unconditional basic sequence) if and only if there exists a constant $K \geqslant 1$ (respectively, $K_u \geqslant 1$) such that

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant K \Big\|\sum_{i=1}^{n+m} a_i x_i\Big\| \tag{1}$$

for any scalars $a_1, \ldots, a_{n+m}$ (respectively, such that

$$\Big\|\sum_{i=1}^{n} \delta_i a_i x_i\Big\| \leqslant K_u \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \tag{2}$$

for any scalars $a_1, \ldots, a_n, \delta_1, \ldots, \delta_n$ with $|\delta_1| \leqslant 1, \ldots, |\delta_n| \leqslant 1$); some authors call this the *K-condition*. The least such constant $C(\{x_n\}) = \min K$ (respectively, $C_u(\{x_n\}) = \min K_u$) is called the *constant* (respectively, the *unconditional constant*) of the basic sequence $\{x_n\}$; obviously we have $1 \leqslant C \leqslant C_u$. In the particular case where $C = 1$ (respectively, $C_u = 1$) $\{x_n\}$ is called a *monotone* (respectively, an *orthogonal* [5]) basic sequence.

It is well known [4] that if $\{x_n\}$ is a basis (respectively, an unconditional basis) of a Banach space E, then the sequence of coefficient functionals $\{f_n\} \subset E^*$ (i.e. for which $f_i(x_j) = \delta_{ij}$) is a basic sequence (respectively, an unconditional basic sequence) in the conjugate space E^* (but, in general, $[f_n] \neq E^*$). Therefore it is natural to ask what are the relations between the constants of $\{x_n\}$ and $\{f_n\}$, and the present note is devoted to this problem. We shall give upper and lower evaluations of $C(\{f_n\})$

by means of $C(\{x_n\})$ and we shall show that although they are the best possible (in the sense that for some bases they all become equalities), they are, in general, strict. We shall also give an example of a biorthogonal system (x_n, f_n) with $[x_n] = E$ and $\{f_n\}$ total on E such that $\{f_n\}$ is an unconditional basic sequence but $\{x_n\}$ is not a basis of E. Finally, we shall show that if $\{x_n\}$ is a basis of E, then the constant (respectively, the unconditional constant) of the sequence of coefficient functionals associated with the basis $\{f_n\}$ of $[f_n]$ coincides with $C(\{f_n\})$ (respectively, $C_u(\{f_n\})$).

2. We recall that the *characteristic* of a linear subspace V of a conjugate Banach space E^* is the greatest number $r = r(V)$ such that the unit cell $\{f\in V \mid \|f\| \leqslant 1\}$ of V is $\sigma(E^*, E)$-dense in the r-cell $\{f\in E^* \mid \|f\| \leqslant r\}$ of E^* (see [2]). Obviously, $0 \leqslant r(V) \leqslant 1$. By [2], theorem 7, we have

$$r(V) = \inf_{\substack{x\in E\\ x\neq 0}} \sup_{\substack{f\in V\\ \|f\|\leqslant 1}} \left| f\left(\frac{x}{\|x\|}\right)\right|. \tag{3}$$

Hence, whenever $r(V) > 0$, the canonical mapping φ of E into V^* (defined by $[\varphi(x)](f) = f(x)$ for all $x\in E$, $f\in V$) is an isomorphism and

$$\|x\|_V \leqslant \|x\| \leqslant \frac{1}{r(V)}\|x\|_V \quad (x\in E), \tag{4}$$

where we use the notation

$$\|x\|_V = \|\varphi(x)\| = \sup_{\substack{f\in V\\ \|f\|\leqslant 1}} |f(x)|.$$

We have proved ([6], theorem 1 and remark 1) that if $\{x_n\}$ is a basis of a Banach space E and $\{f_n\} \subset E^*$ is the associated sequence of coefficient functionals, then $r([f_n]) > 0$ [1] and

$$C(\{x_n\}) \geqslant \frac{1}{r([f_n])}, \tag{5}$$

whence the canonical mapping φ of E into $[f_n]^*$ is an isomorphism and we have (4) with $V = [f_n]$.

This being recalled, we can now prove

THEOREM 1. *Let $\{x_n\}$ be a basis of a Banach space E and let $\{f_n\} \subset E^*$ be the associated sequence of coefficient functionals. Then*

$$1 \leqslant r([f_n])C(\{x_n\}) \leqslant C(\{f_n\}) \leqslant C(\{x_n\}). \tag{6}$$

(1) Independently, Gapoškin and Kadec ([3], theorem 2) have proved, by a different method, that the inequality $r([f_n]) > 0$ is also valid for the more general "operatorial bases" $\{x_n\}$ of E.

If $\{x_n\}$ is an unconditional basis of E, we also have

$$1 \leqslant r([f_n])C_u(\{x_n\}) \leqslant C_u(\{f_n\}) \leqslant C_u(\{x_n\}). \tag{7}$$

Proof. It is well known (and immediate) that $C(\{x_n\}) = \sup_n \|s_n\|$, where s_n is the n-th partial sum operator, i.e.

$$s_n(x) = \sum_{i=1}^{n} f_i(x)x_i \quad (x\in E,\ n = 1, 2, \ldots). \tag{8}$$

Now, for the adjoint s_n^* of s_n we have

$$s_n^*(f) = \sum_{i=1}^{n} f(x_i)f_i \quad (f\in E^*,\ n = 1, 2, \ldots), \tag{9}$$

whence, in particular,

$$s_n^*|_{[f_j]}(f) = \sum_{i=1}^{n} f(x_i)f_i = \sum_{i=1}^{n} [\varphi(x_i)](f)f_i \quad (f\in[f_j],\ n = 1, 2, \ldots), \tag{10}$$

where φ denotes, as before, the canonical mapping of E into $[f_n]^*$. Hence, since $\{\varphi(x_n)\}$ is the sequence of coefficient functionals associated with the basis $\{f_n\}$ of $[f_n]$,

$$C(\{f_n\}) = \sup_n \|s_n^*|_{[f_j]}\| \leqslant \sup_n \|s_n^*\| = \sup_n \|s_n\| = C(\{x_n\}), \tag{11}$$

and, taking also into account (4) for $V = [f_n]$,

$$\begin{aligned} C(\{f_n\}) &= \sup_n \|s^*_{n[f_j]}\| = \sup_n \sup_{\substack{f\in[f_j]\\ \|f\|\leqslant 1}} \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} |[s_n^*(f)](x)| \\ &= \sup_n \sup_{\substack{f\in[f_j]\\ \|f\|\leqslant 1}} \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} |f[s_n(x)]| = \sup_n \sup_{\|x\|\leqslant 1} \|s_n(x)\|_{[f_j]} \\ &\geqslant r([f_j]) \sup_n \sup_{\|x\|\leqslant 1} \|s_n(x)\| = r([f_j])C(\{x_n\}). \end{aligned} \tag{12}$$

Consequently, by (5), (12) and (11) we have (6). The proof of (7) is similar, considering instead of $\{s_n\}$ the operators

$$s_{n,\{\delta_i\}}(x) = \sum_{i=1}^{n} \delta_i f_i(x)x_i \tag{13}$$

$$(x\in E,\ |\delta_1| \leqslant 1, \ldots, |\delta_n| \leqslant 1,\ n = 1, 2, \ldots).$$

COROLLARY 1. *If $\{x_n\}$ is a monotone (respectively, orthogonal) basis of E, then $\{f_n\} \subset E^*$ is a monotone (respectively, orthogonal) basic sequence and $r([f_n]) = 1$.*

COROLLARY 2. *Let $\{x_n\}$ be a basis of a Banach space E and let $\{f_n\} \subset E^*$ be the associated sequence of coefficient functionals. If $r([f_n]) = 1$ (in par-*

ticular, if $[f_n] = E^*$, *i.e. the basis* $\{x_n\}$ *is shrinking) or, equivalently, if the canonical mapping* φ *of* E *into* $[f_n]^*$ *is an isometry, then*

$$C(\{f_n\}) = C(\{x_n\}). \tag{14}$$

If $\{x_n\}$ *is an unconditional basis of* E *and* $r([f_n]) = 1$, *then we also have*

$$C_u(\{f_n\}) = C_u(\{x_n\}). \tag{15}$$

Remark 1. Let us mention separately a part of formula (12) which may be useful for applications:

$$C(\{f_n\}) = \sup_n \sup_{\|x\|\leqslant 1} \|s_n(x)\|_{[f_j]}. \tag{12'}$$

3. Corollary 1 above shows that inequalities (6) and (7) are the best possible. Let us now show that they are, in general, strict inequalities. For this purpose we shall give an example in which simultaneously all of them are strict.

Example 1. Let $\{x_n\}$ be the following (unconditional) basis of the space $E = l$:

$$x_1 = \frac{1}{\lambda} e_1, \quad x_2 = e_2 - e_3, \quad x_n = -\frac{1}{\lambda} e_1 + e_n \quad (n = 3, 4, \ldots); \tag{16}$$

$\{e_n\}$ denotes the natural basis of the space l and λ is an arbitrary number such that $0 < \lambda \leqslant 1$. We have, for the $f_n \epsilon E^*$ and $h_n \epsilon E^*$ satisfying $f_i(x_j) = h_i(e_j) = \delta_{ij}$ $(i, j = 1, 2, \ldots)$,

$$f_1 = \lambda h_1 + \sum_{i=2}^{\infty}{}^* h_i, \quad f_2 = h_2, \quad f_3 = h_2 + h_3, \quad f_n = h_n \quad (n = 4, 5, \ldots), \tag{17}$$

where $\sum^*$ denotes the sum for the weak* topology $\sigma(E^*, E)$, i.e.

$$f_1(x) = \lambda h_1(x) + \sum_{i=2}^{\infty} h_i(x)$$

for all $x \epsilon E = l$. We have proved in [6] that

$$r([f_n]) = \lambda \tag{18}$$

and we shall now show that

$$C(\{x_n\}) = 2 + \frac{1}{\lambda}, \quad C(\{f_n\}) = 3, \tag{19}$$

which will prove our assertion, since for $0 < \lambda < 1$ we have

$$1 < \lambda\left(2 + \frac{1}{\lambda}\right) = 2\lambda + 1 < 3 < 2 + \frac{1}{\lambda}.$$

By (16), for any positive integer n and any scalars $a_1, \ldots, a_n$ we have

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| = \Big\|\Big(\frac{a_1}{\lambda} - \sum_{i=3}^{n} \frac{a_i}{\lambda}\Big) e_1 + a_2 e_2 + (a_3 - a_2) e_3 + \sum_{i=4}^{n} a_i e_i\Big\|$$
$$= \Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n} \frac{a_i}{\lambda}\Big| + |a_2| + |a_3 - a_2| + \sum_{i=4}^{n} |a_i|.$$

Hence, since for $\sum_{i=3}^{n} |a_i| \neq 0$ $(n \geqslant 3)$, $m = 1, 2, \ldots$, we have

$$\Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n} \frac{a_i}{\lambda}\Big| - \frac{1}{\lambda} \sum_{i=n+1}^{n+m} |a_i| \leqslant \Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n+m} \frac{a_i}{\lambda}\Big|$$
$$\leqslant \frac{1}{\lambda}\Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n+m} \frac{a_i}{\lambda}\Big| + \Big(\frac{1}{\lambda} - 1\Big)\Big(|a_2| + |a_3 - a_2| + \sum_{i=4}^{n} |a_i|\Big),$$

we obtain

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| = \Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n} \frac{a_i}{\lambda}\Big| + |a_2| + |a_3 - a_2| + \sum_{i=4}^{n} |a_i| \tag{20}$$
$$\leqslant \frac{1}{\lambda}\Big(\Big|\frac{a_1}{\lambda} - \sum_{i=3}^{n+m} \frac{a_i}{\lambda}\Big| + |a_2| + |a_3 - a_2| + \sum_{i=4}^{n+m} |a_i|\Big) = \frac{1}{\lambda}\Big\|\sum_{i=1}^{n+m} a_i x_i\Big\|$$

whenever $\sum_{i=3}^{n} |a_i| \neq 0$ $(n \geqslant 3)$, $m = 1, 2, \ldots$ Since obviously

$$\|a_1 x_1\| = \frac{|a_1|}{\lambda} \leqslant \frac{|a_1|}{\lambda} + 2|a_2| = \Big\|\sum_{i=1}^{2} a_i x_i\Big\|, \tag{21}$$

$$\|a_1 x_1\| = \frac{|a_1|}{\lambda} \leqslant \frac{1}{\lambda}\Big(\Big|\frac{a_1}{\lambda} - \sum_{i=3}^{p} \frac{a_i}{\lambda}\Big| + |a_2| + |a_3 - a_2| + \sum_{i=4}^{p} |a_i|\Big) \tag{22}$$
$$= \frac{1}{\lambda}\Big\|\sum_{i=1}^{p} a_i x_i\Big\| \quad (p \geqslant 3),$$

it remains to consider $\|\sum_{i=1}^{2} a_i x_i\|$. Taking in the inequalities

$$\Big\|\sum_{i=1}^{2} a_i x_i\Big\| \leqslant K \Big\|\sum_{i=}^{p} a_i x_i\Big\| \quad (p \geqslant 2) \tag{23}$$

the scalars $\alpha_1=\alpha_3=1$, $\alpha_2=(n-1)/n$, $\alpha_4=\ldots=\alpha_p=0$, we get

$$\frac{1}{\lambda}+\frac{2(n-1)}{n}=\Big\|\sum_{i=1}^{2}\alpha_i x_i\Big\|\leqslant K\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\| =K\left(\left|\frac{1}{\lambda}-\frac{1}{\lambda}\right|+\frac{n-1}{n}+\frac{1}{n}\right)=K,$$

whence, for $n\to\infty$, we obtain $1/\lambda+2\leqslant K$, and thus, taking also i account (20), (21) and (22), $C(\{x_n\})=\min K\geqslant 1/\lambda+2$.

However, we have here the equality sign, since from

$$\frac{|\alpha_1|}{\lambda}-\frac{|\alpha_2|}{\lambda}\leqslant\frac{|\alpha_1-\alpha_2|}{\lambda}\leqslant\left|\frac{\alpha_1}{\lambda}-\sum_{i=3}^{p}\frac{\alpha_i}{\lambda}\right|+\frac{|\alpha_3-\alpha_2|}{\lambda}+\sum_{i=4}^{p}\frac{|\alpha_i|}{\lambda} \leqslant\left(\frac{1}{\lambda}+2\right)\left(\left|\frac{\alpha_1}{\lambda}-\sum_{i=3}^{p}\frac{\alpha_i}{\lambda}\right|+|\alpha_3-\alpha_2|+\sum_{i=4}^{p}|\alpha_i|\right)$$

we obtain

$$\Big\|\sum_{i=1}^{2}\alpha_i x_i\Big\|=\frac{|\alpha_1|}{\lambda}+2|\alpha_2| \leqslant\left(\frac{1}{\lambda}+2\right)\left(\left|\frac{\alpha_1}{\lambda}-\sum_{i=3}^{p}\frac{\alpha_i}{\lambda}\right|+|\alpha_2|+|\alpha_3-\alpha_2|+\sum_{i=4}^{p}|\alpha_i|\right) =\left(\frac{1}{\lambda}+2\right)\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\|\quad (p\geqslant 2).$$

Therefore $C(\{x_n\})=1/\lambda+2$.

Finally, let us compute $C(\{f_n\})$. By (17), for any positive integer n and any scalars $\alpha_1,\ldots,\alpha_n$ we have

$$(24)\qquad \Big\|\sum_{i=1}^{n}\alpha_i f_i\Big\|=\Big\|\alpha_1\lambda h_1+(\alpha_1+\alpha_2+\alpha_3)h_2+\sum_{i=3}^{n}(\alpha_1+\alpha_i)h_i+\alpha_1\sum_{i=n+1}^{\infty}{}^{*}h_i\Big\| =\max_{3\leqslant i\leqslant n}(|\alpha_1|,\,|\alpha_1+\alpha_2+\alpha_3|,\,|\alpha_1+\alpha_i|).$$

Hence for $\sum_{i=3}^{n}|\alpha_i|\neq 0$ $(n\geqslant 3)$, $m=1,2,\ldots$, we have

$$(25)\qquad \Big\|\sum_{i=1}^{n}\alpha_i f_i\Big\|=\max_{3\leqslant i\leqslant n}(|\alpha_1|,\,|\alpha_1+\alpha_2+\alpha_3|,\,|\alpha_1+\alpha_i|) \leqslant\max_{3\leqslant i\leqslant n+m}(|\alpha_1|,\,|\alpha_1+\alpha_2+\alpha_3|,\,|\alpha_1+\alpha_i|)=\Big\|\sum_{i=1}^{n+m}\alpha_i f_i\Big\|.$$

Since obviously

$$(26)\qquad \|\alpha_1 f_1\|=|\alpha_1|\leqslant\Big\|\sum_{i=1}^{p}\alpha_i f_i\Big\|\quad (p\geqslant 2),$$

it remains to consider $\|\sum_{i=1}^{2}\alpha_i f_i\|$. Taking in the inequalities

$$\Big\|\sum_{i=1}^{2}\alpha_i f_i\Big\|\leqslant K\Big\|\sum_{i=1}^{p}\alpha_i f_i\Big\|\quad (p\geqslant 2)$$

the scalars $\alpha_1=1$, $\alpha_2=2$, $\alpha_3=-2$, $\alpha_4=\ldots=\alpha_p=0$, we get

$$\max(1,3)=\Big\|\sum_{i=1}^{2}\alpha_i f_i\Big\|\leqslant K\Big\|\sum_{i=1}^{p}\alpha_i f_i\Big\|=K\max(1,1,1),$$

whence $3\leqslant K$ and thus, taking also into account (25) and (26),

$$C(\{f_n\})=\min K\geqslant 3.$$

However, we have here the equality sign, since

$$\Big\|\sum_{i=1}^{2}\alpha_i f_i\Big\|=\max(|\alpha_1|,\,|\alpha_1+\alpha_2|)\leqslant|\alpha_1|+|\alpha_2| \leqslant 3\max\left(|\alpha_1|,\,\frac{|\alpha_2|}{2}\right)\leqslant 3\max_{3\leqslant i\leqslant p}(|\alpha_1|,\,|\alpha_1+\alpha_2+\alpha_3|,\,|\alpha_1+\alpha_i|) =3\Big\|\sum_{i=1}^{p}\alpha_i f_i\Big\|\quad (p\geqslant 2);$$

indeed, if $|\alpha_2|/2\leqslant|\alpha_1|$, then $|\alpha_1|+|\alpha_2|\leqslant|\alpha_1|+2|\alpha_1|=3|\alpha_1|$, while if $|\alpha_2|/2\geqslant|\alpha_1|$, then $|\alpha_1|+|\alpha_2|\leqslant|\alpha_2|/2+|\alpha_2|=3|\alpha_2|/2$ and, on the other hand,

$$\frac{|\alpha_2|}{2}=\frac{1}{2}|(\alpha_1+\alpha_2+\alpha_3)-(\alpha_1+\alpha_3)|\leqslant\max(|\alpha_1+\alpha_2+\alpha_3|,\,|\alpha_1+\alpha_3|).$$

Therefore $C(\{f_n\})=3$, which completes the proof of our assertion. For inequalities (7) one can use similar arguments.

4. Example 1 above shows that there exist (unconditional) bases $\{x_n\}$ with $C(\{f_n\})=3$ and $C(\{x_n\})$ taking any arbitrarily large preassigned value [2]. It is natural to ask whether the limit case $C(\{x_n\})=\infty$ is also possible, i.e. whether there exists a biorthogonal system (x_n, f_n) with

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[2] Let us mention that for the unconditional basis $\{x_n\}$ of l constructed in [6], § 1, section 2, which differs from (16) only in the term $x_2=-\lambda^{-1}e_1+e_2$, we have $r([f_n])=\lambda$, $C(\{x_n\})=1/\lambda$, $C(\{f_n\})=1$ (but $C_u(\{f_n\})>1$).

$[x_n] = E$ and $\{f_n\}$ total on E such that $\{f_n\}$ is an (unconditional) basic sequence, but $\{x_n\}$ is *not* a basis of E. We shall show that the answer is affirmative.

Example 2. Let (3) $E = (E_1 \times E_2 \times \ldots)_l \equiv l$, where $E_j = l$ $(j = 1, 2, \ldots)$ and for each j let $\{x_n^{(j)}\}$ be the basis (16) of $E_j = l$, with $\lambda = 1/j$. Since the set

$$\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} \{\underbrace{0, \ldots, 0}_{j-1}, x_n^{(j)}, 0, \ldots\}$$

in E is countable, let $\{x_n\}$ be an arbitrary numbering of it, and let $\{f_n\}$ be the corresponding numbering of the functionals

$$\{\underbrace{0, \ldots, 0}_{j-1}, f_n^{(j)}, 0, \ldots\} \in (E_1^* \times E_2^* \times \ldots)_m \equiv E^* \equiv m,$$

where $\{f_n^{(j)}\}$ is the basic sequence (17) in $E_j^* = m$, with $\lambda = 1/j$. Then the biorthogonal system (x_n, f_n) has the required properties.

Indeed, it is obvious that $[x_n] = E$ and that $\{f_n\}$ is total on E. Since by (19) we have $C(\{x_n^{(j)}\}) = 2+j$ $(j = 1, 2, \ldots)$, it follows that $\{x_n\}$ is not a basis of E. Finally, since $\{x_n^{(j)}\}$ is an unconditional basis of E_j, $\{f_n^{(j)}\}$ is an unconditional basic sequence in E_j^*, and by (24) its unconditional constant $C_u(\{f_n^{(j)}\})$ does not depend on j. Since for

$$\{g_1, g_2, \ldots\} \in (E_1^* \times E_2^* \times \ldots)_m$$

we have

$$\|\{g_1, g_2, \ldots\}\| = \sup_j \|g_j\|,$$

it follows that $\{f_n\}$ is an unconditional basic sequence in E^* with the same unconditional constant $C_u(\{f_n\}) = C_u(\{f_n^{(j)}\})$, which completes the proof of our assertion. Let us also mention that $r([f_n]) = 0$.

5. If $\{x_n\}$ is a basis of a Banach space E with the associated sequence of coefficient functionals $\{f_n\}$, then, as we have already mentioned above, $r([f_n]) > 0$ and the canonical mapping φ of E into $[f_n]^*$ is an isomorphism. Since $\{f_n\}$ is a basis of $[f_n]$ with the associated sequence of coefficient functionals $\{\varphi(x_n)\}$, by virtue of theorem 1 we have

$$1 \leqslant r([\varphi(x_n)])C(\{f_n\}) \leqslant C(\{\varphi(x_n)\}) \leqslant C(\{f_n\}) \tag{27}$$

and a similar relation for the constants C_u if $\{x_n\}$ is unconditional. It is natural to ask whether one can say more about the constants $C(\{\varphi(x_n)\})$, $C_u(\{\varphi(x_n)\})$. We shall now show that this is indeed the case, namely, that for the couples $\big(C(\{f_n\}), C(\{\varphi(x_n)\})\big)$ and $\big(C_u(\{f_n\}), C_u(\{\varphi(x_n)\})\big)$ we always have the situation of corollary 2 above.

(3) By $\equiv$ we denote the canonical linear isometry.

THEOREM 2. *Let V be a linear subspace* (4) *of a conjugate Banach space E^* and let φ be the canonical mapping of E into V^*. Then*

$$r(\varphi(E)) = 1. \tag{28}$$

Hence, in particular, if $\{x_n\}$ is a basis of a Banach space E and $\{f_n\}$ is the associated sequence of coefficient functionals, we have

$$r([\varphi(x_n)]) = 1, \tag{29}$$

$$C(\{\varphi(x_n)\}) = C(\{f_n\}), \tag{30}$$

and, if $\{x_n\}$ is an unconditional basis of E, then we also have

$$C_u(\{\varphi(x_n)\}) = C_u(\{f_n\}). \tag{31}$$

Proof. By formula (3) applied to $\varphi(E)$ and the relations $\|\varphi(x)\| = \|x\|_V \leqslant \|x\|$ $(x \in E)$ we have

$$r(\varphi(E)) = \inf_{\substack{f\in V\\ f\neq 0}} \sup_{\substack{\psi\in\varphi(E)\\ \|\psi\|\leqslant 1}} \left|\psi\left(\frac{f}{\|f\|}\right)\right| = \inf_{\substack{f\in V\\ f\neq 0}} \sup_{\substack{x\in E\\ \|x\|_V\leqslant 1}} \left|\frac{f}{\|f\|}(x)\right|$$
$$\geqslant \inf_{\substack{f\in V\\ f\neq 0}} \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} \left|\frac{f}{\|f\|}(x)\right| = \inf_{\substack{f\in V\\ f\neq 0}} \left\|\frac{f}{\|f\|}\right\| = 1,$$

whence, since the characteristic of any subspace is always $\leqslant 1$, we infer (28).

The second assertion of theorem 2 follows from the first one if we take into account that $[\varphi(x_n)] = \varphi(E)$ whenever $\{x_n\}$ is a basis of E and φ the canonical mapping of E into $[f_n]^*$ and apply corollary 2 above to the couple $(\{f_n\}, \{\varphi(x_n)\})$. This completes the proof of theorem 2.

Remark 2. Formula (28) is equivalent to the statement that the canonical mapping u of V into $\varphi(E)^*$ defined by

$$u(f)[\varphi(x)] = [\varphi(x)](f) = f(x) \quad (f \in V, \varphi(x) \in \varphi(E))$$

is an isometry, i.e.

$$\sup_{\substack{x\in E\\ \|x\|_V\leqslant 1}} |f(x)| = \|u(f)\| = \|f\| = \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} |f(x)| \quad (f \in V), \tag{32}$$

and this latter formula follows also directly from the relations $\|x\|_V \leqslant \|x\|$ $(x \in E)$ and $\|u\| \leqslant 1$.

Combining theorems 1 and 2, we obtain the following relations between the constants of $\{x_n\}$ and $\{\varphi(x_n)\}$:

(4) We need not assume that $r(V) > 0$. Let us mention that the semi-norm $\|x\|_V$ is a norm on E if and only if V is total on E (or, equivalently, iff φ is one-to-one).

Corollary 3. *Let $\{x_n\}$ be a basis of a Banach space E with the associated sequence of coefficient functionals $\{f_n\}$, and let φ be the canonical mapping of E into $[f_n]^*$. Then*

$$1 \leqslant r([f_n])\, C(\{x_n\}) \leqslant C(\{\varphi(x_n)\}) \leqslant C(\{x_n\}). \tag{33}$$

If $\{x_n\}$ is an unconditional basis of E, we also have

$$1 \leqslant r([f_n])\, C_u(\{x_n\}) \leqslant C_u(\{\varphi(x_n)\}) \leqslant C_u(\{x_n\}). \tag{34}$$

Remark 3. Let us also mention that by (10) and (12′) we have the following formula for the computation of $C(\{\varphi(x_n)\})$:

$$C(\{\varphi(x_n)\}) = \sup_n \sup_{\substack{f\in[f_j]\\ \|f\|\leqslant 1}} \|s_n^*(f)\|_{[\varphi(x_j)]} = \sup_n \sup_{\substack{x\in E\\ \|x\|_{[f_j]}\leqslant 1}} \|s_n(x)\|_{[f_j]}. \tag{35}$$

With the aid of (35) it is easy to obtain again, directly, formula (30). In fact, by (35), $\|x\|_{[f_j]} \leqslant \|x\|$ $(x\in E)$ and (12′) we have

$$C(\{\varphi(x_n)\}) \geqslant \sup_n \sup_{\substack{x\in E\\ \|x\|\leqslant 1}} \|s_n(x)\|_{[f_j]} = C(\{f_n\}),$$

whence, since by theorem 1 we have $C(\{\varphi(x_n)\}) \leqslant C(\{f_n\})$, we obtain (30).

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On compact metric spaces and the group of Baire equivalences*

by

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Dedicated to Professors
S. Mazur and W. Orlicz,
distinguished leaders of the distinguished
Polish school in Functional Analysis

If $(\boldsymbol{E}, \tau)$ is a topological space, the class of Baire sets generated by τ plays a fundamental role in innumerable problems. The present paper investigates questions which arise from the circumstance that many topologies τ on $\boldsymbol{E}$ lead to the same class of Baire sets. The group of Baire equivalences seems to play a fundamental role in this investigation. Clearly, the group is unique for all these topologies. The problems considered lead quickly to deep questions concerning Baire sets and projective sets. For this reason, we limit ourselves to topologies τ which are metric and compact and where classic topology provides some answers to these deep questions. No particular gain would be obtained by considering complete separable metric spaces instead of compact ones and the present procedure has the advantage of setting the stage for the non-metric case. It may also be pointed out that the preponderance of the objects favored in many branches of mathematics (algebraic topology, for instance) have a metric structure. By virtue of classical theorems on generalized homeomorphisms, the present paper presents a background for the comparative study of all compact metric spaces.

The proper structure to be placed on the collection of topologies τ is, paradoxically enough, a topological structure! In fact, at least three such topologies can be introduced, of which one in particular is dominant. As for the group of Baire equivalences, there is a uniform topology assigned to it for each τ. A principal result of this paper is to show that

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