Then there is a relatively dense set of integers $S$ so that $p \in S$ implies each summand in the finite sum is less than $c/2k$. Thus $|x_{n+1} - x_n| < c/k + c/2 = \varepsilon$ and $(x_n)$ is $A-\varepsilon$.

Among $U$-sequences, then, the almost periodic ones are the ones whose basis components are $A-\varepsilon$-scalar sequences, and $U'(x_n) = E\{L(x_n, e_0)\} e_0$, where $L(x_n, e_0)$ is the mean value of the sequence.

An immediate corollary to the above theorem is that a $U$-sequence is $A$-periodic if and only if $(x_n, y_n)$ is $A$-periodic for each $y \in H$. Unfortunately, a complete analog of Theorem 3.2.1 cannot be proved. That is, we cannot drop the hypothesis that the sequence be in $U$ already. Let $x_n = e_0$, whenever $k \equiv 2^n - 1$ (mod $2^{n+1}$), $n = 0, 1, 2, \ldots$ This sequence has component sequences $a_n = (x_n, e_0)$, each with period $2^{n+1}$. The vector sequence is not $A$ periodic, simply because it is not in $U$. It is also easy to see, that for each fixed $y_n (x_n, y_n)$ will be $A$-periodic. Hence, neither Theorem 3.2.4 nor its corollary will be true if we drop the assumption that the sequence has range contained in a compact set.

References


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Uniformly convex and reflexive modulated variation spaces
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§ 1. Introduction. In chapter 11, entitled “Modular Spaces”, of his treatise [3], Professor Hideo Nakano presents a theory of modules on arbitrary (not necessarily semi-ordered) linear spaces. Namely, given any linear space $X$, a functional $m(x)$ defined on $X$ with values $0 \leq m(x) \leq +\infty$ is called a Nakano modular if

M. 1. $m(0) = 0$,
M. 2. $(\forall y \in X) m(-x) = m(x),$
M. 3. $(\forall y \in X) (\exists \lambda > 0) m(\lambda x) < +\infty,$
M. 4. $m(tx) = 0$ for all $t > 0 \Rightarrow x = 0$,
M. 5. $(\forall y \in X) (\forall \alpha, \beta > 0) \alpha m(x + \beta y) \leq \alpha m(x) + \beta m(y),$
M. 6. $(\forall y \in X) m(x) = \sup_{t \in \mathbb{R}} m(tx),$

The space $X$ associated with the functional $m(x)$ is called a Nakano modular space.

It is easy to see that, for example, the $p^\text{th}$ power variation (as basic papers, see [4] or [2]) are special cases of Nakano modules on generalized variation spaces. In this paper we are concerned with a new class of spaces which include the $p^\text{th}$ power variation spaces. Let $x$ be a real function in $[a, b]$ such that $x(a) = 0$, let $p(t, s)$ be a real function of two real variables such that $t, s \in [a, b], t > s$, and $1 \leq p(t, s) < +\infty$; let $n: a = a_0 < a_1 < \ldots < a_n = b$ be a partition of $[a, b]$. Define

$$B_{p(t, s)} = \{x: V_{p(t, s)}(x) = \sup_n \sum_{k \in n} |x(t_k) - x(t_{k-1})|^p(x)(t_k - t_{k-1}) < +\infty\},$$

and denote by $B_{p(t, s)}^\alpha$ the linear space generated by $B_{p(t, s)}$. Here, $V_{p(t, s)}$ is the Nakano modular on the space $B_{p(t, s)}$. If $p(t, s) = p = \text{constant}$ (1 \leq p < +\infty), we have the case of $p^\text{th}$ variation. The spaces $B_{p(t, s)}$ generalize the idea of $p^\text{th}$ variation in the same way as Nakano's $L_p$-spaces (see [3], p. 234-240). In fact, the methods, employed in the present paper, although they are perhaps not widely known, are essentially due to Nakano.
§ 2. Basic results from the Nakano theory. Here we review as much of the theory of Nakano modular spaces as is necessary for comprehension of the exposition in § 3. Proofs of the theorems here quoted and many additional concepts can be found in chapter 11 of [3]. However, Professor Nakano does not discuss explicitly the connection between Banach space and modular space concepts.

From M 2 and M 5 we obtain

\[ m\left(\sum_{n=1}^{\infty} a_n x_n\right) \leq \sum_{n=1}^{\infty} |a_n| m(x_n), \quad \text{provided} \quad \sum_{n=1}^{\infty} |a_n| < 1. \]

**Definition 1.** A sequence \( (a_n) \subset X (n = 1, 2, \ldots) \) is modular convergent to a limit \( a \in X \) (written \( m - \lim n a_n = a \)) if \( \lim_{n \to \infty} \{ (a_n - a) Ù \xi \} = 0 \) for every \( \xi \geq 0. \)

**Definition 2.** If every sequence \( (a_n) \subset X (n = 1, 2, \ldots) \) subject to the condition

\[ \lim_{n \to \infty} m(\lambda a_n - a) = 0 \quad \text{for every} \quad \lambda > 0 \]

is modular convergent, then \( X \) is modular complete.

**Definition 3.** A real valued linear functional \( \varphi \) on \( X \) is modular bounded if \( \sup_{x \in X} |\varphi(x)| < \infty. \)

Let \( X \) be a modularized space and \( X \) the set of modular bounded linear functionals on \( X \). Clearly, \( X \) is a linear space; if we set

\[ (\forall x \in X) \| \varphi(x) \| = \sup_{x \in X} |\varphi(x)|, \]

then we can show that \( \bar{m} \) satisfies the conditions M 1-M 6, so that \( \bar{m} \) is by definition a modular on \( X \).

**Definition 4.** \( \bar{m} \) called the adjoint modular of \( m \) and the linear space \( \bar{X} \) is called the modular adjoint space of \( X \).

For \( \bar{m} \) we have obviously by (2)

\[ (\forall x \in \bar{X})(\forall \varphi \in X) \| \varphi(x) \| \leq \bar{m}(\bar{\varphi}) + m(x). \]

**Theorem 1.** For every \( x \in X \) we have \( m(x) = \sup_{\bar{X}} |\varphi(x) - \bar{\varphi}(x)|. \)

**Theorem 2.** For a subset \( A \) of the modular adjoint space \( X \) of \( X \), if \( \lim x(a_n) = \bar{x}(a) \) for every \( x \in A \) and \( m(a) = \sup \{ (a_n - a) Ù \xi \} \), then \( m(a) = \lim_{n \to \infty} m(a_n). \)

Given a modularized space \( X \), every linear subset \( A \) of \( X \) can be considered as a modularized space associated with the same modular of \( X \). In this sense, \( A \) is termed a subspace of \( X \). Since the modular adjoint space \( \bar{X} \) is also a modular space (with the adjoint modular \( \bar{m} \)), we can also consider the modular adjoint space \( \bar{X} \) of \( X \) with the adjoint modular \( \bar{m} \) of \( m \). Because of Theorem 1, \( X \) may then be considered as a subspace of \( \bar{X} \) under the convention

\[ (\forall x \in X)(\forall \varphi \in \bar{X}) \| \varphi(x) \| = \bar{m}(\varphi). \]

**Definition 5.** If \( X \) coincides with \( \bar{X} \) under the above convention, then both \( X \) and the modular \( m \) of \( X \) are called reflexive. (Nakano uses the term "regular" instead of "reflexive".)

In the modular space \( X \), for every \( \lambda > 0 \), putting

\[ U_\lambda = \{ x : m(x) \leq \lambda \}, \]

we obtain a unique linear topology \( \mathcal{J}^m \) on \( X \) such that \( U_\lambda \) is a basis of \( \mathcal{J}^m \).

This linear topology is the same for every \( \lambda > 0 \).

**Definition 6.** \( \mathcal{J}^m \) is called the modular topology of \( X \).

Since \( U_\lambda \) is symmetric and convex for every \( \lambda > 0 \), each \( U_\lambda, \lambda > 0 \), is a basis of \( \mathcal{J}^m \), and \( \mathcal{J}^m \) is of single vicinity, locally convex and separated, it follows that the pseudo-norm of \( U_\lambda \) is a Banach norm on \( X \) and that the modular topology of \( X \) coincides with the norm topology by this norm.

**Definition 7.** The pseudo-norm of the 1-sphere \( U_1 \) of \( X \) is called the modular norm of \( X \) and is denoted by \( \| x \|_m \) for all \( x \in X \).

It is evident that a linear functional \( \varphi \) on \( X \) is modular bounded iff \( \varphi \) is bounded by the modular norm. Hence

**Theorem 3.** The modular adjoint space of \( X \) coincides with the adjoint space of \( X \) by the modular (Banach) norm.

This very important theorem means that the modular dual of \( \mathcal{B}^m \) coincides with the usual Banach dual of \( \mathcal{B}^m \) by the modular norm.

By investigating modulars on quotient spaces of \( X \) we can prove

**Theorem 4.** For a finite number of elements \( \bar{n}, x \in X \) and real numbers \( a_n (n = 1, 2, \ldots, n) \) if

\[ \sum_{n=1}^{m} \bar{\xi}_n \bar{a}_n = 0 \]

implies

\[ \sum_{n=1}^{m} \xi_n a_n = 0, \]

and (for \( \gamma > 0 \)) if

\[ \sum_{n=1}^{m} \xi_n a_n \leq \gamma + \bar{m}(\sum_{n=1}^{m} \xi_n \bar{a}_n) \]
for every finite number of real numbers \( \xi, (\xi = 1, 2, \ldots, k) \), then for any \( 0 < \epsilon < 1 \) there exists an \( x \in X \) such that

\[
m\{1 - \epsilon\}|x| < \gamma, \quad s_i(x) = a_i (i = 1, 2, \ldots, n),
\]

DEFINITION 8. If \( X \) is a modular space and \( (\forall x \in X) m(x) = 0 \Rightarrow x = 0 \), then \( X \) is called simple. \( X \) is called uniformly simple if

\[
(\forall x > 0) \inf_{m \geq 0} m(\xi x) > 0.
\]

Every space \( B_{\text{mod}}^{\mathcal{R}} \) is uniformly simple.

DEFINITION 9. If \( X \) is simple, a sequence \( \{a_n\} = X (r = 1, 2, \ldots) \) is conditionally modular convergent to a limit \( a \in X \) if \( (\exists a > 0) \lim_{r \to \infty} m(a(a_n - a)) = 0 \).

THEOREM 5. \( X \) is uniformly simple iff conditionally modular convergence coincides with modular convergence.

DEFINITION 10. A modular space \( X \) and its modular \( m \) are uniformly convex if \( (\forall \epsilon > 0, \forall \delta > 0) \forall m(x) \leq \gamma, m(y) \leq \gamma \) and \( m(x - y) \geq \epsilon \Rightarrow \frac{1}{2}(m(x) + m(y)) \leq m\left(\frac{1}{2}(x + y)\right) + \delta \).

This definition provides a modular parallel to the concept of uniform convexity in Banach spaces first considered by Professor James A. Clarkson [1].

§ 3. Main theorems.

THEOREM 6. Every space \( B_{\text{mod}}^{\mathcal{R}} \) is modular complete.

Proof. If

\[
\lim_{\nu \to \infty} V_{\nu}(\xi(a_n - a)) = 0 \quad \text{for every } \xi > 0,
\]

then there exists a subsequence \( a_{\nu_\xi}(\nu = 1, 2, \ldots) \) of \( a_n (r = 1, 2, \ldots) \) such that \( V_{\nu}(\xi(a_{\nu_\xi} - a)) = 0 \). Then we can assert, by (1), that

\[
\sum_{\nu = 1}^{\infty} \frac{1}{2^\nu} \cdot 2^\nu(a_{\nu_\xi} - a) \leq \sum_{\nu = 1}^{\infty} \frac{1}{2^\nu} \cdot V_{\nu}(\xi(a_{\nu_\xi} - a))
\]

\[
\leq \xi < 1 \quad \text{for } \nu = 1, 2, \ldots
\]

Hence \( \sum_{\nu = 1}^{\infty} (a_{\nu_\xi} - a) \) is convergent to a function \( y \) such that \( y(a_n) = 0 \) and generally

\[
y(r) = \sum_{\nu = 1}^{\infty} (a_{\nu_\xi}(r) - a(r)) \quad \text{for all } r \in [a, b].
\]

Setting

\[
x(r) = z(r) + \sum_{\nu = 1}^{\infty} (a_{\nu}(r) - a(r))
\]

we have \( V_{\nu}(\xi(a_n - a)) \leq 1 \) and hence \( \xi \in B_{\text{mod}}^{\mathcal{R}} \). Since \( V_{\nu}(\xi(a_{\nu} - a)) \leq 1 \), we infer, again by (1), that

\[
V_{\nu}(\xi) \leq \sum_{\nu = 1}^{\infty} \frac{1}{2^\nu} \cdot 2^\nu(a_{\nu} - a) \leq \sum_{\nu = 1}^{\infty} \frac{1}{2^\nu} \leq \frac{1}{2^\nu - 1} \quad \text{for } \nu = 1, 2, \ldots
\]

Hence we have

\[
V_{\nu}(\xi) \leq \frac{1}{2^\nu - 1} \quad \text{for every } \nu = 1, 2, \ldots
\]

and thus

\[
V_{\nu}(\xi) - \lim_{\nu \to \infty} \xi = 0.
\]

Given \( \epsilon > 0 \) and \( \xi > 0 \), because

\[
\lim_{\nu \to \infty} V_{\nu}(\xi(a_n - a)) = 0 \quad \text{and} \quad V_{\nu}(\xi - \lim_{\nu \to \infty} \xi) = 0
\]

we can find \( a_\nu \) and \( z \), such that \( V_{\nu}(\xi(a_n - a)) < \epsilon \) and \( V_{\nu}(\xi(a_n - a)) < \epsilon \); hence

\[
V_{\nu}(\xi(z - a)) = V_{\nu}(\xi(a_n - a)) + V_{\nu}(\xi(z - z))
\]

\[
\leq \frac{1}{2} \cdot V_{\nu}(\xi(a_n - a)) + \frac{1}{2} \cdot V_{\nu}(\xi(z - z)) < \epsilon.
\]

This shows that \( V_{\nu}(\xi) - \lim_{\nu \to \infty} \xi = 0 \), i.e., \( B_{\text{mod}}^{\mathcal{R}} \) is modular complete.

THEOREM 7. If

\[
1 < p = \inf_{t > 0} p(t, s) \leq \sup_{t > 0} p(t, s) = p_0 < +\infty,
\]

then \( B_{\text{mod}}^{\mathcal{R}} \) is uniformly convex.

Proof. Let \( x \) be the characteristic function of \( \{(t, s) ; p(t, s) > 2\} \) and \( X \) that of \( \{(t, s) ; p(t, s) < 2\} \). Hence \( (\forall x \in B_{\text{mod}}^{\mathcal{R}}) V_{\nu}(x) + V_{\nu}(X) = V_{\nu}(a) \). Given real numbers \( y \geq 0, z > 0 \), assume \( V_{\nu}(y) < 2 \), \( V_{\nu}(x - z) \geq \epsilon \). Then either

(i) \( V_{\nu}(y - z) \geq 2^\nu \) or

(ii) \( V_{\nu}(a) \geq 2^\nu \).

If (i) holds, then it follows from the inequality (for a proof, see [3], p. 275)

\[
(\forall p \geq 2) \frac{|\xi + \eta|^p}{2} \geq \frac{\xi + \eta}{2} + \frac{\xi - \eta}{2} + \epsilon.
\]
that
\[ \frac{1}{2} \langle V_{\nu, \delta}(c \cdot d) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{2} \nu. \]

Since
\[ \rho_\nu = \sup_{t \geq 0} \{ p(t, s) \} < + \infty, \]
we have
\[ V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{2} \nu. \]

Furthermore, by M.5,
\[ \frac{1}{2} \langle V_{\nu, \delta}(c \cdot d) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{2} \nu. \]

Hence we obtain
\[ \frac{1}{2} \langle V_{\nu, \delta}(c \cdot d) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{2} \nu. \]

If \( (ii) \) holds, set
\[ \varepsilon' = \min \left\{ \frac{\varepsilon}{2}, \frac{1}{4} \right\} \]
and denote by \( \chi \) the characteristic function of
\[ \{ (t, s) : p(t, s) < 2 \} \]
and \( \{ (t, s) : p(t, s) < 2 \} \) then we have
\[ V_{\nu, \delta}(\chi \cdot d) \Rightarrow V_{\nu, \delta}(\chi \cdot d) \Rightarrow V_{\nu, \delta}(\chi \cdot d) \cdot d > \frac{\varepsilon}{4}, \]
and therefore
\[ V_{\nu, \delta}(\chi \cdot d) \cdot d > \frac{\varepsilon}{4}. \]

If we set
\[ \varepsilon = p_\nu - 1 = \inf_{t > 0} \{ p(t, s) \} - 1, \]
it follows from the inequality (for a proof, see \[33\] p. 275-276)
\[ (\forall p : 1 < p < 2) \frac{|A|^p + |B|^p}{2} \geq \frac{\varepsilon + \eta}{2} p + \frac{\varepsilon + \eta}{2} \left( \frac{|A|}{|B|} + \frac{|B|}{|A|} \right), \]
that
\[ \frac{1}{2} \langle V_{\nu, \delta}(c \cdot d) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{2} \nu. \]

On the other hand, we obtain
\[ V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{16}, \]
and, by M.5,
\[ \frac{1}{2} \langle V_{\nu, \delta}(c \cdot d) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c \cdot d) \cdot d) \cdot d > \frac{\varepsilon}{16}. \]

Hence
\[ \frac{1}{2} \langle V_{\nu, \delta}(c) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c + d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c + d) \cdot d) \cdot d > \frac{\varepsilon}{32}, \]
and consequently
\[ \frac{1}{2} \langle V_{\nu, \delta}(c) \rangle \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c + d) \cdot d) \Rightarrow V_{\nu, \delta}(\frac{1}{2} (c + d) \cdot d) + \frac{\varepsilon}{32}. \]

in both cases (i) and (ii), hence \( \mathcal{K}_{\nu, \delta} \) is uniformly convex.

**Theorem 8.** If a Nakaoru modular adjoint space \( \mathcal{X} \) is at once uniformly simple, modular complete and uniformly convex, then \( \mathcal{X} \) is reflexive.

Proof. Let \( \mathcal{X} \) be the modular adjoint space of the modular adjoint space \( \mathcal{X} \). For any \( \varepsilon, \mathcal{X} \) satisfying \( \mu(\varepsilon) < + \infty \), using (2) we can find a sequence \( \langle \xi_n \rangle \subset \mathcal{X}(n = 1, 2, \ldots) \) such that
\[ \sup_{n} \{ \xi_n \} < \mu(\varepsilon) + \mu(\varepsilon) - \frac{\varepsilon}{4}. \]

Since we get by (3) that
\[ \sum_{n} \xi_n \mathcal{E}(\xi_n) \leq \mu(\varepsilon) + \mu(\sum_{n} \xi_n \mathcal{E}(\xi_n) \]
for any finite set of real numbers \( \xi_n (n = 1, 2, \ldots, n) \), we can find by Theorem 4 a sequence \( \xi_n \subset \mathcal{X}(n = 1, 2, \ldots) \) such that \( \mathcal{E}(\xi_n) = \mathcal{E}(\xi_n) \) for every \( n = 1, 2, \ldots \), and such that
\[ m \left( \frac{1}{n-1} \right) \mathcal{E}(\xi_n) \leq \mu(\varepsilon) \]
for every \( n = 1, 2, \ldots \) (3)

For such \( \xi_n (n = 1, 2, \ldots) \) we obtain
\[ \lim_{n \to \infty} \left( 1 - \frac{1}{n} \right) \mathcal{E}(\xi_n) - \left( 1 - \frac{1}{n} \right) \mathcal{E}(\xi_n) = 0. \]

Otherwise, if we could find \( \varepsilon > 0 \) and two subsequences \( \lambda_n (n = 1, 2, \ldots) \) of \( 1, 2, \ldots \) such that
\[ (\forall n = 1, 2, \ldots) m \left( 1 - \frac{1}{n} \right) \mathcal{E}(\xi_n) - \left( 1 - \frac{1}{n} \right) \mathcal{E}(\xi_n) \geq \varepsilon. \]
then, since $X$ is uniformly convex,
\[
\left(\exists \delta > 0\right) \left\{ \left(1 - \frac{1}{\lambda'}\right) s_n + m \left(\left(1 - \frac{1}{\mu'}\right) s_n\right) \right\} \geq m \left\{ \left(1 - \frac{1}{\lambda'}\right) s_n + \left(1 - \frac{1}{\mu'}\right) s_n\right\} + \delta
\]
for every $\nu = 1, 2, \ldots$. On the other hand,
\[
\frac{1}{\nu} \left\{ \left(1 - \frac{1}{\lambda'}\right) s_n + m \left(\left(1 - \frac{1}{\mu'}\right) s_n\right) \right\} \leq \frac{m}{\nu} (\tilde{\nu})
\]
and by (2) for $\varphi \leq \lambda', \mu'$,
\[
m \left\{ \left(1 - \frac{1}{\lambda'}\right) s_n + \left(1 - \frac{1}{\mu'}\right) s_n\right\} \geq m \left\{ \left(1 - \frac{1}{\lambda'}\right) s_n + \left(1 - \frac{1}{\mu'}\right) s_n\right\} - \frac{m}{\nu} (\tilde{\nu})\]
\[
= \frac{1}{2 \lambda'} - \frac{1}{2 \mu'} \frac{m}{\nu} (\tilde{\nu}) - m(\tilde{\nu}).
\]
Hence we obtain for such $\delta$
\[
\frac{m}{\nu} (\tilde{\nu}) \geq \frac{1}{2 \lambda'} - \frac{1}{2 \mu'} \frac{m}{\nu} (\tilde{\nu}) - m(\tilde{\nu}) - \frac{1}{\nu} + \delta
\]
for every $\nu = 1, 2, \ldots$, contradicting $\delta > 0$. Since $X$ is uniformly simple and modular complete by assumption, by Theorem 5 there exists $z \in X$ such that
\[
m - \lim_{v \to \infty} \frac{1}{v} s_n = z,
\]
and thus we have by Theorems 3 and 2,
\[
m(z) = \lim_{v \to \infty} m \left(\left(1 - \frac{1}{v}\right) s_n\right) \leq m(\tilde{\nu}),
\]
and $\tilde{\nu}(z) = \tilde{\nu}(\tilde{\nu})$ for every $\nu = 1, 2, \ldots$. For an arbitrary $z \in X$, the same process can be applied to $\tilde{\nu}, \tilde{\nu}, \tilde{\nu}, \ldots$ instead of $\tilde{\nu}, \tilde{\nu}, \ldots$, and then we obtain similarly $z \in X$ such that $m(z) \leq m(\tilde{\nu}), \tilde{\nu}(z) = \tilde{\nu}(\tilde{\nu})$ for every $\nu = 1, 2, \ldots$. For such $z$, if $m(z - x) > 0$, then, since $X$ is uniformly convex, we can find $\delta > 0$ such that
\[
\frac{1}{\nu} \left\{ m(z) + m(x) \right\} \geq m \left(\frac{1}{2} (z + x)\right) + \delta.
\]
and then, by (3), for every $\nu = 1, 2, \ldots$,
\[
\frac{m}{\nu} (\tilde{\nu}) \geq \frac{1}{2} \frac{m}{\nu} (z + x) + \frac{1}{\nu} + \delta \geq \frac{m}{\nu} (z + x) + \delta
\]
\[
= \tilde{\nu}(x) - \tilde{\nu}(\tilde{\nu}) + \delta \geq \frac{m}{\nu} (\tilde{\nu}) - \frac{1}{\nu} + \delta,
\]
contradicting $\delta > 0$. Hence $m(z - x) = 0$ and therefore $z = x$, because $X$ is uniformly simple. It follows that $\tilde{\nu}(z) = \tilde{\nu}(x) = \tilde{\nu}(\tilde{\nu})$. Since $\tilde{\nu}(X)$ is arbitrary, we have $(\forall \tilde{\nu}(X)) \tilde{\nu}(z) = \tilde{\nu}(\tilde{\nu})$, i.e., $X$ is reflexive by Definition 5.

**Theorem 9.** $B_{\alpha_0}(\tilde{\nu})$ with $p(t, x)$ restricted as in Theorem 7 is reflexive as a Nakano modular space and as a Banach space.

Proof. Because $B_{\alpha_0}(\tilde{\nu})$ is a uniformly simple Nakano modular space, it follows by Theorems 6, 7, and 8 that $B_{\alpha_0}(\tilde{\nu})$ is reflexive in the sense of Definition 5. In view of Theorem 3, we finally obtain that $B_{\alpha_0}(\tilde{\nu})$ is reflexive in the usual Banach space sense.

**References**


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