Then there is a relatively dense set of integers S so that $p \in S$ implies each summand in the finite sum is less than $\varepsilon/2k$. Thus $\|x_{n+p}-x_n\|^2 \leqslant k(\varepsilon/2k)+\varepsilon/2 = \varepsilon$ and (x_n) is AP.

Among U-sequences, then, the almost periodic ones are the ones whose basis components are AP-scalar sequences, and $L^0(x_n) = \Sigma_i[L(x_n, e_i)]e_i$, where $L(x_n, e_i)$ is the mean value of the sequence.

An immediate corollary to the above theorem is that a U-sequence is AP if and only if (x_n, y) is AP for each $y \in H$. Unfortunately, a complete analog of Theorem 3.1.1 cannot be proved. That is, we cannot drop the hypothesis that the sequence be in U already. Let $x_n = e_n$ whenever $k \equiv 2^n-1 \pmod{2^{n+1}}$, $n = 0, 1, 2, \ldots$ This sequence has component sequences $a_k = (x_k, e_n)$, each with period 2^{n+1}. The vector sequence is not AP however, simply because it is not in U. It is also easy to see, that for each fixed y, (x_k, y) will be AP also. Hence, neither Theorem 3.2.4 nor its corollary will be true if we drop the assumption that the sequence has range contained in a compact set.

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Uniformly convex and reflexive modulared variation spaces

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§ 1. Introduction. In chapter 11, entitled "Modular Spaces", of his treatise [3], Professor Hidegorô Nakano presents a theory of modulars on arbitrary (not necessarily semi-ordered) linear spaces. Namely, given any linear space X, a functional $m(x)$ defined on X with values $0 \leqslant m(x) \leqslant +\infty$ is called a *Nakano modular* if

M. 1. $m(0) = 0$,

M. 2. $(\forall x \in X)\, m(-x) = m(x)$,

M. 3. $(\forall x \in X)(\exists \lambda > 0)\, m(\lambda x) < +\infty$,

M. 4. $m(\xi x) = 0$ for all $\xi > 0 \Rightarrow x = 0$,

M. 5. $(\forall x, y \in X)(\forall \alpha, \beta \geqslant 0)\ \alpha+\beta = 1 \Rightarrow m(\alpha x+\beta y) \leqslant \alpha m(x)+\beta m(y)$,

M. 6. $(\forall x \in X)\ m(x) = \sup_{0 \leqslant \xi < 1} m(\xi x)$.

The space X associated with the functional $m(x)$ is called a *Nakano modulared space*.

It is easy to see that, for example, the p^{th} power variations (as basic papers, see [4] or [2]) are special cases of Nakano modulars on generalized variation spaces. In this paper we are concerned with a new class of spaces which include the p^{th} power variation spaces. Let x be a real function in $[a, b]$ such that $x(a) = 0$, let $p(t, s)$ be a real function of two real variables such that $t, s \in [a, b]$, $t > s$, and $1 \leqslant p(t, s) < +\infty$; let π: $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a, b]$. Define

$$B_{p(t,s)} = \Big\{x\colon V_{p(t,s)}(x) = \sup_{\pi} \sum_{i=1}^{n} |x(t_i)-x(t_{i-1})|^{p(t_i,t_{i-1})} < +\infty\Big\},$$

and denote by $B^*_{p(t,s)}$ the linear space generated by $B_{p(t,s)}$. Here, $V_{p(t,s)}$ is the Nakano modular on the space $B^*_{p(t,s)}$. If $p(t, s) \equiv p =$ constant $(1 \leqslant p < +\infty)$, we have the case of p^{th} variation. The spaces $B^*_{p(t,s)}$ generalize the idea of p^{th} variation in the same way as Nakano's $L_{p(t)}$-spaces generalize the classical L_p-spaces (see [3], p. 234-240). In fact, the methods employed in the present paper, although they are perhaps not widely known, are essentially due to Nakano.

§ 2. Basic results from the Nakano theory. Here we review as much of the theory of Nakano modulared spaces as is necessary for comprehension of the exposition in § 3. Proofs of the theorems here quoted and many additional concepts can be found in chapter 11 of [3]. However, Professor Nakano does not discuss explicitly the connection between Banach space and modular space concepts.

From M. 2 and M. 5 we obtain

(1) $$m\Big(\sum_{\nu=1}^{\varkappa} a_\nu x_\nu\Big) \leqslant \sum_{\nu=1}^{\varkappa} |a_\nu| m(x_\nu), \quad \text{provided } \sum_{\nu=1}^{\varkappa} |a_\nu| \leqslant 1.$$

DEFINITION 1. A sequence $\{a_\nu\} \subset X (\nu = 1, 2, \ldots)$ is *modular convergent to a limit* $a \in X$ (written $m-\lim_{\nu\to\infty} a_\nu = a$) if $\lim_{\nu\to\infty} m\{\xi(a_\nu - a)\} = 0$ for every $\xi \geqslant 0$.

DEFINITION 2. If every sequence $\{a_\nu\} \subset X (\nu = 1, 2, \ldots)$ subject to the condition

$$\lim_{\mu,\nu\to\infty} m\{\lambda(a_\mu - a_\nu)\} = 0 \quad \text{for every } \lambda \geqslant 0$$

is modular convergent, then X is *modular complete*.

DEFINITION 3. A real valued linear functional φ on X is *modular bounded* if $\sup_{m(x)\leqslant 1} |\varphi(x)| < +\infty$.

Let X be a modulared space and $\overline{X}$ the set of modular bounded linear functionals on X. Clearly, $\overline{X}$ is a linear space; if we set

(2) $$(\forall \bar{x} \in \overline{X})\ \overline{m}(\bar{x}) = \sup_{x\in X}\{\bar{x}(x) - m(x)\},$$

then we can show that $\overline{m}$ satisfies the conditions M. 1-M. 6, so that $\overline{m}$ is by definition a modular on $\overline{X}$.

DEFINITION 4. $\overline{m}$ is called the *adjoint modular* of m and the linear space $\overline{X}$ is called the *modular adjoint space* of X.

For $\overline{m}$ we have obviously by (2)

(3) $$(\forall \bar{x} \in \overline{X})(\forall x \in X)\, |\bar{x}(x)| \leqslant \overline{m}(\bar{x}) + m(x).$$

THEOREM 1. *For every* $x \in X$ *we have* $m(x) = \sup_{\bar{x}\in\overline{X}}\{\bar{x}(x) - \overline{m}(\bar{x})\}$.

THEOREM 2. *For a subset* $\overline{A}$ *of the modular adjoint space* $\overline{X}$ *of* X, *if* $\lim_{\nu\to\infty} \bar{x}(a_\nu) = \bar{x}(a)$ *for every* $\bar{x} \in \overline{A}$ *and* $m(a) = \sup_{\bar{x}\in\overline{A}}\{\bar{x}(a) - \overline{m}(\bar{x})\}$, *then* $m(a) \leqslant \varliminf_{\nu\to\infty} m(a_\nu)$.

Given a modulared space X, every linear subset A of X can be considered as a modulared space associated with the same modular of X. In this sense, A is termed a *subspace* of X. Since the modular adjoint space $\overline{X}$ of X is also a modulared space (with the adjoint modular $\overline{m}$), we can also consider the modular adjoint space $\overline{\overline{X}}$ of $\overline{X}$ with the adjoint modular $\overline{\overline{m}}$ of $\overline{m}$. Because of Theorem 1, X may then be considered as a subspace of $\overline{\overline{X}}$ under the convention

$$(\forall x \in X)(\forall \bar{x} \in \overline{X})\ x(\bar{x}) = \bar{x}(x).$$

DEFINITION 5. If X coincides with $\overline{\overline{X}}$ under the above convention, then both X and the modular m of X are called *reflexive*. (Nakano uses the term "regular" instead of "reflexive".)

In the modulared space X, for every $\lambda > 0$, putting

$$U_\lambda = \{x: m(x) \leqslant \lambda\},$$

we obtain a unique linear topology $\mathscr{I}^m$ on X such that U_λ is a basis of $\mathscr{I}^m$. This linear topology is the same for every $\lambda > 0$.

DEFINITION 6. $\mathscr{I}^m$ is called the *modular topology* of X.

Since U_λ is symmetric and convex for every $\lambda > 0$, each U_λ, $\lambda > 0$, is a basis of $\mathscr{I}^m$, and $\mathscr{I}^m$ is of single vicinity, locally convex and separated, it follows that the pseudo-norm of U_λ is a Banach norm on X and that the modular topology of X coincides with the norm topology by this norm.

DEFINITION 7. The pseudo-norm of the 1-sphere U_1 of X is called the *modular norm* of X and is denoted by $|||x|||$ for all $x \in X$.

It is evident that a linear functional φ on X is modular bounded iff φ is bounded by the modular norm. Hence

THEOREM 3. *The modular adjoint space of* X *coincides with the adjoint space of* X *by the modular (Banach) norm.*

This very important theorem means that the modular dual of $B^*_{p(t,s)}$ coincides with the usual Banach dual of $B^*_{p(t,s)}$ by the modular norm.

By investigating modulars on quotient spaces of X we can prove

THEOREM 4. *For a finite number of elements* $\bar{a}_\nu \in \overline{X}$ *and real numbers* $a_\nu (\nu = 1, 2, \ldots, \varkappa)$, *if*

$$\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu = 0$$

implies

$$\sum_{\nu=1}^{\varkappa} \xi_\varkappa a_\nu = 0,$$

and (for $\gamma > 0$*) if*

$$\sum_{\nu=1}^{\varkappa} \xi_\nu a_\nu \leqslant \gamma + \overline{m}\Big(\sum_{\nu=1}^{\varkappa} \xi_\nu \bar{a}_\nu\Big)$$

for every finite number of real numbers $\xi_\nu (\nu = 1, 2, \ldots, \varkappa)$, *then for any* $0 < \varepsilon < 1$ *there exists* $x \in X$ *such that* $m\{(1-\varepsilon)x\} \leqslant \gamma$, $\bar{a}_\nu(x) = a_\nu$ $(\nu = 1, 2, \ldots, \varkappa)$.

DEFINITION 8. If X is a modulared space and $(\forall x \in X)$ $m(x) = 0 \Rightarrow x = 0$, then X is called *simple*. X is called *uniformly simple* if

$$(\forall \xi > 0) \inf_{m(x) \geqslant 1} m(\xi x) > 0.$$

Every space $B^*_{p(t,s)}$ is uniformly simple.

DEFINITION 9. If X is simple, a sequence $\{a_\nu\} \subset X (\nu = 1, 2, \ldots)$ is *conditionally modular convergent to a limit* $a \in X$ if $(\exists \alpha > 0) \lim_{\nu \to \infty} m\{\alpha(a_\nu - a)\} = 0$.

THEOREM 5. *X is uniformly simple iff conditional modular convergence coincides with modular convergence.*

DEFINITION 10. A modulared space X and its modular m are *uniformly convex* if $(\forall \varepsilon > 0)(\forall \gamma > 0)(\exists \delta > 0) m(x) \leqslant \gamma$, $m(y) \leqslant \gamma$ and $m(x-y) \geqslant \varepsilon \Rightarrow \frac{1}{2}\{m(x)+m(y)\} \geqslant m\{\frac{1}{2}(x+y)\} + \delta$.

This definition provides a modular parallel to the concept of uniform convexity in Banach spaces first considered by Professor James A. Clarkson [1].

§ 3. Main theorems.

THEOREM 6. *Every space $B^*_{p(t,s)}$ is modular complete.*

Proof. If

$$\lim_{\mu,\nu \to \infty} V_{p(t,s)}\{\xi(x_\mu - x_\nu)\} = 0 \quad \text{for every } \xi \geqslant 0,$$

then there exists a subsequence $x_{\nu_\mu} (\mu = 1, 2, \ldots)$ of $x_\nu (\nu = 1, 2, \ldots)$ such that $V_{p(t,s)}\{2^\mu(x_{\nu_{\mu+1}} - x_{\nu_\mu})\} \leqslant 1$. Then we can assert, by (1), that

$$V_{p(t,s)}\left(\sum_{\mu=1}^{\varkappa} \frac{1}{2^\mu} \cdot 2^\mu |x_{\nu_{\mu+1}} - x_{\nu_\mu}|\right) \leqslant \sum_{\mu=1}^{\varkappa} \frac{1}{2^\mu} \cdot V_{p(t,s)}(2^\mu |x_{\nu_{\mu+1}} - x_{\nu_\mu}|)$$

$$\leqslant \sum_{\mu=1}^{\varkappa} \frac{1}{2^\mu} < 1 \quad \text{for } \varkappa = 1, 2, \ldots$$

Hence $\sum_{\mu=1}^{\varkappa} |x_{\nu_{\mu+1}}(r) - x_{\nu_\mu}(r)|$ is convergent to a function y such that $y(a) = 0$ and generally

$$y(r) = \sum_{\mu=1}^{\infty} |x_{\nu_{\mu+1}}(r) - x_{\nu_\mu}(r)| \quad \text{for all } r \in [a, b].$$

Setting

$$x(r) = x_{\nu_1}(r) + \sum_{\mu=1}^{\infty} \{x_{\nu_{\mu+1}}(r) - x_{\nu_\mu}(r)\},$$

we have $V_{p(t,s)}(x - x_{\nu_1}) \leqslant 1$ and hence $x \in B^*_{p(t,s)}$. Since $V_{p(t,s)}\{2^\mu(x_{\nu_{\mu+1}} - x_{\nu_\mu})\} \leqslant 1 (\mu = 1, 2, \ldots)$, we infer, again by (1), that

$$V_{p(t,s)}\left\{2^\varrho \sum_{\mu=2\varrho}^{\varkappa} \frac{1}{2^\mu} \cdot 2^\mu (x_{\nu_{\mu+1}} - x_{\nu_\mu})\right\} \leqslant \sum_{\mu=2\varrho}^{\varkappa} \frac{2^\varrho}{2^\mu} \leqslant \frac{1}{2^{\varrho-1}} \quad \text{for } \varkappa = 1, 2, \ldots$$

Hence we have

$$V_{p(t,s)}\{2^\varrho(x - x_{\nu_{2\varrho}})\} \leqslant \frac{1}{2^{\varrho-1}} \quad \text{for every } \varrho = 1, 2, \ldots,$$

and thus

$$V_{p(t,s)} - \lim_{\mu \to \infty} x_{\nu_\mu} = x.$$

Given $\varepsilon > 0$ and $\xi > 0$, because

$$\lim_{\nu,\mu \to \infty} V_{p(t,s)}\{\xi(x_\nu - x_\mu)\} = 0 \quad \text{and} \quad V_{p(t,s)} - \lim_{\mu \to \infty} x_{\nu_\mu} = x,$$

we can find x_{ν_μ} and x_ν such that $V_{p(t,s)}\{\xi(x_\nu - x_{\nu_\mu})\} < \varepsilon$ and $V_{p(t,s)}\{\xi(x_{\nu_\mu} - x)\} < \varepsilon$; hence

$$V_{p(t,s)}\left\{\frac{\xi}{2}(x - x_\nu)\right\} = V_{p(t,s)}\left\{\frac{\xi}{2}[(x_{\nu_\mu} - x) + (x_\nu - x_{\nu_\mu})]\right\}$$

$$\leqslant \tfrac{1}{2} V_{p(t,s)}\{\xi(x_{\nu_\mu} - x)\} + \tfrac{1}{2} V_{p(t,s)}\{\xi(x_\nu - x_{\nu_\mu})\} < \varepsilon.$$

This shows that $V_{p(t,s)} - \lim_{\nu \to \infty} x_\nu = x$, i.e., $B^*_{p(t,s)}$ is modular complete.

THEOREM 7. *If*

$$1 < p_1 = \inf_{t > s \in [a,b]} p(t,s) \leqslant \sup_{t > s \in [a,b]} p(t,s) = p_0 < +\infty,$$

then $B^*_{p(t,s)}$ is uniformly convex.

Proof. Let χ_0 be the characteristic function of $\{(t,s): p(t,s) \geqslant 2\}$ and χ_1 that of $\{(t,s): p(t,s) < 2\}$. Hence $(\forall x \in B^*_{p(t,s)})$ $V_{p(t,s)}(x\chi_0) + V_{p(t,s)}(x\chi_1) = V_{p(t,s)}(x)$. Given real numbers $\gamma > 0$, $\varepsilon > 0$, assume $V_{p(t,s)}(c) \leqslant \gamma$, $V_{p(t,s)}(d) \leqslant \gamma$, and $V_{p(t,s)}(c-d) \geqslant \varepsilon$. Then either

(i) $V_{p(t,s)}\{(c-d)\chi_0\} \geqslant \varepsilon/2$ or

(ii) $V_{p(t,s)}\{(c-d)\chi_1\} \geqslant \varepsilon/2$. If (i) holds, then it follows from the inequality (for a proof, see [3], p. 275)

$$(\forall p \geqslant 2) \frac{|\xi|^p + |\eta|^p}{2} \geqslant \left|\frac{\xi+\eta}{2}\right|^p + \left|\frac{\xi-\eta}{2}\right|^p$$

that

$$\tfrac{1}{2}\{V_{p(t,s)}(c\chi_0)+V_{p(t,s)}(d\chi_0)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\chi_0\}+V_{p(t,s)}\{\tfrac{1}{2}(c-d)\chi_0\}.$$

Since

$$p_0 = \sup_{t>s\in[a,b]} p(t,s) < +\infty,$$

we have

$$V_{p(t,s)}\{\tfrac{1}{2}(c-d)\chi_0\} \geqslant \frac{1}{2^{p_0}}\cdot V_{p(t,s)}\{(c-d)\chi_0\} \geqslant \frac{\varepsilon}{2^{p_0+1}}.$$

Furthermore, by M.5,

$$\tfrac{1}{2}\{V_{p(t,s)}(c\chi_1)+V_{p(t,s)}(d\chi_1)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\chi_1\}.$$

Hence we obtain

$$\tfrac{1}{2}\{V_{p(t,s)}(c)+V_{p(t,s)}(d)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\}+\frac{\varepsilon}{2^{p_0+1}}.$$

If (ii) holds, set

$$\varepsilon' = \min\left\{\frac{\varepsilon}{8\gamma}, \tfrac{1}{2}\right\}$$

and denote by χ_2 the characteristic function of

$$\{(t,s): p(t,s)<2 \text{ and } (\exists r: t\geqslant r\geqslant s)\,|c(r)-d(r)|\geqslant \varepsilon'(|c(r)|+|d(r)|)\};$$

then we have

$$V_{p(t,s)}\{(c-d)(\chi_1-\chi_2)\} \leqslant V_{p(t,s)}\{\varepsilon'(|c|+|d|)\}$$
$$\leqslant \tfrac{1}{2}\{V_{p(t,s)}(2\varepsilon' c)+V_{p(t,s)}(2\varepsilon' d)\} \leqslant \tfrac{1}{2}\cdot 2\varepsilon'\{V_{p(t,s)}(c)+V_{p(t,s)}(d)\} \leqslant 2\varepsilon'\gamma \leqslant \frac{\varepsilon}{4},$$

and therefore

$$V_{p(t,s)}\{(c-d)\chi_2\} \geqslant \frac{\varepsilon}{2}-\frac{\varepsilon}{4} = \frac{\varepsilon}{4}.$$

If we set

$$\sigma = p_1-1 = \{\inf_{t>s\in[a,b]} p(t,s)\}-1,$$

it follows from the inequality (for a proof, see [3], p. 275-276)

$$(\forall p: 1\leqslant p\leqslant 2)\ \frac{|\xi|^p+|\eta|^p}{2} \geqslant \left|\frac{\xi+\eta}{2}\right|^p + \frac{p(p-1)}{2}\cdot\left|\frac{\xi-\eta}{|\xi|+|\eta|}\right|^{2-p}\cdot\left|\frac{\xi-\eta}{2}\right|^p$$

that

$$\tfrac{1}{2}\{V_{p(t,s)}(c\chi_2)+V_{p(t,s)}(d\chi_2)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\chi_2\}+\frac{\sigma}{2}\varepsilon' V_{p(t,s)}\{\tfrac{1}{2}(c-d)\chi_2\}.$$

On the other hand, we obtain

$$V_{p(t,s)}\{\tfrac{1}{2}(c-d)\chi_2\} \geqslant \tfrac{1}{4}V_{p(t,s)}\{(c-d)\chi_2\} \geqslant \frac{\varepsilon}{16}$$

and, by M.5,

$$\tfrac{1}{2}\{V_{p(t,s)}[c(1-\chi_2)]+V_{p(t,s)}[d(1-\chi_2)]\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)(1-\chi_2)\}.$$

Hence

$$\tfrac{1}{2}\{V_{p(t,s)}(c)+V_{p(t,s)}(d)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\}+\frac{\sigma\varepsilon'\varepsilon}{32}$$

and consequently

$$\tfrac{1}{2}\{V_{p(t,s)}(c)+V_{p(t,s)}(d)\} \geqslant V_{p(t,s)}\{\tfrac{1}{2}(c+d)\}+\min\left\{\frac{\varepsilon}{2^{p_0+1}}, \frac{\sigma\varepsilon'\varepsilon}{32}\right\}$$

in both cases (i) and (ii). Hence $B^*_{p(t,s)}$ is uniformly convex.

THEOREM 8. *If a Nakano modulared space X is at once uniformly simple, modular complete and uniformly convex, then X is reflexive.*

Proof. Let $\overline{\overline{X}}$ be the modular adjoint space of the modular adjoint space $\overline{X}$. For any $\overline{\overline{x}}\in\overline{\overline{X}}$ satisfying $\overline{\overline{m}}(\overline{\overline{x}})<+\infty$, using (2) we can find a sequence $\{\overline{x}_\nu\}\subset\overline{X}$ $(\nu=1,2,\ldots)$ such that

$$\overline{\overline{x}}(\overline{x}_\nu) \geqslant \overline{\overline{m}}(\overline{\overline{x}})+\overline{m}(\overline{x}_\nu)-\frac{1}{\nu}.$$

Since we get by (3) that

$$\sum_{\nu=1}^{\varkappa}\xi_\nu\overline{\overline{x}}(\overline{x}_\nu) \leqslant \overline{\overline{m}}(\overline{\overline{x}})+\overline{m}\Big(\sum_{\nu=1}^{\varkappa}\xi_\nu\overline{x}_\nu\Big)$$

for any finite set of real numbers ξ_ν $(\nu=1,2,\ldots,\varkappa)$, we can find by Theorem 4 a sequence $\{x_\varrho\}\subset X$ $(\varrho=1,2,\ldots)$ such that $\overline{x}_\nu(x_\varrho)=\overline{\overline{x}}(\overline{x}_\nu)$ for every $\nu=1,2,\ldots,\varrho$, and such that

$$m\left\{\left(1-\frac{1}{\varrho}\right)x_\varrho\right\} \leqslant \overline{\overline{m}}(\overline{\overline{x}}) \quad \text{for every } \varrho=1,2,\ldots$$

For such x_ϱ $(\varrho=1,2,\ldots)$ we obtain

$$\lim_{\mu,\nu\to\infty} m\left\{\left(1-\frac{1}{\nu}\right)x_\nu-\left(1-\frac{1}{\mu}\right)x_\mu\right\}=0.$$

Otherwise, if we could find $\varepsilon>0$ and two subsequences λ_ν,μ_ν $(\nu=1,2,\ldots)$ of $\{1,2,\ldots\}$ such that

$$(\forall\nu=1,2,\ldots)\,m\left\{\left(1-\frac{1}{\lambda_\nu}\right)x_{\lambda_\nu}-\left(1-\frac{1}{\mu_\nu}\right)x_{\mu_\nu}\right\}\geqslant\varepsilon,$$

then, since X is uniformly convex,

$$(\exists\delta>0)\tfrac{1}{2}\Big\{m\Big[\Big(1-\frac{1}{\lambda_\nu}\Big)x_{\lambda_\nu}\Big]+m\Big[\Big(1-\frac{1}{\mu_\nu}\Big)x_{\mu_\nu}\Big]\Big\}$$
$$\geqslant m\Big\{\tfrac{1}{2}\Big[\Big(1-\frac{1}{\lambda_\nu}\Big)x_{\lambda_\nu}+\Big(1-\frac{1}{\mu_\nu}\Big)x_{\mu_\nu}\Big]\Big\}+\delta$$

for every $\nu=1,2,\ldots$ On the other hand,

$$\tfrac{1}{2}\Big\{m\Big[\Big(1-\frac{1}{\lambda_\nu}\Big)x_{\lambda_\nu}\Big]+m\Big[\Big(1-\frac{1}{\mu_\nu}\Big)x_{\mu_\nu}\Big]\Big\}\leqslant\bar{\bar{m}}(\bar{\bar{x}}),$$

and by (2) for $\varrho\leqslant\lambda_\nu,\mu_\nu$

$$m\Big\{\tfrac{1}{2}\Big[\Big(1-\frac{1}{\lambda_\nu}\Big)x_{\lambda_\nu}+\Big(1-\frac{1}{\mu_\nu}\Big)x_{\mu_\nu}\Big]\Big\}\geqslant\bar{x}_\varrho\Big\{\tfrac{1}{2}\Big[\Big(1-\frac{1}{\lambda_\nu}\Big)x_{\lambda_\nu}+\Big(1-\frac{1}{\mu_\nu}\Big)x_{\mu_\nu}\Big]\Big\}-\bar{m}(\bar{x}_\varrho)$$
$$=\Big(1-\frac{1}{2\lambda_\nu}-\frac{1}{2\mu_\nu}\Big)\bar{\bar{x}}(\bar{x}_\varrho)-\bar{m}(\bar{x}_\varrho).$$

Hence we obtain for such δ

$$\bar{\bar{m}}(\bar{\bar{x}})\geqslant\Big(1-\frac{1}{2\lambda_\nu}-\frac{1}{2\mu_\nu}\Big)\bar{\bar{x}}(\bar{x}_\varrho)-\bar{m}(\bar{x}_\varrho)+\delta\quad\text{for every }\varrho\leqslant\lambda_\nu,\mu_\nu.$$

Now, letting $\nu\to\infty$, we conclude

$$\bar{\bar{m}}(\bar{\bar{x}})\geqslant\bar{\bar{x}}(\bar{x}_\varrho)-\bar{m}(\bar{x}_\varrho)+\delta\geqslant\bar{\bar{m}}(\bar{\bar{x}})-\frac{1}{\varrho}+\delta\quad\text{for every }\varrho=1,2,\ldots,$$

contradicting $\delta>0$. Since X is uniformly simple and modular complete by assumption, by Theorem 5 there exists $x\in X$ such that

$$m-\lim_{\nu\to\infty}\Big(1-\frac{1}{\nu}\Big)x_\nu=x,$$

and thus we have by Theorems 3 and 2,

$$m(x)\leqslant\varliminf_{\nu\to\infty}m\Big\{\Big(1-\frac{1}{\nu}\Big)x_\nu\Big\}\leqslant\bar{\bar{m}}(\bar{\bar{x}}),$$

and $\bar{x}_\nu(x)=\bar{\bar{x}}(\bar{x}_\nu)$ for every $\nu=1,2,\ldots$ For an arbitrary $\bar{x}\in\bar{X}$, the same process can be applied to $\bar{x},\bar{x}_1,\bar{x}_2,\ldots$ instead of $\bar{x}_1,\bar{x}_2,\ldots$, and then we obtain similarly $x_0\in X$ such that $m(x_0)\leqslant\bar{\bar{m}}(\bar{\bar{x}})$, $\bar{x}(x_0)=\bar{\bar{x}}(\bar{x})$, $\bar{x}_\nu(x_0)=\bar{\bar{x}}(\bar{x}_\nu)$ for every $\nu=1,2,\ldots$ For such x_0, if $m(x-x_0)>0$, then, since X is uniformly convex, we can find $\delta>0$ such that

$$\tfrac{1}{2}\{m(x)+m(x_0)\}\geqslant m\{\tfrac{1}{2}(x+x_0)\}+\delta,$$

and then, by (3), for every $\nu=1,2,\ldots$,

$$\bar{\bar{m}}(\bar{\bar{x}})\geqslant m\{\tfrac{1}{2}(x+x_0)\}+\delta\geqslant\bar{x}_\nu\{\tfrac{1}{2}(x+x_0)\}-\bar{m}(\bar{x}_\nu)+\delta$$
$$=\bar{\bar{x}}(\bar{x}_\nu)-\bar{m}(\bar{x}_\nu)+\delta\geqslant\bar{\bar{m}}(\bar{\bar{x}})-\frac{1}{\nu}+\delta,$$

contradicting $\delta>0$. Hence $m(x-x_0)=0$ and therefore $x=x_0$, because X is uniformly simple. It follows that $\bar{x}(x)=\bar{x}(x_0)=\bar{\bar{x}}(\bar{x})$. Since $\bar{x}\in\bar{X}$ is arbitrary, we have $(\forall\bar{x}\in\bar{X})\ \bar{x}(x)=\bar{\bar{x}}(\bar{x})$, i.e., X is reflexive by Definition 5.

THEOREM 9. *$B^*_{p(t,s)}$ with $p(t,s)$ restricted as in Theorem 7 is reflexive as a Nakano modulared space and as a Banach space.*

Proof. Because $B^*_{p(t,s)}$ is a uniformly simple Nakano modulared space, it follows by Theorems 6, 7, and 8 that $B^*_{p(t,s)}$ is reflexive in the sense of Definition 5. In view of Theorem 3, we finally obtain that $B^*_{p(t,s)}$ is reflexive in the usual Banach space sense.

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