Soit

$$\frac{1}{b_n}\frac{a_n^{p+1}-b_n^{p+1}}{a_n^p-b_n^p}\to\frac{a_0}{a_2};$$

donc

$$\frac{1}{b_n}\frac{a_1}{a_0}\to\frac{a_0}{a_2}$$

ce qui est impossible puisque b_n tend vers zéro.

UNIVERSITÉ DE PARIS
FACULTÉ DES SCIENCES, D'ORSAY

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A characterization of multiplicative linear functionals in complex Banach algebras

by

W. ŻELAZKO (Warszawa)

It was shown[1] that if A is a commutative complex Banach algebra with unit element, then a functional $f\in A^*$ is a multiplicative linear functional on A if (and only if)

$$f(x)\in\sigma(x) \tag{1}$$

for every $x\in A$, where $\sigma(x)$ denotes the spectrum of an element x. In this paper we extent this result onto non-commutative Banach algebras. Our result is based upon the following, purely algebraic fact:

THEOREM 1. *Let A be a real or complex algebra with unit e. Let f be a linear functional on A such that its restriction to any subalgebra of A generated by single element and containing e is a multiplicative linear functional. Then f is a multiplicative and linear functional on the algebra A.*

Proof. By our assumptions we have

$$f(e)=1 \tag{2}$$

and

$$f(x^2)=f(x)^2 \tag{3}$$

for every $x\in A$. Consequently,

$$f[(x+y)^2]=[f(x)+f(y)]^2,$$

or

$$f(xy+yx)=2f(x)f(y) \tag{4}$$

for every $x,y\in A$. It follows that if we set

$$x\circ y=\tfrac{1}{2}(xy+yx)$$

we obtain a (non-associative) multiplication on A such that

$$f(x\circ y)=f(x)f(y).$$

(1) See J.-P. Kahane and W. Żelazko, *A characterization of maximal ideals in commutative Banach algebras*, Studia Math. 29 (1968), p. 339-340.

Consequently,

$$f[(x\circ y)\circ z] = f(x)f(y)f(z),$$

which is equivalent with

(5) $$f(xyz+zxy+yxz+zyx) = 4f(x)f(y)f(z),$$

$x, y, z \in A$. It is also

$$f[y\circ(z\circ x)-(y\circ z)\circ x] = 0,$$

which is equivalent with

(6) $$f(xyz+zyx) = f(zxy+yxz)$$

for any $x, y, z \in A$. From (5) and (6) it follows

(7) $$f(xyz+zyx) = 2f(x)f(y)f(z),$$

$x, y, z \in A$. Substituting in (7) $x = z$, we obtain

(8) $$f(xyx) = f(x)^2f(y), \quad x, y \in A.$$

We are going to show that

(9) $$f(xy) = f(yx)$$

for every $x, y \in A$. Suppose, to the contrary, that for some $x_0, y_0 \in A$ we have

$$f(x_0y_0-y_0x_0) = C \neq 0.$$

Taking here y_0/C instead of y_0, we may ssume

(10) $$f(x_0y_0-y_0x_0) = 1.$$

It is clear that relation (10) also holds true if we take instead of x_0 any element of the from x_0+ae, where a is a scalar. We may, therefore, assume also that

(11) $$f(x_0) = 0.$$

By (4) and (11) we have

$$f(x_0y_0)+f(y_0x_0) = 0,$$

and so, by (10),

(12) $$f(x_0y_0) = \tfrac{1}{2}, \quad f(y_0x_0) = -\tfrac{1}{2}.$$

On the other hand, by (3), (8), (10), (11) and (12) we have

$$\begin{aligned} 1 &= f[(x_0y_0-y_0x_0)^2] \\ &= f[(x_0y_0)^2+(y_0x_0)^2-x_0y_0^2x_0-y_0x_0^2y_0] \\ &= \tfrac{1}{4}+\tfrac{1}{4}-2f(x_0)^2f(y_0)^2 = \tfrac{1}{2}, \end{aligned}$$

which is a contradiction proving formula (9). By (4) and (9) we obtain now the desired result

$$f(xy) = f(x)f(y)$$

for every $x, y \in A$.

If A is an algebra without unit, then considering the algebra A_1 obtained from A by adjunction of an identity, we see that if a functional f satisfies on A the assumptions of theorem 1, then its extension onto A_1, defined by $f(e) = 1$, also satisfies these assumptions. So we have

COROLLARY 1. *The conclusion of theorem 1 is also true for an algebra without unit element.*

We may formulate now our main result:

THEOREM 2. *Let A be a complex Banach algebra. Then a functional $f \in A^*$ is a multiplicative and linear functional on A if (and only if) (1) holds.*

Suppose that (1) holds true. We may assume that there is a unit element in A. Otherwise we would consider an algebra obtained from A by adjunction of a unit element e, and an extension of f onto this algebra, given by $f(e) = 1$, which clearly satisfies relation (1). Since for any subalgebra $A_0 \subset A$ we have $\sigma(x) \subset \sigma_0(x)$ for any $x \in A_0 \subset A$, where $\sigma_0(x)$ denotes the spectrum of x in A_0, we infer, by theorem 2 of cited paper, that the restriction of f to any commutative subalgebra of A is a multiplicative functional on this subalgebra. The conclusion is now a consequence of theorem 1.

The same reasoning as in cited paper leads to the following

COROLLARY 2. *Let A be a complex Banach algebra with unit element. A subspace $X \subset A$, codim $X = 1$, is a maximal two-sided ideal in A if it consists of non-invertible elements.*

This condition characterizes all maximal two-sided ideals of codimension 1.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK

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