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On mappings of sequence spaces

by

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In [2], p. 242, Banach defines, for two Banach spaces E and F, the number (E, F) by

$$(E, F) = \text{g.l.b.}\{\log(\|\varphi\|\,\|\varphi^{-1}\|)\},$$

where φ runs through all isomorphisms of E onto F. He calls E and F *nearly isometric* if $(E, F) = 0$, and inquires whether the spaces c of convergent sequences and c_0 of sequences convergent to zero (with the usual sup norm) are nearly isometric.

It follows readily from a result of McWilliams [7] that $(c, c_0) \geqslant \log\frac{3}{2}$, and hence that these spaces are not nearly isometric. In [4], p. 397, the sharper estimate $(c, c_0) \geqslant \log 2$ is given. This latter estimate was found quite independently by Gurariĭ in [6]. Indeed, the fact that $(c, c_0) \geqslant \log 2$ was previously proved by A. Pełczyński, who did not publish his result. Thus it would seem to be implied by [3], p. 55, (2.1), that actually we have equality: $(c, c_0) = \log 2$. In this paper, however, we establish that the exact value of (c, c_0) is, in fact, $\log 3$.

Aside from giving a precise answer to Banach's question, this fact is of interest in the following context. If X is a locally compact Hausdorff space, let us denote by $C_0(X)$ the space of continuous, complex-valued functions on X which are zero at infinity, with norm given by

$$\|f\| = \sup_{x\in X}|f(x)|, \quad f\in C_0(X).$$

(If X is actually compact, then $C_0(X) = C(X)$, the space of all continuous, complex-valued functions on X). In [4] and [5] it is shown that if X and Y are any two locally compact Hausdorff spaces which are not homeomorphic, and φ is any isomorphism of $C_0(X)$ onto $C_0(Y)$, then $\|\varphi\|\,\|\varphi^{-1}\| \geqslant 2$. The analogous result for spaces of real-valued functions defined on compact X and Y was established by Amir in [1]. Now if we denote by N the set of positive integers, and by N^* the one-point compactification of N, then $c = C(N^*)$ and $c_0 = C_0(N)$. The fact that $(c, c_0) \geqslant \log 2$ thus follows from the more general result about spaces of continuous functions just cited.

In [1] Amir notes that there exist various examples of non-homeomorphic compact Hausdorff spaces X and Y, such that $C(X)$ and $C(Y)$ are isomorphic under a map φ with $\|\varphi\|\|\varphi^{-1}\| = 3$. However, no such examples with $\|\varphi\|\|\varphi^{-1}\| < 3$ seem to be known. Amir thus poses the problem: Do there exist non-homeomorphic compact Hausdorff spaces X and Y, and an isomorphism of $C(X)$ onto $C(Y)$ with $2 \leqslant \|\varphi\|\|\varphi^{-1}\| < 3$? One may, of course, formulate this question for locally compact spaces, or one may ask a somewhat simpler variant: If X is a compact Hausdorff space, and Y is locally compact but not compact, must any isomorphism φ of $C(X)$ onto $C_0(Y)$ satisfy $\|\varphi\|\|\varphi^{-1}\| \geqslant 3$? The result of this paper suggests to the author that the answer to this latter question may well be affirmative.

THEOREM. $(c, c_0) = \log 3$.

Proof. Throughout the proof we consider c as $C(N^*)$ and c_0 as $C_0(N)$. As was noted in [4], pp. 396-397, in order to prove that $(c, c_0) \geqslant \log 3$, it suffices to show that if φ is any given norm-increasing isomorphism of c onto c_0 (i.e., $\|f\| \leqslant \|\varphi(f)\|$, $f \in c$), then $\|\varphi\| \geqslant 3$. Thus let φ be a norm-increasing isomorphism of c onto c_0. We assume that $\|\varphi\| < 3$, and show that this assumption leads to a contradiction.

If $\|\varphi\| < 3$, we first choose a real number ε with $0 < \varepsilon < 1$ and such that

$$\|\varphi\| < \frac{3-\varepsilon}{1+\varepsilon}. \tag{1}$$

Next, let l denote that element of c which is identically equal to 1 on N^*, and consider the element $\varphi(l)$ of c_0. Since $\varphi(l)$ is zero at infinity on N, there exists an integer K such that $|(\varphi(l))(n)| < \varepsilon$ for all $n \in N$ with $n > K$. We now define the element f of c_0 by

$$f(n) = \begin{cases} (\varphi(l))(n), & n \leqslant K, \\ 0 \quad , & n > K. \end{cases}$$

Then define $g \in c_0$ by $g(n) = (\varphi(l))(n) - f(n)$, $n \in N$. Note that since $\|g\| < \varepsilon$, and φ^{-1} is norm-decreasing, we have

$$\|\varphi^{-1}(g)\| < \varepsilon. \tag{2}$$

Also, since $f = \varphi(l) - g$, $\varphi^{-1}(f) = l - \varphi^{-1}(g)$ and consequently for each $n \in N$,

$$\operatorname{Re}(\varphi^{-1}(f))(n) = 1 - \operatorname{Re}(\varphi^{-1}(g))(n) \geqslant 1 - |(\varphi^{-1}(g))(n)| > 1-\varepsilon. \tag{3}$$

Note also that

$$\|\varphi^{-1}(f)\| \leqslant \|l\| + \|\varphi^{-1}(g)\| < 1+\varepsilon. \tag{4}$$

Now for each $n \in N$, let f_n be that element of c such that $f_n(k) = \delta_{nk}$, $k \in N$. With each $n \in N$ we associate two subsets S_n and T_n of N as follows. S_n is the maximum set of the function $|\varphi(f_n)|$: $S_n = \{k \in N : |(\varphi(f_n))(k)| = \|\varphi(f_n)\|\}$. In order to define T_n, we first fix a positive integer p such that

$$\frac{1}{p} < \frac{3-(\varepsilon+\|\varphi\|)}{2},$$

and then set $T_n = \{k \in N : |(\varphi(f_n))(k)| > 1/p\}$. Again recalling the fact that φ is norm-increasing, we observe that $[3-(\varepsilon+\|\varphi\|)]/2 < 1$, and that on S_n, $|(\varphi(f_n))(k)| \geqslant 1$. Hence $S_n \subseteq T_n$ for all n.

We claim that there can be at most a finite number of integers n such that $T_n \cap \{1, 2, \ldots, K\}$ is non-void. For if more than $(3p-1)K$ of the T_n have non-void intersection with $\{1, 2, \ldots, K\}$, then at least one integer $m \in \{1, 2, \ldots, K\}$ must belong to $3p$ of the sets T_n, say $T_{n_1}, T_{n_2}, \ldots, T_{n_{3p}}$. We might then define the complex numbers λ_i, $i = 2, 3, \ldots, 3p$, by $|\lambda_i| = 1$ and $\arg \lambda_i = \arg(\varphi(f_{n_1}))(m) - \arg(\varphi(f_{n_i}))(m)$. Consequently,

$$h = f_{n_1} + \sum_{i=2}^{3p} \lambda_i f_{n_i}$$

would be an element of c with $\|h\| = 1$, and

$$\|\varphi(h)\| \geqslant |(\varphi(h))(m)| = \Big|(\varphi(f_{n_1}))(m) + \sum_{i=2}^{3p} \lambda_i (\varphi(f_{n_i}))(m)\Big|$$
$$= |(\varphi(f_{n_1}))(m)| + \sum_{i=2}^{3p} |\lambda_i (\varphi(f_{n_i}))(m)| > 3p\left(\frac{1}{p}\right) = 3,$$

which contradicts the fact that $\|\varphi\| < 3$.

We thus may define the integer M as follows. If the set $\{n \in N : T_n \cap \{1, 2, \ldots, K\} \neq \emptyset\}$ is void, let $M = 0$. If this set is non-void, let $M = \max\{n \in N : T_n \cap \{1, 2, \ldots, K\} \neq \emptyset\}$. Now let n be an integer with $n > M$, and consider the element $f + 2\varphi(f_n)$ of c_0. Since $S_n \cap \{1, 2, \ldots, K\} = \emptyset$, and $f(k) = 0$ for all $k > K$, it follows that for all $k \in N$ with $k > K$, we have $|f(k) + 2(\varphi(f_n))(k)| = 2|(\varphi(f_n))(k)| \leqslant 2\|\varphi(f_n)\|$, with equality holding for $k \in S_n$. Thus

$$\|f + 2\varphi(f_n)\| = \max\{2\|\varphi(f_n)\|, |f(k) + 2(\varphi(f_n))(k)| : k = 1, 2, \ldots K\}.$$

If we now apply φ^{-1} to this element, we obtain

$$\|\varphi^{-1}(f + 2\varphi(f_n))\| = \|\varphi^{-1}(f) + 2f_n\| \geqslant |(\varphi^{-1}(f))(n) + 2f_n(n)|$$
$$\geqslant \operatorname{Re}[(\varphi^{-1}(f))(n) + 2f_n(n)],$$

a quantity which, by (3) and the definition of f_n, is greater than $3-\varepsilon$. Since φ^{-1} is a norm-decreasing map, we conclude that $\|f+2\varphi(f_n)\|$ is greater than $3-\varepsilon$, and thus either

(a) $2\|\varphi(f_n)\| > 3-\varepsilon$,

or

(b) $|f(k)+2(\varphi(f_n))(k)| > 3-\varepsilon$, for some $k\in\{1,2,\ldots,K\}$. (Or (a) and (b) may both be valid.)

Let us suppose that (a) is true, and consider the element $\varphi^{-1}(f)-2f$ of c. For $k\in N$, $k\neq n$, $f_n(k)=0$, and thus

$$|(\varphi^{-1}(f))(k)-2f_n(k)| = |(\varphi^{-1}(f))(k)| \leqslant \|\varphi^{-1}(f)\| < 1+\varepsilon,$$

by (4). Moreover, we have

$$|(\varphi^{-1}(f))(n)-2f_n(n)| = |1-(\varphi^{-1}(g))(n)-2| \leqslant |(\varphi^{-1}(g))(n)|+1 < 1+\varepsilon,$$

by (2). Hence $\|\varphi^{-1}(f)-2f_n\| < 1+\varepsilon$.

But again employing the fact that $f(k)=0$ for $k>K$, and that $S_n \cap \{1,2,\ldots,K\} = \emptyset$, for $k\in S_n$ we have

$$|(\varphi(\varphi^{-1}(f)-2f_n))(k)| = |f(k)-2(\varphi(f_n))(k)| = |-2(\varphi(f_n))(k)| = 2\|\varphi(f_n)\| > 3-\varepsilon.$$

Consequently $(\varphi^{-1}(f)-2f_n)/(1+\varepsilon)$ is an element of c with norm less than 1, and

$$\left\|\varphi\left(\frac{\varphi^{-1}(f)-2f_n}{1+\varepsilon}\right)\right\| > \frac{3-\varepsilon}{1+\varepsilon},$$

which contradicts (1).

Now suppose that (b) is true. Then for some $k\in\{1,2,\ldots,K\}$ we would have

$$|f(k)|+2|(\varphi(f_n))(k)| \geqslant |f(k)+2(\varphi(f_n))(k)| > 3-\varepsilon,$$

so that

$$|f(k)| > 3-\varepsilon-2|(\varphi(f_n))(k)| > 3-\varepsilon-2\left(\frac{1}{p}\right) > 3-\varepsilon-2\left(\frac{3-(\varepsilon+\|\varphi\|)}{2}\right) = \|\varphi\|.$$

But since $\varepsilon<1$ and φ is norm-increasing, the maximum set of $|\varphi(l)|$ is necessarily contained in $\{1,2,\ldots,K\}$. And as $\varphi(l)=f$ on this latter set, we have $\|\varphi(l)\| = \|f\| \geqslant |f(k)| > \|\varphi\|$, which again is a contradiction. Therefore, we must conclude that our initial assumption that $\|\varphi\|<3$ is false. Hence $(c, c_0) \geqslant \log 3$.

Finally, in order to show that $(c, c_0) = \log 3$, we exhibit a norm-increasing isomorphism φ of c onto c_0 with $\|\varphi\| = 3$. We denote the point at infinity of N^* by n_∞, and define φ as follows. For $f\in c$,

$$(\varphi(f))(1) = 3f(n_\infty),$$

$$(\varphi(f))(n) = \frac{3}{2}(f(n-1)-f(n_\infty)), \quad n\in N,\ n>1.$$

Then φ maps c onto c_0 and is clearly of norm 3. The fact that it is norm-increasing may readily be seen by noting that for $g\in c_0$,

$$(\varphi^{-1}(g))(n) = \frac{2}{3}g(n+1)+\frac{1}{3}g(1), \quad n\in N,$$

so that $\|\varphi^{-1}\| \leqslant 1$.

References

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