On mappings of sequence spaces

by

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In [2], p. 242, Banach defines, for two Banach spaces $E$ and $F$, the number $(E, F)$ by

$$(E, F) = \text{g.l.b.} \left\{ \log (\|p\| \|p^{-1}\|) \right\},$$

where $p$ runs through all isomorphisms of $E$ onto $F$. He calls $E$ and $F$ nearly isometric if $(E, F) = 0$, and inquires whether the spaces $c$ of convergent sequences and $\ell_2$ of sequences convergent to zero (with the usual sup norm) are nearly isometric.

It follows readily from a result of McWilliams [7] that $(c, \ell_2) \geq \log \frac{3}{2}$, and hence that these spaces are not nearly isometric. In [4], p. 397, the sharper estimate $(c, \ell_2) \geq \log 2$ is given. This latter estimate was found quite independently by Gurarij in [6]. Indeed, the fact that $(c, \ell_2) \geq \log 2$ was previously proved by A. Pełczyński, who did not publish his result.

Thus it would seem to be implied by [3], p. 55, (2.1), that actually we have equality: $(c, \ell_2) = \log 2$. In this paper, however, we establish that the exact value of $(c, \ell_2)$ is, in fact, $\log 3$.

Aside from giving a precise answer to Banach’s question, this fact is of interest in the following context. If $X$ is a locally compact Hausdorff space, let us denote by $C_0(X)$ the space of continuous, complex-valued functions on $X$ which are zero at infinity, with norm given by

$$\|f\| = \sup_{x \in X} |f(x)|, \quad f \in C_0(X).$$

(If $X$ is actually compact, then $C_0(X) = C(X)$, the space of all continuous, complex-valued functions on $X$). In [4] and [5] it is shown that if $X$ and $Y$ are any two locally compact Hausdorff spaces which are not homeomorphic, and $\varphi$ is any isomorphism of $C_0(X)$ onto $C_0(Y)$, then $\|\varphi\| \|\varphi^{-1}\| \geq 2$. The analogous result for spaces of real-valued functions defined on compact $X$ and $Y$ was established by Amir in [1]. Now if we denote by $N$ the set of positive integers, and by $N^*$ the one-point compactification of $N$, then $c = C(N^*)$ and $c_0 = C_0(N)$. The fact that $(c_0, \ell_2) \geq \log 2$ thus follows from the more general result about spaces of continuous functions just cited.
In [1], Amir notes that there exist various examples of non-homeomorphic compact Hausdorff spaces $X$ and $Y$, such that $C(X)$ and $C(Y)$ are isomorphic under a map $\varphi$ with $||\varphi|| < 3$. However, no such examples with $||\varphi|| < 3$ seem to be known. Amir thus poses the problem: Do there exist non-homeomorphic compact Hausdorff spaces $X$ and $Y$, and an isomorphism of $C(X)$ onto $C(Y)$ with $2 < ||\varphi|| < 3$?

One may, of course, formulate this question for locally compact spaces, or one may ask a somewhat simpler variant: If $X$ is a compact Hausdorff space, and $Y$ is locally compact but not compact, must any isomorphism $\varphi$ of $C(X)$ onto $C(Y)$ satisfy $||\varphi|| ||\varphi^{-1}|| > 3$? The result of this paper suggests to the author that the answer to this latter question may well be affirmative.

**Theorem.** $(c, c_0) = \log 3$.

**Proof.** Throughout the proof we consider $c$ as $C(N^*)$ and $c_0$ as $C(N^*)$. As was noted in [4], pp. 396–397, in order to prove that $(c, c_0) > \log 3$, it suffices to show that if $\varphi$ is any given norm-increasing isomorphism of $c$ onto $c_0$ (i.e., $||\varphi|| = ||\varphi(f)||$, $f \in c$), then $||\varphi|| > 3$. Thus let $\varphi$ be a norm-increasing isomorphism of $c$ onto $c_0$. We assume that $||\varphi|| < 3$, and show that this assumption leads to a contradiction.

If $||\varphi|| < 3$, we first choose a real number $\varepsilon$ with $0 < \varepsilon < 1$ such that

$$||\varphi|| < 3 - \varepsilon < 1.$$  

(1)

Next, let $l$ denote that element of $c$ which is identically equal to 1 on $N^*$, and consider the element $\varphi(l)$ of $c_0$. Since $\varphi(l)$ is zero at infinity on $N$, there exists an integer $K$ such that $||\varphi(l)|| < \varepsilon$ for all $n \in N$ with $n > K$. We now define the element $f$ of $c_0$ by

$$f(n) = \begin{cases} \varphi(l)(n) & n \leq K, \\ 0 & n > K. \end{cases}$$

Then define $g_n$ by $g(n) = \varphi(l)(n) - f(n)$, $n \in N$. Note that since $||\varphi|| < 3$, and $\varphi^{-1}$ is norm-decreasing, we have

$$||\varphi^{-1}(g)|| < \varepsilon.$$  

(2)

Also, since $f = \varphi(l) - g$, $\varphi^{-1}(f) = 1 - \varphi^{-1}(g)$ and consequently for each $n \in N$,

$$\text{Re}(\varphi^{-1}(f)(n)) = 1 - \text{Re}(\varphi^{-1}(g)(n)) > 1 - ||\varphi^{-1}(g)||(n) > 1 - \varepsilon.$$  

(3)

Note also that

$$||\varphi^{-1}(f)|| \leq ||g|| + ||\varphi^{-1}(g)|| < 1 + \varepsilon.$$  

(4)

Now for each $\alpha \in N^*$, let $f_\alpha$ be that element of $c$ such that $f_\alpha(k) = \delta_{k, \alpha}$, $k \in N$. With each $\alpha \in N$ we associate two subsets $S_\alpha$ and $T_\alpha$ of $N$ as follows. $S_\alpha$ is the maximum set of the function $\varphi(f_\alpha): S_\alpha = \{k \in N : ||\varphi(f_\alpha)||(k) = ||\varphi(f_\alpha)||\}$. In order to define $T_\alpha$, we first fix a positive integer $p$ such that

$$\frac{1}{p} < \frac{3 - (x + ||\varphi||)}{2},$$

and then set $T_\alpha = \{k \in N : ||\varphi(f_\alpha)||(k) > \frac{1}{p}\}$. Again recalling the fact that $\varphi$ is norm-increasing, we observe that $3 - (x + ||\varphi||)/2 < 1$, and that on $S_\alpha$, $||\varphi(f_\alpha)||(k) > 1$. Hence $S_\alpha \subseteq T_\alpha$ for all $\alpha$.

We claim that there can be at most a finite number of integers $\alpha$ such that $T_\alpha \cap \{1, 2, \ldots, K\}$ is non-empty. For if more than $3p - 2$ of the $T_\alpha$ have non-empty intersection with $\{1, 2, \ldots, K\}$, then at least one integer $m(1, 2, \ldots, K)$ must belong to $3p$ of the sets $T_\alpha$, say $T_{n_1}, T_{n_2}, \ldots, T_{n_3}$, $n_m$. We may then define the complex numbers $\lambda_i$, $i = 2, 3, \ldots, 3p$, by $|\lambda_i| = 1$ and $\arg \lambda_i = \arg \varphi(f_\alpha)(m) - \arg \varphi(f_\alpha)(m)$. Consequently,

$$\lambda = f_{\alpha_1} + \sum_{i=2}^{3p} \lambda_i f_{\alpha_i}$$

would be an element of $c$ with $||\lambda|| = 1$, and

$$||\varphi(\lambda)|| = ||\varphi(\lambda)|| = ||\varphi(f_{\alpha_1})(m) + \sum_{i=2}^{3p} \lambda_i ||\varphi(f_\alpha)||(m)||$$

$$= ||\varphi(f_{\alpha_1})(m)|| + \sum_{i=2}^{3p} ||\varphi(f_\alpha)||(m) > 3p\left(\frac{1}{p}\right) = 3,$$

which contradicts the fact that $||\varphi|| < 3$.

We thus may define the integer $M$ as follows. If the set $\{n \in N : T_n \cap \{1, 2, \ldots, K\} \neq \emptyset\}$ is void, let $M = 0$. If this set is non-empty, let $M = \max\{n \in N : T_n \cap \{1, 2, \ldots, K\} \neq \emptyset\}$. Now let $n$ be an integer with $n > M$, and consider the element $f + 2\varphi(f_\alpha)$ of $c$. Since $S_\alpha \cap \{1, 2, \ldots, K\} = \emptyset$, and $f(k) = 0$ for all $k > K$, it follows that for all $k > K$, we have $f(k) + 2||\varphi(f_\alpha)||(k) = 2||\varphi(f_\alpha)||(k) > 2||\varphi(f_\alpha)||$ with equality holding for $k \in S_\alpha$. Thus

$$||f + 2\varphi(f_\alpha)|| = \max\{2||\varphi(f_\alpha)||, ||f(k) + 2||\varphi(f_\alpha)||(k) : k = 1, 2, \ldots, K\}.$$  

If we now apply $\varphi^{-1}$ to this element, we obtain

\[
||\varphi^{-1}(f + 2\varphi(f_\alpha))|| = ||\varphi^{-1}(f)|| + 2\varphi(f_\alpha) \geq ||\varphi^{-1}(f)||(n) + 2\varphi(f_\alpha)(n)
\]

$$\geq \text{Re}(||\varphi^{-1}(f)||(n) + 2\varphi(f_\alpha)(n)).$$
a quantity which, by (3) and the definition of $f_{n_{0}}$, is greater than $3 - \epsilon$. Since \( \varphi^{-1} \) is a norm-decreasing map, we conclude that \( \| f + 2\varphi(f_{n_{0}}) \| \) is greater than $3 - \epsilon$, and thus either

(a) \( 2\| \varphi(f_{n_{0}}) \| > 3 - \epsilon \),

or

(b) \( \| f(k) + 2\varphi(f_{n_{0}})(k) \| > 3 - \epsilon \), for some \( k \in \{ 1, 2, \ldots, K \} \). (Or (a) and (b) may both be valid.)

Let us suppose that (a) is true, and consider the element \( \varphi^{-1}(f) - 2f\) of \( e \). For \( k \in \mathbb{N} \), \( k \neq n_{0} \), \( f_{n_{0}}(k) = 0 \), and thus

\[
\| \varphi^{-1}(f)(k) - 2f_{n_{0}}(k) \| = \| \varphi^{-1}(f)(k) \| < 1 + \epsilon,
\]

by (4). Moreover, we have

\[
\| \varphi^{-1}(f)(n) - 2f_{n_{0}}(n) \| = 1 - \| \varphi^{-1}(f)(n) \| - 2 < \| \varphi^{-1}(f)(n) \| + 1 < 1 + \epsilon,
\]

by (2). Hence \( \| \varphi^{-1}(f) - 2f_{n_{0}} \| < 1 + \epsilon \).

But again employing the fact that \( f(k) = 0 \) for \( k > K \), and that \( S_{n} \cap \{ 1, 2, \ldots, K \} = \emptyset \), for \( k \in S_{n} \), we have

\[
\| \varphi^{-1}(f)(k) - 2f_{n_{0}}(k) \| = | f(k) - 2\varphi(f_{n_{0}})(k) | = 2\| \varphi(f_{n_{0}}) \| > 3 - \epsilon.
\]

Consequently \( \varphi^{-1}(f) - 2f_{n_{0}}/(1 + \epsilon) \) is an element of \( e \) with norm less than 1, and

\[
\| \varphi^{-1}(f)(k) - 2f_{n_{0}}(k) \| > \frac{3 - \epsilon}{1 + \epsilon},
\]

which contradicts (1).

Now suppose that (b) is true. Then for some \( k \in \{ 1, 2, \ldots, K \} \) we would have

\[
| f(k) + 2\varphi(f_{n_{0}})(k) | > | f(k) + 2\varphi(f_{n_{0}})(k) | > 3 - \epsilon,
\]

so that

\[
| f(k) | > 3 - \epsilon - 2\| \varphi(f_{n_{0}})(k) \| > 3 - \epsilon - 2\left( \frac{1}{2} \right) > 3 - \epsilon - 2\left( \frac{1}{2} - \epsilon \right) = \| \varphi \| .
\]

But since \( \epsilon < 1 \) and \( \varphi \) is norm-increasing, the maximum set of \( | \varphi(k) \| \) is necessarily contained in \( \{ 1, 2, \ldots, K \} \). As \( \varphi(k) = f \) on this latter set, we have \( | \varphi(k) | = | f(k) | > | \varphi(k) | \), which again is a contradiction. Therefore, we must conclude that our initial assumption that \( | \varphi \| < 3 \) is false. Hence \( (\epsilon, c_{0}) = \log 3 \).

Finally, in order to show that \( (\epsilon, c_{0}) = \log 3 \), we exhibit a norm-increasing isomorphism \( \varphi \) of \( e \) onto \( e_{0} \) with \( | \varphi \| = 3 \). We denote the point at infinity of \( N^{\ast} \) by \( n_{0} \), and define \( \varphi \) as follows. For \( f \in e \),

\[
\varphi(f)(1) = 3f(n_{0}),
\]

\[
\varphi(f)(n) = \frac{3}{2} f(n - 1) - f(n_{0}), \quad n \in \mathbb{N}, \quad n > 1.
\]

Then \( \varphi \) maps \( e \) onto \( e_{0} \) and is clearly of norm 3. The fact that it is norm-increasing may readily be seen by noting that for \( g \in e_{0} \),

\[
\| \varphi^{-1}(g)(n) \| = \frac{2}{3} g(n + 1) + \frac{1}{3} g(1), \quad n \in \mathbb{N},
\]

so that \( | \varphi^{-1} \| \leq 1 \).

**References**


