Now $\{T_k = 0\}$ occurs with probability (in Y)

$$2^{-2s}\binom{2s}{s} \leqslant As^{-1/2},$$

and in the complementary case $|T_k| \geqslant 2$. Thus

$$(nb_k)E(\theta\,|nT_k|) \leqslant As^{-1/2} + As^{-1/2}\sum_{p=1}^{2s} p^{-1} \leqslant Bs^{-1/2}\log s.$$

Here we used the fact that for any p, $\{T_k = p\}$ occurs with a smaller probability than $\{T_k = 0\}$.

Define $s = s(n)$ for $n > 3$ by the inequality

$$s \leqslant \log n/\log\log n < s+1,$$

so that for large n,

$$\log\left(Bs(n)^{-1/2}\log s(n)\right) \leqslant -\frac{1}{3}\log\log n.$$

Also

$$1 \leqslant nb_k \quad \text{for} \quad 1 \leqslant k \leqslant 5\log n/\log\log n,$$

whence

$$\log\int |\hat{\mu}(n)|^{2s(n)}P(dx) \leqslant -\frac{5}{3}\log n, \qquad n > n_0.$$

Still following Salem [4], we use the fact that for every M

$$\sum_{n=1}^{\infty} M^{2s(n)}\int |\hat{\mu}(n)|^{2s(n)}P(dx) < \infty,$$

so

$$\limsup_{|n|\to\infty} |M\hat{\mu}(n)| \leqslant 1$$

for almost all x, and the proof is complete.

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Modular spaces of generalized variation

by

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In this paper the author continues investigations by J. Musielak and W. Orlicz ([2] and, especially, [3]) about modular functionals of generalized variation. Some fundamental lemmas are first established. Ratio characterizations of inclusions among variation spaces, convex and concave variation, and modular spaces of generalized variation are then treated in succession. The last topic includes the study of locally bounded and locally convex linear topological spaces of generalized variation having the Musielak-Orlicz F-norm topology. Finally, some examples are listed. It is my pleasure here to express my thanks to Dr. Takashi Itô for his excellent advice.

1. M^{th} variation and some fundamental lemmas. Given a real, even, right-continuous function $M(u)$, non-decreasing for $u \geqslant 0$, with $M(0) = 0$ and $M(u) > 0$ for $u > 0$ (such a function will be referred to as a *variation function*) and a real function $x(t)$ defined in a finite closed interval $[a, b]$, the value

$$V_M(x) = \sup_{\pi}\sum_{i=1}^{m} M[x(t_i) - x(t_{i-1})],$$

where $\pi\colon a = t_0 < t_1 < \ldots < t_m = b$ is an arbitrary partition of the interval $[a, b]$, is called the M^{th} *variation* of $x(t)$ in $[a, b]$. It can be shown that

$$M(\alpha x + \beta y) \leqslant \alpha M(x) + \beta M(y) \quad \text{iff} \quad V_M(\alpha x + \beta y) \leqslant \alpha V_M(x) + \beta V_M(y),$$

while

$$M(\alpha x + \beta y) \geqslant \alpha M(x) + \beta M(y) \quad \text{iff} \quad V_M(\alpha x + \beta y) \geqslant \alpha V_M(x) + \beta V_M(y),$$

for $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$; that is, M is convex (concave) iff V_M is convex (concave). For a more detailed discussion, see [2]. Let X be the class of all real functions defined on $[a, b]$ which vanish at a. For $x, y \in X$, it is easy to verify that $V_M(x) = 0$ iff $x = 0$, $V_M(-x) = V_M(x)$, and if $\alpha, \beta \geqslant 0$ and $\alpha + \beta = 1$, then $V_M(\alpha x + \beta y) \leqslant V_M(x) + V_M(y)$. Define

$$B_M = \{x \in X\colon V_M(x) < +\infty\};$$

B_M is convex and symmetric. Also define

$B_M^* = \{x \in X \colon V_M(ax) < +\infty$ for some $a > 0$, where a depends on $x\}$.

B_M^* is the linear space generated by the space B_M of functions of bounded M^{th} variation. If $M(x) = |x|$, denote B_M by B and V_M by V. B is the linear space of functions of ordinary bounded variation. Space B should not be confused with condition B defined below.

LEMMA 1. *Let M and N be variation functions. If*

$$(\exists a > 0) \lim_{u \to 0+} \frac{M(u)}{N(u)} < a,$$

then $(\forall K > 0)$ *there exists a sequence* $\{u_\nu\} \downarrow 0$ *such that*

$$\sum_{\nu=1}^{\infty} M(u_\nu) = K \quad \text{and} \quad \sum_{\nu=1}^{\infty} N(u_\nu) \geqslant \frac{K}{a}.$$

Proof. We have

$$(\forall v > 0) \inf_{0 < u \leqslant v} \frac{M(u)}{N(u)} < a.$$

Define $B = \{u > 0 \colon M(u)/N(u) < a\}$. Then $0 \in B^-$ because B contains arbitrarily small positive numbers. Choose from B a sequence $\{v_n\}$ such that $v_n \downarrow 0$. Then $M(v_n) \downarrow 0$ because M is continuous at 0. Then there exist positive integers m_ν and a subsequence v_{n_ν} of v_n such that $K = \sum_{\nu=1}^{\infty} m_\nu M(v_{n_\nu})$. By a suitable relabeling, we get $K = \sum_{\nu=1}^{\infty} M(u_\nu)$, where $u_\nu \in B$ and $u_\nu \downarrow 0$. On the other hand, $K = \sum_{\nu=1}^{\infty} M(u_\nu) \leqslant \sum_{\nu=1}^{\infty} a \cdot N(u_\nu)$, so that $\sum_{\nu=1}^{\infty} N(u_\nu) \geqslant K/a$.

LEMMA 2. *Let M and N be variation functions. If*

$$\lim_{u \to 0+} \frac{M(u)}{N(u)} = 0,$$

then there exists a sequence $\{u_m\} \downarrow 0$ *such that*

$$\sum_{m=1}^{\infty} M(u_m) = K \quad \text{and} \quad \sum_{m=1}^{\infty} N(u_m) = +\infty.$$

Proof. By Lemma 1, for each $n > 0$ there exists a positive sequence $u_{i,n} \downarrow_i 0$ such that

$$\sum_{i=1}^{\infty} M(u_{i,n}) = K/2^n \quad \text{and} \quad \sum_{i=1}^{\infty} N(u_{i,n}) \geqslant n.$$

Then

$$\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} M(u_{i,n}) = K \quad \text{and} \quad \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} N(u_{i,n}) = +\infty.$$

Each of these iterated series has only non-negative terms, so that any rearrangement of the first converges to the limit K and any rearrangement of the second diverges. Because the first iterated series converges, the sets

$$C_0 = \{u_{i,n} \colon M(u_{i,n}) \geqslant 1\} \quad \text{and} \quad C_\nu = \left\{u_{i,n} \colon \frac{1}{\nu+1} \leqslant M(u_{i,n}) < \frac{1}{\nu}\right\},$$

$\nu = 1, 2, 3, \ldots$, are all finite. They are also mutually disjoint and if $a \in C_\nu$ and $\beta \in C_{\nu+1}$, then $a > \beta$. Thus, the elements in each C_ν, $\nu = 0, 1, 2, \ldots$, can be arranged in descending order and then listed in that order beginning with those from C_0, then those from C_1, from C_2, etc., thus forming a rearrangement $\{u_m\}$ of $\{u_{i,n}\}$ for which $u_m \downarrow 0$. Hence

$$\sum_{m=1}^{\infty} M(u_m) = K \quad \text{and} \quad \sum_{m=1}^{\infty} N(u_m) = +\infty.$$

LEMMA 3. *Given any sequence* $\{u_m\} \downarrow 0$,

$$(\exists x \in X)(\forall \beta \geqslant 0)\, V_M(\beta x) = \sum_{m=1}^{\infty} M(\beta u_m)$$

for all variation functions M.

Proof. Set

$$0 \leqslant \sum_{m=1}^{\infty} (-1)^{m+1} u_m = a < +\infty.$$

Now, define the following step function $x(t)$ for $a \leqslant t \leqslant b$:

$$x(t) = a \quad \text{for} \quad \frac{a+b}{2} < t \leqslant b,$$

$$x(t) = a - \sum_{m=1}^{k} (-1)^{m+1} u_m \quad \text{for} \quad \frac{a(2^{k+1}-1)+b}{2^{k+1}} < t \leqslant \frac{a(2^k-1)+b}{2^k},$$

$$k = 1, 2, 3, \ldots,$$

and $x(a) = 0$. Clearly, $\lim_{t \to a^+} x(t) = 0$. For this function x any refinement of a given partition can only increase the M^{th} variation of x or leave it unchanged, because

$$\left|\sum_{m=s}^{t} (-1)^{m+1} u_m\right| \leqslant u_s$$

for any s, t with $s \leqslant t$. Hence

$$V_M(x) = \sum_{m=1}^{\infty} M(u_m)$$

for any variation function M. Working instead with the function $\beta x(t)$ and the sequence $\{\beta u_m\}$, which does not change the nature of the argument, one obtains the original assertion

$$V_M(\beta x) = \sum_{m=1}^{\infty} M(\beta u_m).$$

2. Ratio characterizations of inclusions among generalized variation spaces. Now we prove

THEOREM 1. *If M and N are variation functions, then*

$$\varliminf_{u\to 0+} \frac{M(u)}{N(u)} > 0$$

iff $B_M \subset B_N$.

Proof. $\leftarrow$: If

$$\varliminf_{u\to 0+} \frac{M(u)}{N(u)} = 0,$$

then by Lemma 2 there exists a sequence $\{u_m\}\downarrow 0$ such that

$$\sum_{m=1}^{\infty} M(u_m) = 1 \quad \text{and} \quad \sum_{m=1}^{\infty} N(u_m) = +\infty.$$

By Lemma 3,

$$(\exists x \in X)\, V_M(x) = \sum_{m=1}^{\infty} M(u_m) = 1,$$

so that $x \in B_M$, but

$$V_N(x) = \sum_{m=1}^{\infty} N(u_m) = +\infty,$$

so that $x \notin B_N$ and $B_M \not\subset B_N$.

$\rightarrow$: We have

$$(\exists v_0) \inf_{0<u\leqslant v_0} \frac{M(u)}{N(u)} = K > 0.$$

Hence $(\forall u, 0 < u \leqslant v_0) N(u) \leqslant M(u)/K$. Let $x \in B_M$. Then x is a bounded function on $[a, b]$, say $|x| \leqslant T$. Moreover, there is an upper bound $A = [V_M(x)/M(v_0)]$ (independent of the partition π) on the number of expressions $|x(t_i)-x(t_{i-1})| > v_0$ in the sum $\sum_{i=1}^{m} N[x(t_i)-x(t_{i-1})]$ for

any partition π of $[a, b]$. Let $i = 1', 2', \ldots, m'$ be that subset of the indices in the sum for which $|x(t_i)-x(t_{i-1})| \leqslant v_0$. Note also that $|x(t_i)-x(t_{i-1})| \leqslant 2T$ for arbitrary $i = 1, 2, \ldots, m$. Then

$$\sum_{i=i}^{m} N[x(t_i)-x(t_{i-1})] \leqslant \sum_{i=1'}^{m'} N[x(t_i)-x(t_{i-1})] + A\cdot N(2T)$$
$$\leqslant \frac{\sum_{i=1'}^{m'} M[x(t_i)-x(t_{i-1})]}{K} + A\cdot N(2T),$$

so that $V_N(x) \leqslant V_M(x)/K + A\cdot N(2T) < +\infty$ and $x \in B_N$. Hence $B_M \subset B_N$.

COROLLARY 1. $0 < \varliminf_{u\to 0+} \frac{M(u)}{N(u)} \leqslant \varlimsup_{u\to 0+} \frac{M(u)}{N(u)} < +\infty$ *iff* $B_M = B_N$.

COROLLARY 2. $\varliminf_{u\to 0+} \frac{M(u)}{M(2u)} > 0$ *iff* $B_M = B_M^*$.

Proof. $\rightarrow$: Let $N(u) = M(2u)$ in Theorem 1, which implies $B_{M(\cdot)} \subset B_{M(2\cdot)} = \frac{1}{2}B_{M(\cdot)}$. Hence $2B_M \subset B_M$, and thus

$$B_M^* = \bigcup_{n=1}^{\infty} 2^n B_M \subset B_M.$$

$\leftarrow$: If $B_M = B_M^*$, then $V_M(x) < +\infty \Rightarrow V_M(2x) < +\infty$, which means that

$$B_{M(\cdot)} \subset B_{M(2\cdot)} \quad \text{or} \quad \varliminf_{u\to 0+} \frac{M(u)}{M(2u)} > 0$$

by Theorem 1.

THEOREM 2. *If M and N are variation functions, then*

$$\varliminf_{u\to 0+} \frac{M(u)}{N\left(\frac{1}{n}u\right)} > 0$$

for some positive integer n iff $B_M^* \subset B_N^*$.

Proof. $\rightarrow$: By Theorem 1,

$$\varliminf_{u\to 0+} \frac{M(u)}{N\left(\frac{1}{n}u\right)} > 0$$

implies $B_{M(\cdot)} \subset B_{N(1/n\cdot)} = nB_{N(\cdot)} \subset B_{N(\cdot)}^*$, so that $B_M \subset B_N^*$ which implies $B_M^* \subset B_N^*$.

$\leftarrow$: If
$$\lim_{u\to 0^+}\frac{M(u)}{N\left(\frac{1}{n}u\right)}=0 \quad \text{for all } n,$$
then by Lemma 2, there exists a double sequence $\{u_{n,i}\}$ such that $u_{n,i}\downarrow_i 0$ for fixed n,
$$\sum_{i=1}^{\infty} M(u_{n,i})=1/2^n \quad \text{for each } n,$$
and
$$\sum_{i=1}^{\infty} N\left(\frac{1}{n}u_{n,i}\right)=+\infty \quad \text{for each } n.$$
But
$$\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} M(u_{n,i})=1,$$
so that (as in the proof of Lemma 2) a rearrangement $\{u_m\}\downarrow 0$ of $\{u_{n,i}\}$ can be constructed, and
$$\sum_{m=1}^{\infty} M(u_m)=1$$
but
$$(\forall n)\sum_{m=1}^{\infty} N\left(\frac{1}{n}u_m\right)\geqslant \sum_{i=1}^{\infty} N\left(\frac{1}{n}u_{n,i}\right)=+\infty.$$
By Lemma 3
$$(\exists x\in X)\, V_M(x)=\sum_{m=1}^{\infty} M(u_m)=1,$$
but
$$(\forall n)\, V_N\left(\frac{1}{n}x\right)=\sum_{m=1}^{\infty} N\left(\frac{1}{n}u_m\right)=+\infty.$$
Hence $x\in B_M\subset B_M^*$ but $x\notin nB_N$ for any n, so that $x\notin B_N^*$. Hence $B_M^*\not\subset B_N^*$.

COROLLARY. $B_M^*\subset B_N^*$ *iff* $(\exists n>0)\, B_M\subset nB_N$.

3. Convex variation and concave variation. Two classes of variation functions, each element of which satisfies one of the following two conditions, are studied in this section:

I. $(\exists K\geqslant 1)(\forall 0<u\leqslant v<v_0)\,\dfrac{M(u)}{u}\leqslant K\dfrac{M(v)}{v}$;

D. $(\exists K\geqslant 1)(\forall 0<u\leqslant v<v_0)\,K\dfrac{M(u)}{u}\geqslant\dfrac{M(v)}{v}$.

THEOREM 3. (i) *Condition* I *holds iff*
$$(\exists v_0>0)\inf_n\inf_{0<u\leqslant v_0}\frac{\frac{1}{n}M(u)}{M\left(\frac{1}{n}u\right)}>0,$$
and (ii) *condition* D *holds iff*
$$(\exists v_0>0)\sup_n\sup_{0<u\leqslant v_0}\frac{\frac{1}{n}M(u)}{M\left(\frac{1}{n}u\right)}<+\infty.$$

Proof. (i) $\rightarrow$:
$$(\exists K\geqslant 1)(\forall 0<u\leqslant v<v_0)\,\frac{M(u)}{u}\leqslant K\frac{M(v)}{v}$$
means
$$M\left(\frac{1}{n}u\right)\leqslant K\cdot\frac{u}{n}\cdot\frac{M(u)}{u}=\frac{K}{n}M(u) \quad \text{for} \quad u<v_0.$$
Hence
$$\inf_n\inf_{0<u\leqslant v_0}\frac{\frac{1}{n}M(u)}{M\left(\frac{1}{n}u\right)}\geqslant\inf_n\inf_{0<u\leqslant v_0}\frac{\frac{1}{n}\cdot\frac{n}{K}\cdot M\left(\frac{1}{n}u\right)}{M\left(\frac{1}{n}u\right)}=\frac{1}{K}>0.$$
$\leftarrow$: If
$$\inf_n\inf_{0<u\leqslant v_0}\frac{\frac{1}{n}M(u)}{M\left(\frac{1}{n}u\right)}=Q>0, \quad \text{and} \quad 0<u\leqslant v<v_0,$$
then
$$(\exists n\geqslant 1)\,\frac{1}{n+1}<\frac{u}{v}\leqslant\frac{1}{n},$$
hence
$$\frac{\frac{1}{n}M(v)}{M(u)}\geqslant\frac{\frac{1}{n}M(v)}{M\left(\frac{1}{n}v\right)}\geqslant Q,$$

and

$$\frac{M(v)}{M(u)} \geqslant \frac{Q}{2}\cdot 2n \geqslant \frac{Q}{2}(n+1) \geqslant \frac{Q}{2}\frac{v}{u}.$$

Putting $K = 2/Q$, we get

$$\frac{M(u)}{u} \leqslant K\cdot\frac{M(v)}{v}.$$

The proof of (ii) requires the same types of calculations and is omitted.

A variation function M is said to *satisfy condition* B iff

$$(\exists K \geqslant 1)(\exists v_0 > 0)\Big(\forall u_1, \ldots, u_n > 0\colon \sum_{i=1}^{n} u_i < v_0\Big)\, M\Big(\sum_{i=1}^{n} u_i\Big) \leqslant K\sum_{i=1}^{n} M(u_i).$$

THEOREM 4. *Condition* D *holds iff condition* B *holds.*

Proof. $\to$: By D,

$$\Big(\forall u_i > 0,\ i = 1, 2, \ldots, n\colon \sum_{i=1}^{n} u_i < v_0\Big)(\exists K \geqslant 1)K\frac{M(u_i)}{u_i} \geqslant \frac{M\big(\sum_{i=1}^{n} u_i\big)}{\sum_{i=1}^{n} u_i},$$

so that

$$K\sum_{i=1}^{n} M(u_i) \geqslant M\Big(\sum_{i=1}^{n} u_i\Big),$$

which is condition B.

$\leftarrow$: In the statement of condition B, let $u_i = u$, $i = 1, 2, \ldots, n$, so that $nu < v_0$, yielding $M(nu) \leqslant nKM(u)$, or

$$K\frac{M(u)}{u} \geqslant \frac{M(nu)}{nu} \quad \text{for} \quad nu < v_0.$$

Now, suppose generally that $0 < u \leqslant v < v_0/2$. Then there exists a unique integer $m \geqslant 1$ such that $mu < v \leqslant (m+1)u$, and since $mu < v < v_0/2$, $(m+1)u \leqslant 2mu < v_0$, or $(m+1)u < v_0$. Hence

$$\frac{M(v)}{v} \leqslant \frac{M\big((m+1)u\big)}{mu} = \frac{m+1}{m}\,\frac{M\big((m+1)u\big)}{(m+1)u} \leqslant \frac{m+1}{m}K\frac{M(u)}{u} \leqslant 2K\frac{M(u)}{u},$$

so

$$\frac{M(v)}{v} \leqslant 2K\frac{M(u)}{u}$$

and condition D holds with constant $K' = 2K$.

In a quite similar way one can prove

THEOREM 5. *Condition* I *holds iff*

$$(\exists K \geqslant 1)(\exists v_0 > 0)\Big(\forall u_1, u_2, \ldots, u_n > 0\colon \sum_{i=1}^{n} u_i < v_0\Big)\sum_{i=1}^{n} M(u_i) \leqslant KM\Big(\sum_{i=1}^{n} u_i\Big).$$

THEOREM 6. (i) *If $B_M = B_{\hat{M}}$ for some convex function $\hat{M}$, then condition* I *holds*; (ii) *if condition* I *holds, then there exists a convex function $\check{M}$ such that $B_M^* = B_{\check{M}}^*$.*

Proof. (i) By Corollary 1 of Theorem 1, $B_M = B_{\hat{M}}$ implies

$$0 < K = \varliminf_{u\to 0^+}\frac{M(u)}{\hat{M}(u)} \leqslant \varlimsup_{u\to 0^+}\frac{M(u)}{\hat{M}(u)} = L < +\infty.$$

This means

$$(\exists v_0 > 0)(\forall 0 < u < v_0)K\hat{M}(u) \leqslant M(u) \leqslant L\hat{M}(u).$$

Hence

$$(\forall 0 < u \leqslant v < v_0)\frac{M(u)}{u} \leqslant L\frac{\hat{M}(u)}{u} \leqslant L\frac{\hat{M}(v)}{v} \leqslant \frac{L}{K}\,\frac{M(v)}{v};$$

the middle inequality holds because $\hat{M}(u)/u\uparrow$ for convex functions $\hat{M}$, and hence condition I holds for M with constant $L/K \geqslant 1$.

(ii) If

$$(\exists K \geqslant 1)(\forall 0 < u \leqslant v < v_0)\frac{M(u)}{u} \leqslant K\frac{M(v)}{v},$$

define

$$(\forall u < v_0)\varphi(u) = \sup_{0<t\leqslant u}\frac{M(t)}{t},$$

and

$$(\forall u \geqslant v_0)\varphi(u) = \sup_{0<t\leqslant v_0}\frac{M(t)}{t}.$$

$\varphi(u)$ is well-defined because $0 < \varphi(u) \leqslant KM(u)/u$ for $u < v_0$ and $\varphi(u)$ is a positive constant for $u \geqslant v_0$. $\varphi(u)\uparrow$ by the definition. Now define

$$\check{M}(u) = \int_0^u \varphi(t)\,dt;$$

the function $\check{M}$ is convex because $\varphi(u)\uparrow$. By definition of $\varphi(u)$, if $u < v_0$, then $M(\tfrac{1}{2}u)/\tfrac{1}{2}u \leqslant \varphi(\tfrac{1}{2}u)$, hence

$$M\Big(\frac{u}{2}\Big) \leqslant \frac{u}{2}\varphi\Big(\frac{u}{2}\Big) \leqslant \check{M}(u) \leqslant u\varphi(u) \leqslant KM(u);$$

the inner two inequalities hold because of the integral definition of $\hat M$ and because $\varphi(u)\uparrow$ and is positive. $M(u/2) \leqslant \hat M(u)$ for $u < v_0$ means

$$\varliminf_{u\to 0^+} \frac{\hat M(u)}{M(\frac{1}{2}u)} \geqslant 1 > 0,$$

or $B^*_{\hat M} \subset B^*_M$ by Theorem 2. $\hat M(u) \leqslant KM(u)$ for $u < v_0$ means

$$\varliminf_{u\to 0^+} \frac{M(u)}{\hat M(u)} \geqslant \frac{1}{K} > 0,$$

or $B_M \subset B_{\hat M}$ by Theorem 1, hence $B^*_M = B^*_{\hat M}$.

Similarly, one can prove

THEOREM 7. *$B^*_M = B^*_{\hat M}$ for some convex function $\hat M$ iff*

$$(\exists K \geqslant 1)(\exists v_0 > 0)(\exists n \geqslant 1)(\forall 0 < u \leqslant v < v_0)\frac{M(u)}{u} \leqslant K\frac{M(nv)}{v}.$$

THEOREM 8. (i) *If $B_M = B_{\check M}$ for some concave function $\check M$, then condition* D *holds*; (ii) *if condition* D *holds for M, then there exists a concave function $\check M$ such that $B^*_M = B^*_{\check M}$.*

Proof. (i) This proof is quite similar to that of Theorem 6 (i) and is therefore omitted.

(ii) a) If N is a variation function, then the function N^{-1} defined for $u \geqslant 0$ in the usual way, except that any interval on which N is constant is replaced by a discontinuity of N^{-1} and any discontinuity of N is replaced by an interval on which N^{-1} is constant, in such a way that N^{-1} is also right-continuous for $u \geqslant 0$ and N^{-1} is defined for $u < 0$ by requiring that it be an even function, is also a variation function.

b) Let M be a variation function satisfying D; it will be shown that M^{-1} satisfies I. Define

$$(\forall u > 0)M(u-) = \sup_{0<w<u} M(w).$$

D implies

$$(\exists K \geqslant 1)(\forall 0 < u \leqslant v < v_0)K\frac{M(u-)}{u} \geqslant \frac{M(v)}{v}.$$

Let $M^{-1}(s) = u$, $M^{-1}(t) = v$. Then

$$(\forall 0 < u \leqslant v < v_0)M(u-) \leqslant s \leqslant M(u), \quad M(v-) \leqslant t \leqslant M(v).$$

Now, for $t_0 = \min\{t\colon M^{-1}(t) \geqslant v_0\}$, and $0 < s \leqslant t < t_0$,

$$\frac{M^{-1}(s)}{s} = \frac{u}{s} \leqslant \frac{u}{M(u-)} \leqslant K\frac{v}{M(v)} \leqslant K\frac{v}{t} = K\frac{M^{-1}(t)}{t}.$$

Hence M^{-1} satisfies I.

c) Next we show that $B^*_M = B^*_N$ implies $B^*_{M^{-1}} = B^*_{N^{-1}}$. By symmetry, it suffices to prove that $B^*_M \subset B^*_N$ implies $B^*_{N^{-1}} \subset B^*_{M^{-1}}$; but this is true by Theorem 2 if

$$\varliminf_{u\to 0^+} \frac{M(u)}{N\left(\frac{1}{n}u\right)} > \frac{1}{m}$$

implies

$$\varliminf_{s\to 0^+} \frac{N^{-1}(s)}{M^{-1}\left(\frac{1}{m}s\right)} \geqslant \frac{1}{n},$$

which is now proved:

$$\varliminf_{u\to 0^+} \frac{M(u)}{N\left(\frac{1}{n}u\right)} > \frac{1}{m}$$

means

$$(\exists v_0 > 0)(\forall 0 < u < v_0)M(u) > \frac{1}{m}N\left(\frac{1}{n}u\right),$$

hence

$$(\forall 0 < u < v_0)M(u-) \geqslant \frac{1}{m}N\left(\frac{1}{n}u-\right).$$

Also, $(\exists t_0 > 0)M^{-1}(t_0) < v_0$, and $(\forall 0 < s < t_0)M^{-1}(s) = u < v_0$, so that $M(u-) \leqslant s \leqslant M(u)$. Now,

$$N^{-1}(ms) \geqslant N^{-1}\big(mM(u-)\big) \geqslant N^{-1}\left(m\frac{1}{m}N\left(\frac{1}{n}u-\right)\right)$$

$$= N^{-1}\left(N\left(\frac{1}{n}u-\right)\right) \geqslant \frac{1}{n}u = \frac{1}{n}M^{-1}(s).$$

Hence,

$$(\forall 0 < s < t_0)\frac{N^{-1}(ms)}{M^{-1}(s)} \geqslant \frac{1}{n}, \quad \text{or} \quad \varliminf_{s\to 0^+} \frac{N^{-1}(s)}{M^{-1}\left(\frac{1}{m}s\right)} \geqslant \frac{1}{n}.$$

d) Finally, if condition D holds for M, b) shows that condition I holds for M^{-1}. By Theorem 6, (ii), there exists a convex function $\hat M$ such that $B^*_{\hat M} = B^*_{M^{-1}}$. By c), $B^*_{\hat M^{-1}} = B^*_M$. But $\hat M^{-1}$ is a concave function $\check M$.

4. Modular spaces of generalized variation. Given any linear space X, a functional $\varrho(x)$ defined on X with values $-\infty < \varrho(x) \leqslant +\infty$ is called a *modular* if

(i) $\varrho(x) = 0$ iff $x = 0$,

(ii) $\varrho(-x) = \varrho(x)$, and

(iii) if $a, \beta \geqslant 0$ and $a+\beta = 1$, then $\varrho(ax+\beta y) \leqslant \varrho(x)+\varrho(y)$.

On the linear subspace

$$X_\varrho^* = \{x \in X\colon \varrho(ax) < +\infty \text{ for some } a > 0 \text{ depending on } x\}$$

one can define the *Musielak-Orlicz F-norm*

$$\|x\|_\varrho = \inf\left\{\varepsilon > 0\colon \varrho\left(\frac{x}{\varepsilon}\right) \leqslant \varepsilon\right\}$$

which has the fundamental properties $0 \leqslant \|x\|_\varrho < +\infty$, $\|x\|_\varrho = 0$ iff $x = 0$, $\|x+y\|_\varrho \leqslant \|x\|_\varrho + \|y\|_\varrho$, and $\|x_n\|_\varrho \to 0$ iff $\varrho(ax_n) \to 0$ for all $a > 0$. $\|\cdot\|_\varrho$ induces an addition-invariant metric topology on X_ϱ^*. In our case, X is the class of real functions defined on $[a, b]$ which vanish at a, V_M is the modular (which is convex or concave iff M is), and B_M^* is the subspace X_ϱ^* (for details, see [3]). The Musielak-Orlicz F-norm defined by V_M on B_M^* will be denoted by $\|\cdot\|_M$.

THEOREM 9. *$B_M^* \subset B_N^*$ implies $\|x_n\|_M \to 0 \Rightarrow \|x_n\|_N \to 0$.*

Proof. $\|x_n\|_M \to 0$ implies $(\forall a > 0)\, V_M(ax_n) \to 0$. By theorem 2, $B_M^* \subset B_N^*$ implies

$$(\exists m \geqslant 1)(\exists K > 0)(\exists v_0 > 0)(\forall 0 < u < v_0)\ M(u) \geqslant KN\left(\frac{1}{m}u\right).$$

But $(\forall n \geqslant n_0)\,|ax_n| < v_0$ because $V_M(ax_n) \to 0$. Hence

$$(\forall n \geqslant n_0)\ V_M(ax_n) \geqslant KV_N\left(\frac{a}{m}x_n\right)$$

and thus

$$V_N\left(\frac{a}{m}x_n\right) \to 0 \quad \text{as} \quad n \to \infty.$$

But m is fixed and a is arbitrary, and therefore $\|x_n\|_N \to 0$.

COROLLARY. *$B_M^* = B_N^*$ implies $\|\cdot\|_M \sim \|\cdot\|_N$.*

This corollary shows that if $B_M^* = B_N^*$, then the topologies on these spaces are also the same, i.e., do not depend on the particular form of M or N.

The $\|\cdot\|_M$-topology is a linear topology on B_M^* iff $(\forall x \in B_M^*)\, a_n \to 0$ implies $V_M(a_n x) \to 0$. This fact prompts the definition of condition

B1. $a_n \to 0$ implies $V_M(a_n x) \to 0$ for some element $x \in B_M^*$.

Remark. Taking $a_n x = x_n$ in Theorem 9, it is evident that if B1 is satisfied throughout B_M^* (with respect to M^{th} variation), then, if $B_M^* \subset B_N^*$, B1 is also satisfied in $B_M^* \cap B_N^*$ with respect to N^{th} variation.

No characterization of variation functions M satisfying condition B1 in B_M^* is known; for this reason it is useful to consider the following stronger condition:

B1*. $(\exists \delta > 0)(\exists 0 < a < 1)(\forall x \in B_M^*)\, |x| < \delta \Rightarrow V_M(ax) \leqslant \frac{1}{2}V_M(x)$.

Condition B1 applies generally to elements of B_M^*, but condition B1* applies only to the whole space B_M^*.

As a final condition on variation functions M we define

C. $\sup\limits_n \left\{\varliminf\limits_{u\to 0^+} \dfrac{M(u)}{M\left(\frac{1}{n}u\right)}\right\} > 1.$

THEOREM 10. *Condition* C *holds iff condition* B1* *holds.*

Proof. $\to$: C means

$$(\exists N \geqslant 1)(\exists v_0 > 0) \inf_{0<u\leqslant v_0} \frac{M(u)}{M\left(\frac{1}{N}u\right)} = K > 1.$$

Hence

$$(\exists N \geqslant 1)(\exists v_0 > 0)(\forall 0 < u \leqslant v_0) M(u) \geqslant KM\left(\frac{1}{N}u\right)$$

or, putting $k = 1/K < 1$,

$$M\left(\frac{1}{N}u\right) \leqslant kM(u).$$

Now, let x be a real-valued function in $[a, b]$ with $x(a) = 0$ and such that $|x| \leqslant v_0/2 = \delta$. Then $|x(t_i) - x(t_{i-1})| \leqslant v_0$ for arbitrary points $t_i, t_{i-1} \in [a, b]$. Hence, for any partition sum,

$$\sum_{i=1}^m M\left[\frac{1}{N}\big(x(t_i) - x(t_{i-1})\big)\right] \leqslant k \sum_{i=1}^m M[x(t_i) - x(t_{i-1})],$$

so that

$$V_M\left(\frac{1}{N}x\right) \leqslant kV_M(x) \quad \text{with } k < 1.$$

By induction,

$$V_M\left(\frac{1}{N^p}x\right) \leqslant k^p V_M(x) \quad \text{for} \quad p = 1, 2, 3, \ldots$$

For a sufficiently large p_0, $k^{p_0} \leqslant \frac{1}{2}$; take $\alpha = 1/N^{p_0}$, thus getting $V_M(\alpha x) \leqslant k^{p_0} V_M(x) \leqslant \frac{1}{2} V_M(x)$ which shows that condition B1* holds.

$\leftarrow$: If

$$\sup_n \left\{ \varlimsup_{u\to 0^+} \frac{M(u)}{M\left(\frac{1}{n}u\right)} \right\} = 1,$$

then

$$(\forall n \geqslant 1) \varlimsup_{u\to 0^+} \frac{M(u)}{M\left(\frac{1}{n}u\right)} = 1 < \frac{3}{2}.$$

Now construct a double sequence $\{u_{i,n}\}$ by Lemma 1, such that $u_{i,n}\downarrow_i 0$, $0 < u_{i,n} \leqslant \delta$, and for each n,

$$0 < \sum_{i=1}^{\infty} M(u_{i,n}) = 1 \quad \text{but} \quad \sum_{i=1}^{\infty} M\left(\frac{1}{n} u_{i,n}\right) \geqslant \frac{2}{3}.$$

For each n, by Lemma 3, use the sequence $\{u_{i,n}\}_{i=1}^{\infty}$ to construct a function x_n, where $|x_n(t)| \leqslant \delta$ for $t \in [a, b]$ and

$$V_M(x_n) = \sum_{i=1}^{\infty} M(u_{i,n}) = 1, \quad \text{while} \quad V_M\left(\frac{1}{n} x_n\right) = \sum_{i=1}^{\infty} M\left(\frac{1}{n} u_{i,n}\right) \geqslant \frac{2}{3}.$$

Thus $|x_n| \leqslant \delta$ and

$$\frac{2}{3} V_M(x_n) \leqslant V_M\left(\frac{1}{n} x_n\right)$$

as well as $V_M(x_n) = 1$ for any fixed n. Assuming now that condition B1* holds with a certain constant α, $0 < \alpha < 1$, choose n_0 a positive integer such that $1/n_0 < \alpha$. By B1*,

$$V_M\left(\frac{1}{n_0} x_{n_0}\right) \leqslant \tfrac{1}{2} V_M(x_{n_0})$$

since $|x_{n_0}| < \delta$. But from above,

$$\frac{2}{3} V_M(x_{n_0}) \leqslant V_M\left(\frac{1}{n_0} x_{n_0}\right).$$

Hence $\frac{2}{3} V_M(x_{n_0}) \leqslant \frac{1}{2} V_M(x_{n_0})$, which is a contradiction because $V_M(x_{n_0}) = 1$. Thus, if condition C does not hold, then condition B 1* also fails to hold.

THEOREM 11. *If condition* I *holds, then condition* B1* *holds.*

Proof. If

$$(\exists K \geqslant 1)(\forall 0 < u \leqslant v < v_0)\, \frac{M(u)}{u} \leqslant K \frac{M(v)}{v},$$

then

$$(\forall x < v_0)(\forall \alpha\colon 0 < \alpha < 1)\, \frac{M(\alpha x)}{\alpha x} \leqslant K \frac{M(x)}{x},$$

or $M(\alpha x) \leqslant \alpha K M(x)$. Then, for $\alpha \leqslant 1/2K < 1$ and $x < v_0$, $M(\alpha x) \leqslant \frac{1}{2} M(x)$, and thus for $|x| < v_0/2$,

$$\sum_{i=1}^{m} M[\alpha(x(t_i) - x(t_{i-1}))] \leqslant \tfrac{1}{2} \sum_{i=1}^{m} M[x(t_i) - x(t_{i-1})].$$

Hence $V_M(\alpha x) \leqslant \frac{1}{2} V_M(x)$ for $\alpha \leqslant 1/2K$ and $|x| < v_0/2$, and thus condition B1* holds.

Next it will be shown that condition D for M implies condition B1 throughout B_M^*. For this purpose, the following lemma, a sharpened version of Musielak-Orlicz's Theorem 2.21 ([3], p. 58) is useful. This lemma may also be interesting in its own right.

LEMMA 4. *If*

$$\varlimsup_{u\to 0^+} \frac{M(u)}{u} = +\infty, \qquad x \in B_M,$$

and $x(t)$ *is continuous in* $[\alpha, \beta] \subset [a, b]$, *then* $x(t)$ *is constant in* $[\alpha, \beta]$.

Proof. Assume that $x(t)$ is continuous in $[\alpha, \beta]$, $x(\alpha) = c$, $x(\beta) = d$, where (without loss of generality) $c < d$, and take for each positive integer n a partition $\alpha = t_0 < t_1 < \ldots < t_{2^n} = \beta$ of $[\alpha, \beta]$ such that $x(t_i) = \{i \cdot 2^{-n}(d-c)\} + c$. Since

$$\varlimsup_{u\to 0^+} \frac{M(u)}{u} = +\infty,$$

there exists a sequence $\{v_j\}$ such that $v_j \downarrow 0$ and

$$\lim_{j\to\infty} \frac{M(v_j)}{v_j} = +\infty.$$

Now let $u_j = 2v_j/(d-c)$, so that also $u_j \downarrow 0$ for the sequence $\{u_j\}$. Also,

$$(\forall u_j)(\exists n_j \geqslant 1)\, 2^{-n_j} < u_j \leqslant 2^{-n_j+1}.$$

Construct the above-mentioned partition for $n = n_j$. Then

$$V_M(x) \geqslant \sum_{i=1}^{2^{n_j}} M[x(t_i) - x(t_{i-1})] = 2^{n_j} \cdot M\{2^{-n_j}(d-c)\}$$
$$= \frac{(d-c) M\{2^{-n_j}(d-c)\}}{2^{-n_j}(d-c)} \geqslant \frac{\frac{1}{2}(d-c) M\{\frac{1}{2}u_j(d-c)\}}{(\frac{1}{2}u_j(d-c))}$$
$$= \frac{(d-c)}{2} \cdot \frac{M(v_j)}{v_j} \to +\infty \quad \text{as} \quad j \to +\infty,$$

contradicting the hypothesis $x \in B_M$.

COROLLARY. *$B \subset B_M$ iff there is a function $x \in B_M$ such that x is continuous and non-constant on some interval $[\alpha, \beta] \subset [a, b]$.*

The non-trivial part of this corollary immediately follows by application of Theorem 1.

THEOREM 12. *If condition* D *holds for a variation function M, then condition* B1 *holds with respect to M^{th} variation throughout B_M^*.*

Proof. D implies

$$\varliminf_{u\to 0^+} \frac{M(u)}{u} > 0,$$

and hence by Theorem 1, $B_M \subset B$. Now, if

$$\text{(i)} \qquad \varlimsup_{u\to 0^+} \frac{M(u)}{u} = +\infty,$$

then use of Lemma 4 together with Musielak-Orlicz's Theorems 2.22 and 2.23 ([3], p. 59-60), which may be used here because a) $B_M \subset B$ (so that $x \in B_M$ implies that x is of ordinary bounded variation); b) condition D implies our condition B, which is their condition (B) for $\sum_{i=1}^{n} u_i < v_0$, by Theorem 4 (in order to justify applying B instead of (B) it suffices to replace s_x by $v_0 s_x/3V(x)$ in the proof of Theorem 2.22 and to replace $\bar{x}$ by $v_0\bar{x}/2V(\bar{x})$ in the proof of Theorem 2.23, obtaining their original conclusions); and c) the weaker hypothesis

$$\varlimsup_{u\to 0^+} \frac{M(u)}{u} = +\infty$$

can everywhere be used instead of their condition (A) ([3], p. 58), shows that the subset of B_M^* on which condition B1 holds with respect to M^{th} variation coincides with B_M^*. If

$$\text{(ii)} \qquad \varlimsup_{u\to 0^+} \frac{M(u)}{u} < +\infty,$$

then (since also $\varliminf_{u\to 0^+} \frac{M(u)}{u} > 0$) Corollary 1 of Theorem 1 shows that $B_M^* = B_M = B$; by the Remark following the definition of the B1 condition, condition B1 which trivially holds throughout B with respect to ordinary variation also holds with respect to M^{th} variation throughout B_M^*.

The final pair of theorems concerns locally bounded and locally convex topological linear spaces of generalized variation having the Musielak-Orlicz F-norm topology.

THEOREM 13. *The Musielak-Orlicz F-norm topology introduced in B_M^* is a locally bounded linear topology iff condition* B1* *holds in B_M^*.*

Proof $\leftarrow$: Condition B1* means: $(\exists\delta_1 > 0)(\exists a\colon 0 < a < 1)\,|x(t)| < \delta_1$ implies $V_M(ax) \leqslant \frac{1}{2}V_M(x)$. Since M is a variation function, $(\exists\delta_0 > 0)M(u) < \delta_0$ implies $u < \delta_1$. Fix such a δ_0 throughout this proof; then $V_M(x) < \delta_0$ implies $M(|x(t)|) < \delta_0$, which implies $|x(t)| < \delta_1$. Hence, $V_M(x) < \delta_0$ implies $V_M(ax) \leqslant \frac{1}{2}V_M(x) < \frac{1}{2}\delta_0$. By induction,

$$(\forall\varepsilon > 0)\big(\exists n = n(\varepsilon) \geqslant 1\big)\, V_M(a^n x) \leqslant (\tfrac{1}{2})^n V_M(x) < (\tfrac{1}{2})^n \delta_0 < \varepsilon.$$

Fix such an n, and let $\delta = \varepsilon a^n/\delta_0$, so that

$$V_M\left(\frac{\delta\delta_0 x}{\varepsilon}\right) = V_M(a^n x) < \varepsilon.$$

Therefore $V_M(x) < \delta_0$ implies $V_M(\delta\delta_0 x/\varepsilon) < \varepsilon$. Writing $x' = \delta_0 x$, we have

$$(\exists\delta_0 > 0)(\forall\varepsilon > 0)\big(\exists\delta = \delta(\varepsilon) > 0\big)\, V_M\left(\frac{x'}{\delta_0}\right) < \delta_0 \text{ implies } V_M\left(\frac{\delta x'}{\varepsilon}\right) < \varepsilon,$$

which holds iff

$$(\exists\delta_0 > 0)(\forall\varepsilon > 0)\big(\exists\delta = \delta(\varepsilon) > 0\big) \sup_{\|x'\|_M < \delta_0} \|\delta x'\|_M < \varepsilon,$$

and this is true iff $(\exists\delta_0 > 0)\{x'\colon \|x'\|_M < \delta_0\}$ is a bounded neighborhood of zero, i.e., the norm topology (which is linear because B1* implies B1 throughout B_M^*) is locally bounded.

$\rightarrow$: Assume that the norm topology is a locally bounded linear topology and that $A = \{x\colon \|x\|_M < \delta_0\}$ is a bounded neighborhood of zero. Then

$$(\forall\varepsilon > 0)\big(\exists\delta = \delta(\varepsilon) > 0\big)(\forall x \in A)\,\|\delta x\|_M < \varepsilon.$$

Now assume that condition B1* does not hold. By Theorem 10, this means

$$(\forall n \geqslant 1) \varliminf_{u\to 0^+} \frac{M(u)}{M\left(\frac{1}{n}u\right)} = 1 < 2.$$

By Lemma 1, there exists a double sequence $\{u_{i,n}\}$ such that $u_{i,n}\downarrow_i 0$ for each n,

$$\sum_{i=1}^{\infty} M(u_{i,n}) = \frac{\delta_0}{2} \quad \text{for each} \quad n,$$

and

$$\sum_{i=1}^{\infty} M\left(\frac{1}{n}u_{i,n}\right) \geqslant \frac{\delta_0}{4} \quad \text{for each } n.$$

By Lemma 3, for each n there exists a function x_n such that

$$V_M(x_n) = \sum_{i=1}^{\infty} M(u_{i,n}) = \frac{\delta_0}{2} < \delta_0,$$

while

$$\sum_{i=1}^{\infty} M\left(\frac{1}{n} u_{i,n}\right) = V_M\left(\frac{1}{n} x_n\right) \geqslant \frac{\delta_0}{4}.$$

Now, choose ε so that $0 < \varepsilon < \delta_0/4$. Then

$$(\exists \delta = \delta(\varepsilon))(\forall x \in A)\, \|\delta x\|_M < \varepsilon.$$

$V_M(x_n) < \delta_0$ implies $\|\delta_0 x_n\|_M < \delta_0$, so that $(\forall n \geqslant 1)\, \delta_0 x_n \in A$. Then

$$(\forall n \geqslant 1) \|\delta\delta_0 x_n\|_M < \varepsilon, \quad \text{or} \quad V_M\left(\frac{\delta\delta_0 x_n}{\varepsilon}\right) < \varepsilon.$$

Now choose n_0 so large that $1/n_0 < \delta\delta_0/\varepsilon$. Then

$$V_M\left(\frac{1}{n_0} x_{n_0}\right) < \varepsilon < \frac{\delta_0}{4}.$$

But, by the above reasoning, also

$$V_M\left(\frac{1}{n_0} x_{n_0}\right) \geqslant \frac{\delta_0}{4},$$

which is a contradiction. Hence, if the norm topology is a locally bounded linear topology, then condition B1* holds.

THEOREM 14. *The following conditions are necessary and sufficient for the Musielak-Orlicz F-norm topology introduced in B_M^* to be* (i) *a locally convex linear topology*:

(ii) *there exists at least one neighborhood $A = \{x: \|x\|_M < \varepsilon_0\}$ of zero which contains a convex neighborhood of zero*;

(iii) *the topology is a normable linear topology*;

(iv) *there exists a convex function $\hat{M}$ such that $B_M^* = B_{\hat{M}}^*$*;

(v) $(\exists K \geqslant 1)(\exists v_0 > 0)(\exists n \geqslant 1)(\forall 0 < u \leqslant v < v_0) \dfrac{M(u)}{u} \leqslant K \dfrac{M(nv)}{v}$.

Proof. (ii) $\Rightarrow$ (iv). Suppose $A = \{x: \|x\|_M < \varepsilon_0\}$ contains a convex neighborhood of zero; then

$$(\exists \delta_0 > 0) \|x_i\|_M < \delta_0 \quad \text{for} \quad i = 1, 2, \ldots, p$$

implies

$$\left\|\frac{x_1 + x_2 + \ldots + x_p}{p}\right\|_M < \varepsilon_0.$$

This in turn implies

$$(*) \qquad (\forall u_i > 0) \sum_{i=1}^{m} M\left(\frac{u_i}{\delta_0}\right) < \frac{\delta_0}{2} \Rightarrow \sum_{i=1}^{m} n_i M\left(\frac{u_i}{\varepsilon_0 n_i}\right) < \frac{\varepsilon_0}{2}$$

for $n_i \geqslant 1$ arbitrary, as follows. Divide the interval $[a, b]$ into $2\sum_{k=1}^{m} n_k + 1$ equal parts. Define a step function which is zero on every odd part and whose value on the even parts is successively u_1 for the first n_1 even parts $I_{1,1}$ to I_{1,n_1}, u_2 for the next n_2 even parts $I_{2,1}$ to $I_{2,n_2}, \ldots, u_m$ for the last n_m even parts $I_{m,1}$ to I_{m,n_m}.

Now, set

$$y_{\nu_1,\ldots,\nu_m}(t) = \sum_{i=1}^{m} u_i \chi_{I_{i,\nu_i}}(t) \quad \text{for all} \quad t \in [a, b].$$

Since for this step function any refinement of a partition can only increase the M^{th} variation,

$$V_M\left(\frac{y_{\nu_1,\ldots,\nu_m}}{\delta_0}\right) = \sum_{i=1}^{m} 2M\left(\frac{u_i}{\delta_0}\right) < \delta_0;$$

the last inequality is the assumption of (*). It is easy to verify that $(\forall t \in [a, b])$

$$\frac{1}{n_1 \cdot n_2 \cdot \ldots \cdot n_m} \sum_{\substack{1 \leqslant \nu_i \leqslant n_i \\ i=1,2,\ldots,m}} y_{\nu_1,\ldots,\nu_m}(t)$$

$$= \sum_{i=1}^{n_1} \frac{u_1}{n_1} \chi_{I_{1,i}}(t) + \sum_{i=1}^{n_2} \frac{u_2}{n_2} \chi_{I_{2,i}}(t) + \ldots + \sum_{i=1}^{n_m} \frac{u_m}{n_m} \chi_{I_{m,i}}(t).$$

Since $\|y_{\nu_1,\ldots,\nu_m}\|_M < \delta_0$ and the left term in the last equation represents the average of the $y_{\nu_1,\ldots,\nu_m}$, application of the fact that A contains a convex neighborhood of zero yields

$$\left\|\frac{1}{n_1 \cdot n_2 \cdot \ldots \cdot n_m} \sum_{\substack{1 \leqslant \nu_i \leqslant n_i \\ i=1,2,\ldots,m}} y_{\nu_1,\ldots,\nu_m}\right\|_M < \varepsilon_0,$$

so that

$$\sum_{i=1}^{m} 2n_i M\left(\frac{u_i}{n_i \varepsilon_0}\right) = V_M\left(\frac{1}{n_1 \cdot n_2 \cdot \ldots \cdot n_m} \sum_{\substack{1 \leqslant \nu_i \leqslant n_i \\ i=1,2,\ldots,m}} \frac{y_{\nu_1,\ldots,\nu_m}}{\varepsilon_0}\right) < \varepsilon_0.$$

This is the conclusion of (*). Because M is continuous at zero, $(\exists v_0 > 0)\, M(v_0/\delta_0) < \delta_0/2$. Now define

$$(\forall u \geqslant 0)\ \ L(u) = \sup_{0<t\leqslant 1} \frac{M(tu/\varepsilon_0)}{t};$$

L is not necessarily a variation function. Using (*) and the definition of L, we obtain

$$M\left(\frac{u}{\delta_0}\right) < \frac{\delta_0}{2} \Rightarrow L(u) \leqslant \varepsilon_0.$$

Directly from the definition of L, we obtain

$$(\forall 0 < u < v)\frac{L(u)}{u} \leqslant \frac{L(v)}{v}.$$

Now let

$$(\forall u \geqslant 0)\hat{M}(u) = \int_0^u \frac{L(t)}{t}\, dt.$$

$\hat{M}$ is a convex variation function, and

$$(\forall u \geqslant 0) L(u) \geqslant \hat{M}(u) \geqslant L\left(\frac{u}{2}\right) \geqslant M\left(\frac{u}{2\varepsilon_0}\right).$$

Hence $B^*_{\hat{M}} \subset B^*_M$. Conversely,

$$(\forall x \in B^*_M)(\exists a > 0)\, \|ax\|_M < \frac{\delta_0}{2}, \quad \text{or} \quad V_M\left(\frac{2a}{\delta_0}x\right) < \frac{\delta_0}{2}.$$

Therefore, for all partitions $\pi\colon a = t_0 < t_1 < \ldots < t_m = b$,

$$\sum_{i=1}^m M\left[\frac{2a\{x(t_i) - x(t_{i-1})\}}{\delta_0}\right] < \frac{\delta_0}{2}.$$

By (*),

$$(\forall 0 < s_i \leqslant 1) \sum_{i=1}^m s_i^{-1} M\left[\frac{2as_i\{x(t_i) - x(t_{i-1})\}}{\varepsilon_0}\right] < \frac{\varepsilon_0}{2},$$

hence by definition of L,

$$\sum_{i=1}^m L[2a(x(t_i) - x(t_{i-1}))] \leqslant \frac{\varepsilon_0}{2},$$

so that also

$$\sum_{i=1}^m \hat{M}[2a(x(t_i) - x(t_{i-1}))] \leqslant \frac{\varepsilon_0}{2} \quad \text{and} \quad V_{\hat{M}}(2ax) \leqslant \frac{\varepsilon_0}{2}.$$

Hence $x \in B^*_{\hat{M}}$, and thus $B^*_M \subset B^*_{\hat{M}}$.

Note that condition (ii) is analogous to condition b′) of Itô's Theorem 3 ([1], p. 230).

(iv) $\Rightarrow$ (iii). If there exists a convex function $\hat{M}$ such that $B^*_M = B^*_{\hat{M}}$, then define the homogeneous B-norm

$$|||x|||_{\hat{M}} = \inf\left\{a > 0\colon V_{\hat{M}}\left(\frac{x}{a}\right) \leqslant 1\right\}$$

on B^*_M (see [2], p. 32, and also [4], p. 192). We have $\|x\|_M \sim \|x\|_{\hat{M}}$ by Theorem 9, and $\|x\|_{\hat{M}} \sim |||x|||_{\hat{M}}$ by direct calculation using the convexity of $\hat{M}$; hence the topology introduced by $\|\cdot\|_M$ in B^*_M is a normable linear topology.

(iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii) are obvious, and (iv) $\Leftrightarrow$ (v) is Theorem 7.

5. Examples of variation functions.

Example 1. Define $M(u)$ as follows:

$$M(u) = \begin{cases} 1 & \text{for} \quad 1 \leqslant u < +\infty; \\ \dfrac{1}{n!} & \text{for} \quad \dfrac{1}{n!} \leqslant u \leqslant \dfrac{n(n!-1)+1}{(n!)^2}, \\ n!\,u - \dfrac{n(n!-1)}{n!} & \text{for} \quad \dfrac{n(n!-1)+1}{(n!)^2} \leqslant u \leqslant \dfrac{n}{n!}, \qquad n = 2, 3, \ldots, \end{cases}$$

and $M(-u) = M(u)$ for $u < 0$. It is clear that

$$\overline{\lim_{u\to 0^+}} \frac{M(u)}{u} = 1 \quad \text{and} \quad \underline{\lim_{u\to 0^+}} \frac{M(u)}{u} = 0.$$

This function does not satisfy conditions B1* or D but does satisfy condition B1; no variation function which does not satisfy condition B1 is known. By Theorems 4, 10 and 11, conditions B, C, and I do not hold for M either.

Example 2. Consider $M(u) = e^{-1/u}$ and $N(u) = -1/\ln u$ $(u \geqslant 0)$, and $M(-u) = M(u)$, $N(-u) = N(u)$. M is convex but tends to zero faster than any power p of u, since

$$\lim_{u\to 0^+} \frac{M(u)}{u^p} = 0;$$

similarly, N is concave but tends to zero slower than any power p of u, because

$$\lim_{u\to 0^+} \frac{u^p}{N(u)} = 0.$$

For $u \geqslant 0$, $M^{-1} = N$, yet M satisfies the B1* condition but N does not. N satisfies D, but M does not; and M satisfies I, but N does not.

EXAMPLE 3. Let $M(u) = |u|^{1/2}$. M satisfies both conditions C and D, but not condition I.

References

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Sur des théorèmes de S. Banach et de L. Schwartz concernant le graphe fermé

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Il s'agit ici d'une généralisation du théorème du graphe fermé dans la direction suggérée par A. Grothendieck dans sa thèse [6].

La première solution à ce problème a été fournie par W. Słowikowski [14] suivi de D. A. Raikov [11] une autre solution a été donnée par Laurent Schwartz [13] qui s'appuie sur sa théorie de l'intégration et un lemme de A. Douady, mais qui ne recouvre pas exactement la conjecture de Grothendieck. Le premier pas dans le sens de la conjecture de Grothendieck a été fait par M. Słowikowski, puis M. Raikov a donné une solution complète. L'énoncé de Schwartz est particulièrement suggestif et ma principale contribution [9] a été d'en fournir une nouvelle démonstration puisée dans l'ouvrage de Banach [1]. Je donne ici la plus grande extension possible à cette méthode.

1. Remarques sur la théorie de la catégorie. Dans la suite, sauf mention expresse du contraire, tous les espaces que je considère sont séparés (terminologie Bourbaki). Je suis généralement la terminologie et les notations de cet auteur.

Une partie Y d'un espace topologique X est dite *rare* si elle est incluse dans un fermé sans point intérieur; elle est *maigre* si elle est réunion dénombrable de parties rares (1-ère catégorie chez Baire et chez les Polonais). L'espace X est dit *non maigre* s'il n'est pas un sous-ensemble maigre de lui-même. L'espace X est dit *espace de Baire* si tout ouvert non vide de X est non maigre.

THÉORÈME 1. *Soit Y une partie de X. On désigne par $D(Y)$ l'ensemble des points x de X tels que pour tout voisinage V de x l'ensemble $V \cap Y$ soit non maigre. On a*:

(α) $D(Y) \subset \overline{Y}$.

(β) *Si* $Y_1 \subset Y_2$, $D(Y_1) \subset D(Y_2)$.

(γ) $D(Y)$ *est fermé.*

(δ) *Pour que* $O(Y) = \overset{\circ}{\widehat{D(Y)}} = \emptyset$ *il faut et il suffit que* Y *soit un ensemble maigre.*