STUDIA MATHEMATICA, T. XXX. (1968)

Summability of vector sequences

by

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1.1. Introduction. In his paper [6] G. G. Lorentz defined the notion of almost convergence for scalar sequences, and proved the theorem characterizing them in terms of their averages. It is the purpose of this paper to extend Lorentz's results to sequences of vectors in a Hilbert space and to obtain component-wise criteria for sequences to be periodic and almost periodic. It turns out that the results for scalar sequences cannot be extended to all types of vector sequences, and it is convenient to classify the "good" and "bad" sequences in an enlarged Hilbert space. A further justification for employing this structure (although it is not heavily relied on in the paper), is that there are several methods of extending a generalized limit from scalar to vector sequences. In the enlarged space, the formal distinctions are identified with a single projection operator. Details of the brief description of H_L in 1.3 may be found in [4].

1.2 Definitions and notation. The symbol H will denote a fixed separable Hilbert space over the complex numbers. Let m denote the space of bounded complex sequences. A positive linear functional $L \in m^*$ is called a *generalized limit* provided it preserves ordinary limits. That is, if $\lim(a_n) = a$, $L\big((a_n)\big) = a$ also. If $L\big((a_{n+1})\big) = L\big((a_n)\big)$ for all $(a_n) \in m$, L is said to be *translation invariant* (T. I.) The existence of T. I. generalized limits was proved by Banach [1].

In order to avoid nested parentheses, we will drop them whenever possible. Thus, if (x_n) and (y_n) are two sequences of vectors in H, then $((x_n, y_n))$ is their "inner product" sequence. The value of L on this sequence (provided it is bounded) is $L\big(((x_n, y_n))\big)$, which we write as $L(x_n, y_n)$.

1.3. The space H_L. Let $\Sigma(H)$ be the set of all norm bounded sequences of vectors in H. If operations are defined point-wise, $\Sigma(H)$ becomes a vector space. A semi-definite bi-linear form may be defined on $\Sigma(H)$ as follows: For $x = (x_n)$, $y = (y_n)$ let $[x, y] = L(x_n, y_n)$. If we let

$$K = \{(x_n) \in \Sigma(H) \colon L(\|x_n\|^2) = 0\},$$

then $\Sigma(H)/K$ is a pre-Hilbert space. Denote by H_L the completion of $\Sigma(H)/K$. We will usually view the elements of $\Sigma(H)/K$ as though they were sequences, rather than equivalence classes, and we persist in this viewpoint when $\Sigma(H)/K$ is regarded as a subset of H_L. This construction is due to Berberian [2].

Clearly, the map which sends each $h \in H$ into (h), the constant sequence whose every term is h, is an isometric embedding of H in H_L. The sequences having range contained in a strongly compact set in H form a linear manifold, U, in H_L. The closure, $\langle U\rangle$, of U in H_L is a subspace, and we can write $H_L = \langle U\rangle \oplus U^{\perp}$.

The algebra m is isomorphic to the algebra $C(\beta N)$, where βN is the Stone-Čech compactification of the integers. If the isomorphism is denoted by β, we can view each sequence (a_n) as being associated with an extension $a^{\beta} \in C(\beta N)$ so that $a^{\beta}(n) = a_n$ for all $n \in N \subset \beta N$. Likewise, each element $L \in m^*$ is associated with a measure on βN, say a, so that

$$L(a_n) = \int_{\beta N} a^{\beta}(t)\, da(t) \quad \text{for each } (a_n) \in m.$$

It follows from the fact that L is a generalized limit that a is supported on $\beta N - N$. It is possible to define a similar extension for vector sequences $x = (x_n) \in \Sigma(H)$, and it is seen [4] that the extension $(x_n) \to x^{\beta}$ induces a partial isometry on H_L. Under it, $\langle U\rangle$ is isometric to an L_2-space of vector functions on βN and $U^{\perp}$ goes into zero.

1.4. Extensions of L. (a) Let L be a given generalized limit. For each element $(x_n) \in \Sigma(H)$, the equation $F(y) = L(y, x_n)$ defines a continuous linear functional on H. Thus there is a unique $h \in H$ so that $F(y) = (y, h)$. Define $L_a(x_n) = h$.

The operation of L_a on $\Sigma(H)$ is evidently linear. If $(x_n) \to k \in H$ in the weak topology,

$$L(y, x_n) = (y, k) = \big(y, L_a(x_n)\big)$$

for every $y \in H$. Thus $L_a(x_n) = k$. In this sense, L_a is still a generalized limit. Also, for $y \in H$, $x = (x_n) \in \Sigma(H)$

$$\big|\big(y, L_a(x_n)\big)\big| = |L(y, x_n)| \leqslant |L(\|y\|\cdot\|x_n\|)| \leqslant \|y\|\cdot\|x\|_{H_L}.$$

Thus L_a may be extended (by linearity and continuity) to H_L.

(b) Bourbaki [3] develops a theory of integration for vector functions f with respect to a scalar measure a. This theory is especially simple when applied to Hilbert space valued functions continuous on and supported in a compact set. Then

$$\Big(\int f da, h\Big) = \int \big(f(t), h\big) da(t) \quad \text{for all } h \in H.$$

Applying the Stone-Čech extension to $x = (x_n)$, we define

$$L_b(x_n) = \int_{\beta N} x^{\beta} da,$$

where a is the measure on βN associated with the generalized limit L in the sense of the previous section. Again, if $(x_n) \to h$ weakly, then $(x_n, k) \to (h, k)$ for each $k \in H$. On the other hand, x^{β} is at least weakly continuous on βN, and therefore the scalar sequence (x_n, k) has the extension $(x^{\beta}(t), k)$. It is not difficult to see that a scalar sequence which converges to zero must have an extension on βN which vanishes on $\beta N - N$. Thus $\big(x^{\beta}(t), k\big) = (h, k)$ for all $t \in \beta N - N$. Since a must be supported on $\beta N - N$, $x^{\beta} = h$ almost everywhere and so $L_b(x_n) = h$. As before, we may extend L_b to H_L.

(c) Let us consider only U-sequences for a moment. If we write $x_n = \Sigma_i (x_n, e_i) e_i$ (where (e_i) is a basis sequence), then the series must converge uniformly in n. It is reasonable to expect that an extension L_c should satisfy

$$L_c(x_n) = \Sigma_i [L(x_n, e_i)] e_i.$$

Thus, for finite rank sequences

$$x_n = \sum_{i=0}^{k} (x_n, e_i) e_i$$

in U, we define

$$L_c(x_n) = \sum_{i=0}^{k} [L(x_n, e_i)] e_i.$$

(L is evaluated on the variable n, not i.) This map from a subset of U into H is bounded because

$$\big|\big(L_c(x_n), y\big)\big| = \Big|\sum_{i=0}^{k} [L(x_n, e_i)](e_i, y)\Big| \leqslant L\Big|\sum_{i=0}^{k} (x_n, e_i)(e_i, y)\Big|$$
$$\leqslant L(\|x_n\|\cdot\|y\|) \leqslant \|y\|\cdot\|x\|_{H_L}.$$

Since the finite rank sequences are dense in $\langle U\rangle$, the extension by linearity and continuity can be accomplished. Since $H_L = \langle U\rangle \oplus U^{\perp}$, we define L_c to be zero on $U^{\perp}$.

1.4.1. THEOREM. *On H_L, $L_a = L_b = L_c = L^0$ and L^0 is the projection of H_L onto H (viewed as the constant sequences).*

Proof. Since

$$\big(L_a(x_n), y\big) = L(x_n, y) = \int_{\beta N} \big(x^{\beta}(t), y\big) da(t) = \Big(\int_{\beta N} x^{\beta}(t) da(t), y\Big)$$
$$= \big(L_b(x_n), y\big) \quad \text{for all } y \in H,$$

L_a and L_b coincide on $\Sigma(H)$ and therefore on H_L. Because x^β vanishes when $x \in U^\perp$, both L_a and L_b are zero on $U^\perp$, as is L_c by definition. It is clear that $L_a = L_c$ on the finite rank sequences, and this is enough for the equality of L_a and L_c throughout $\langle U \rangle$. Thus $L_a = L_b = L_c$ on H_L and we denote their common value by L^0.

Since L^0 is identically zero on $U^\perp$ and is evidently the identity on $H \subset H_L$, it suffices to show that L^0 is selfadjoint on $\langle U \rangle$. For this purpose, it is enough to consider the action of L_c on the finite rank sequences

$$x_n = \sum_{i=0}^{k} (x_n, e_i) e_i, \qquad y_m = \sum_{i=0}^{j} (y_m, e_i) e_i.$$

(We will temporarily use a superscript on L to indicate which variable L operates on). For $x = (x_n)$, $y = (y_n)$:

$$[L_c(x), y]_{H_L} = L^m\big(L_c(x), y_m\big) = L^m\Big(\sum_{i=0}^{k} L^n(x_n, e_i) e_i, \sum_{i=0}^{j} (y_m, e_i) e_i\Big)$$
$$= \Big(\sum_{i=0}^{k} L^n(x_n, e_i) e_i, \sum_{i=0}^{j} L^m(y_m, e_i) e_i\Big),$$

which, by symmetry, is equal to $[x, L_c(y)]_{H_L}$. Thus $L_c = L^0$ must be self-adjoint on $\langle U \rangle$. This shows that L^0 is the projection on H.

2.1. Almost convergence. Throughout the remainder of the paper, L will denote a T. I. generalized limit. The following definition and its consequences are due to Lorentz [6]. Let $(a_n) \in m$. If for every L we have $L(a_n) = b$, where b is a number independent of L, we say that (a_n) is *almost convergent to* b, and write $(a_n) \gg b$.

THEOREM. *Let*

$$S_n^p(a) = \frac{a_{n+1} + \ldots + a_{n+p}}{p}.$$

Then $(a_n) \gg b$ *if and only if* $\lim_p S_n^p(a) = b$ *uniformly in* n.

We observe that L produces a T. I. extension L^0 in H_L. Specifically, for each $(x_n) \in \Sigma(H)$, let $T(x_n) = (x_{n+1})$. Then T is a norm preserving map and extends to a unitary transformation of H_L onto itself. We have for $x = (x_n) \in H_L$, $y \in H$

$$[L^0(Tx), y]_{H_L} = L(x_{n+1}, y) = L(x_n, y) = [L^0(x), y]_{H_L}.$$

That is to say $L^0(Tx) = L^0(x)$. Thus we could speak of almost convergence of vector sequences in terms of L^0 commuting with the operator T. Since we will deal with sequences, rather than elements of H_L, we will say that (x_n) is *almost convergent to* $h \in H$ provided $L^0(x_n) = h$ for every L. It is evident that for any L, the sequences which converge weakly will be almost convergent. Looking at H_L for a moment, any sequence which lies in $H^\perp$ regardless of L will be almost convergent to zero. This is the source of the counter-examples that follow. In U, however, we have a well behaved set of sequences; as the following illustrates.

2.1.1. THEOREM. *Let* $x = (x_n)$ *be a* U*-sequence. Define* $S_n^p(x)$ *as above. Then the following statements are equivalent*:

1) $(x_n) \gg h$;

2) $(x_n, e) \gg (h, e)$ *for every* e *in a basis set*;

3) $\|S_n^p(x) - h\|_H \to 0$, *as* $p \to \infty$, *uniformly in* n.

Proof. That 1) implies 2) is clear from the definition of $L^0(x_n)$.

2) implies 3). Write

$$(*) \qquad S_n^p(x) - h = \sum_{j=1}^{k} \big(S_n^p(x), e_j\big) e_j - \sum_{j=1}^{k} (h, e_j) e_j + \sum_{j=k+1}^{\infty} \big(S_n^p(x), e_j\big) e_j - \sum_{j=k+1}^{\infty} (h, e_j) e_j.$$

Let $\varepsilon > 0$ be given. We can surely choose k so that the norm of the fourth term on the right is less than ε. Also, the uniform convergence of the series $x_n = \Sigma_i (x_n, e_i) e_i$ allows us to choose k so that the third term has norm less than ε also. We fix k large enough to satisfy both these requirements and apply the hypothesis to the difference of the first two terms on $(*)$. We may (by Lorentz's theorem) choose p so that this difference is less than ε/k, independent of n. For this choice of p:

$$\|S_n^p(x) - h\| < 3\varepsilon, \qquad \text{independent of } n.$$

3) implies 1). Clearly 3) implies 2) because $|(y, e)| \leqslant \|y\|$ for every $y \in H$. Thus, because $(x_n) \in U$,

$$L^0(x_n) = \Sigma_i L(x_n, e_i) e_i = \Sigma_i (h, e_i) e_i = h$$

for every L.

Among almost convergent scalar sequences are the convergent ones, the periodic ones, and the almost periodic ones. We examine the latter notions in some detail for vector sequences. For the moment, however, we consider the topological properties of almost convergence.

It follows from fairly general principles that the generalized limit of a vector sequence will lie in the closed convex hull of the terms of the sequence. By the preceding theorem, if (x_n) has period p, (x_n) is almost convergent to its mean value. That is, if $x_{n+p} = x_n$ for all n, then

$$(x_n) \gg \frac{x_1 + \ldots + x_p}{p}.$$

From this, it is evident that none of the usual notions of proximity apply. If T is a bounded linear map on H, the relation $L(Tx_n, y) = L(x_n, T^*y)$ shows that T will preserve almost convergence. The following theorem shows that a sort of converse is true.

2.1.2. THEOREM. *Let T be a map from H into H, continuous in the norm topology, and suppose $T(0) = 0$. Then a necessary and sufficient condition that T preserve almost convergence is that T be additive and real homogeneous.*

Proof. Sufficiency. Write $M(x) = iT(x)+T(ix)$, $N(x) = iT(x)-T(ix)$. Then M and N are additive; $M(ax) = aM(x)$ and $N(ax) = \bar{a}N(x)$ for all complex a. Clearly, M, being an ordinary operator, will preserve almost convergence. On the other hand, associated with the conjugate homogeneous transformation N there is a well defined conjugate homogeneous transformation $N^{\sim}$ which satisfies $(Nx, y) = (N^{\sim}y, x)$ for every $x, y \in H$. Thus for any L and $y \in H$

$$L(Nx_n y) = L[\overline{(x_n, N^{\sim}y)}] = \overline{L(x_n, N^{\sim}y)}$$
$$= \overline{(h, N^{\sim}y)} = (Nh, y),$$

whence $Nx_n \gg Nh$, so N preserves almost convergence also. Since $T(x) = (1/2i)[M(x)+N(x)]$, T will preserve almost convergence.

Necessity. The sequence

$$x_n = \begin{cases} 0 & \text{for } n \neq kp, \\ h & \text{for } n = kp, h \neq 0 \text{ in } H \end{cases}$$

is a U-sequence of period p. We have $(x_n) \gg h/p$, and so $T(x_n) \gg T(h/p)$. However,

$$T(x_n) = \begin{cases} 0 & \text{for } n \neq kp, \\ T(h) & \text{for } n = kp, \end{cases}$$

so $T(x_n) \gg T(h)/p$. We have thus shown that $T(h/p) = T(h)/p$ when p is a positive integer. The sequence $x, y, x, y, \ldots$ where $x, y \in H$ is almost convergent to $(1/2)[x+y]$. But then $T(x), T(y), \ldots$ is almost convergent to $T\big((1/2)(x+y)\big)$ by the preservation assumption. We always have $T(x), T(y), T(x), \ldots \gg (1/2)[T(x)+T(y)]$, and so $(1/2)[T(x)+T(y)] = T\big((1/2)(x+y)\big) = (1/2)T(x+y)$, whence T is additive. Now the additivity and continuity combine to give real homogeneity.

2.2. Averages of non-U sequences. We have observed that the generalized limit of every weakly convergent to zero sequence is zero. Part 3 of Theorem 2.1.1 fails in the case of such sequences. For instance, if $[\cdot]$ denotes the greatest integer function, the sequence $x_n = e_{[\sqrt{n}]}$ converges weakly to zero and the averages converges in norm to zero

also — but not in a translation invariant fashion. An even stronger example is provided by the following sequence whose norm averages do not converge to zero. We define a sequence of integers n_i as follows. Let $n_1 = 1$, n_2 chosen so that

$$\left\| \frac{n_1 e_1 + n_2 e_2}{n_1+n_2} \right\|^2 > \tfrac{1}{2}.$$

In general, let $S_i = \sum_{j=1}^{i} n_j$. Then if we have determined $n_1, \ldots, n_{i-1}$, we choose n_i so that

$$\left\| \frac{n_1 e_1 + \ldots + n_i e_i}{S_{i-1}+n_i} \right\|^2 > \tfrac{1}{2}.$$

This can be done because the norm term is at least as large as

$$\left\| \frac{n_i e_i}{S_{i-1}+n_i} \right\|^2 = \frac{(n_i)^2}{(S_{i-1}+n_i)^2}$$

and the fraction approaches 1 as n_i increases without bound. It is thus eventually greater than 1/2, and the sequence (n_i) is defined. Now let

$$x_1 = e_1,$$
$$x_n = e_{i+1} \text{ when } S_i < n \leqslant S_{i+1}.$$

Then

$$\left\| \frac{x_1+\ldots+x_{S_k}}{S_k} \right\|^2 = \left\| \frac{n_1 e_1 + \ldots + n_k e_k}{S_k} \right\| > \tfrac{1}{2}$$

and so the averages cannot converge to zero in norm.

3.1. Periodic sequences. Any periodic vector sequence is in U because its range is finite-dimensional. It will have periodic basis component sequences: $(x_{n+p}, e_i) = (x_n, e_i)$ for $i = 1, 2, \ldots$ Thus the generalized limits of such are computable from the averages of their components by the formula

$$(*) \qquad L^0(x_n) = \sum_{i=0}^{\infty} [L(x_n, e_i)] e_i.$$

Conversely, if we are given that (x_n, e_i) is periodic with period p_i for $i = 1, 2, \ldots$, we might expect that the generalized limit is computable according to $(*)$. We now consider conditions under which this is possible.

3.1.1. THEOREM. *A vector sequence (x_n) is periodic if and only if (x_n, y) is periodic for every $y \in H$.*

Proof. We need to establish only the sufficiency. Let

$$E_k = \{h \in H: (x_{n+k} - x_n, h) = 0 \text{ for all } n\}.$$

Then E_k is a closed set and every $h \in H$ belongs to some E_k. Thus $H = \bigcup_{k=1}^{\infty} E_k$ and by the Baire Category Theorem, at least one E_k contains a non-void open set, i.e. a sphere S of radius $r > 0$. When $h \in S \subset E_k$, $(x_{n+k} - x_n, h) = 0$. Let g be interior to S so that 0 is interior to $S - g$. If k' is chosen so that $(x_{n+k'} - x_n, g) = 0$ for all n, then $k \cdot k' = K$ has the property that $(x_{n+K} - x_n, f - g) = 0$ for every $f \in S$. Thus $(x_{n+K} - x_n, z) = 0$ for all z in a neighborhood of zero. It now follows that $(x_{n+k} - x_n, z) = 0$ for all $z \in H$, and so K is a period for the sequence (x_n).

The hypothesis of this theorem cannot be weakened to assert that (x_n) is periodic whenever (x_n, e) is periodic for each e in a basis set. In the following computation, L always operates on the variable n.

Let C_n^i be the characteristic function of multiples of i, that is $C_n^i = 1$ if i divides n, and zero otherwise. Then $i^{-1} C_n^i$ has period i (as a function of n). We may define

$$x_n = \Sigma_i \frac{1}{i} C_n^i e_i = \Sigma_i (x_n, e_i) e_i.$$

The straightforward verification that this is a U-sequence depends on the fact that $\sum i^{-2}$ converges. For $x = (x_n)$, we have

$$\|x\|_{H_L}^2 = L(\|x_n\|^2) = \Sigma_i L\big(|(x_n, e_i)|^2\big) = \Sigma_i L\left(\left|\frac{1}{i} C_n^i\right|^2\right) = \Sigma_i \frac{1}{i^3},$$

so the sequence is non-trivial. It is not periodic, however. If we view it in the spatial context of $U \subset H_L$, even more is true. Let T be the shift operator defined in 2.1. Then there is no p so that $T^p x = x$ in H_L (regardless of which L is chosen). Thus, we assert $L(\|x_{n+p} - x_n\|^2) \neq 0$ for every p.

We have

$$L(\|x_{n+p} - x_n\|^2) = \Sigma_i \frac{1}{i^2} L(|C_{n+p}^i - C_n^i|^2) = 0$$

if and only if every term vanishes.

$$|C_{n+p}^i - C_n^i|^2 = \begin{cases} 1 & \text{if} \quad i \mid n+p \text{ and } i \nmid n, \\ 1 & \text{if} \quad i \mid n \text{ and } i \nmid n+p, \\ 0 & \text{otherwise.} \end{cases}$$

If p is given, we choose i so that $i \nmid p$. Then for $n = ki$, $i \mid n$, but $i \nmid n+p$. Thus, for $n = ki$, the value of the sequence is 1, so

$$L(|C_{n+p}^i - C_n^i|^2) \geqslant L(C_n^i) = 1/i > 0.$$

It is reasonable to expect, however, that this sequence inherits some sort of periodicity, and it does turn out to be an almost periodic sequence. We establish this in the next section.

3.2. Almost periodicity. The generalized limits of almost periodic scalar sequences [6] may be obtained from their mean value:

$$\lim_N \frac{1}{N} \sum_{i=1}^{N} a_{n+i}.$$

Our object in extending this notion to vector sequences is to characterize the latter in terms of their components and to show (by virtue of the fact that such sequences are in U) that the component-wise computation of their generalized limits is valid.

Definition. A subset S of the positive integers is said to be *relatively dense* in N if there is a number m so that in each interval $[k, k+m]$ there is at least one member of S.

Definition. Let (a_n) be a sequence in a metric space whose metric is denoted by $d(\cdot,\cdot)$. The integer p is called an *ε-translation number* of (a_n) provided

$$\sup_n d(a_{n+p}, a_n) \leqslant \varepsilon.$$

We denote the set of ε-translation numbers of (a_n) by $E[\varepsilon, a_n]$.

Definition. A sequence (a_n) in a metric space will be called *almost periodic* (AP) provided for each $\varepsilon > 0$ the set $E[\varepsilon, a_n]$ is relatively dense in N.

This definition is different from the one usually given for functions on semi-groups in that it excludes sequences which converge to zero. In more general circumstances, if almost periodicity is defined in terms of compactness of translates, the set of AP semi-group functions is the direct sum of functions almost periodic on a group and functions which vanish at infinity. This is, in particular, true for the semi-group of positive reals and the positive integers [5]. For our purposes, the sequences which converge to zero pose no question as to summability — they are merely superfluous. It should be noted also that our definitions are quite similar to H. Bohr's classic formulation of an AP real function. Some of his proofs will apply with minor modifications. Thus it is almost immediate that an almost periodic sequence is bounded. On the other hand the discrete topology of the integers eliminates the possibility of employing uniform continuity to produce an "interval" of translation numbers about zero. For this reason, it is not immediately apparent that two almost periodic sequences will have any ε-translation numbers in common—hence their sum is not clearly AP. The converse of the following lemma is the definition of almost periodicity frequently employed.

We will denote by $\Sigma(H)_\infty$ the space $\Sigma(H)$ with supremum norm:

$$\|x\|_\infty = \sup_n \|x_n\|_H.$$

If M is a positive integer (or zero) ${}_Mx$ is the M-th translate of the sequence x, so ${}_Mx_n = x_{n+M}$.

3.2.1. Lemma. *If $x = (x_n)$ is AP, then the translates $\{{}_Mx\}_{M=1}^\infty$ form a totally bounded set in $\Sigma(H)_\infty$.*

Proof. For $\varepsilon > 0$ given, let $m(\varepsilon)$ be the least length of an interval in which some member of $E[\varepsilon, x_n]$ must lie. Then the sequences $x, {}_1x, {}_2x, \ldots {}_{m(\varepsilon)}x$ are an ε-mesh for the set $\{{}_Mx\}_{M=1}^\infty$.

3.2.2. Lemma. *If (x_n) is AP, then $(x_n) \in U$.*

Proof. We need to show that (x_n) is contained in a totally bounded set in H. But if j and k are greater than one,

$$\|x_k - x_j\|_H \leqslant \sup_n \|{}_{(k-1)}x_n - {}_{(j-1)}x_n\|_H = \|{}_{(k-1)}x - {}_{(j-1)}x\|_\infty.$$

Now the assertion follows from the previous lemma.

We can now complete the example of 3.1 by showing the non-periodic sequence defined therein is almost periodic. Write

$$\|x_{n+p} - x_n\|^2 = \sum_{i=0}^{k} |(x_{n+p}, e_i) - (x_n, e_i)|^2 + \\ + \sum_{i=k+1}^{\infty} |(x_{n+p}, e_i) - (x_n, e_i)|^2.$$

Since (x_n) and (x_{n+p}) are both U-sequences, the second term can be made smaller than ε for large enough k, independent of n or p. Having thus determined k, we consider (x_n, e_i) for $i \leqslant k$. These sequences have period i and so the first k summands above will vanish if p is any number divisible by all $i \leqslant k$. It follows that $j \cdot k!$, where $j = 1, 2, \ldots$ is a relatively dense set of ε-translation numbers for (x_n).

In order to prove the following theorem, we need three additional statements, the verification of which we leave to the reader:

1) If (x_n) is AP, then so is $({}_Kx_n)$ for each K, and for each $\varepsilon > 0$, $E[\varepsilon, x_n] \subseteq E[\varepsilon, {}_Kx_n]$.

2) If $x = (x_n)$ is AP, then $\|x\|_\infty = \|{}_Kx\|_\infty$.

3) If x is AP, then $x - {}_Kx$ is also.

3.2.3. Theorem. *Let (x_n) and (y_n) be AP. Then for $\varepsilon > 0$, there is a relatively dense set of translation numbers which is contained in $E[\varepsilon, x_n] \cap E[\varepsilon, y_n]$. Thus $(x_n) + (y_n)$ is AP also.*

Proof. Because the sets $\{{}_Mx\}_{M=1}^\infty$ and $\{{}_My\}_{M=1}^\infty$ are both totally bounded in $\Sigma(H)_\infty$, for $\varepsilon > 0$ there is a set $\{n_1, \ldots, n_k\}$ and a set $\{m_1, \ldots, m_r\}$ so that for each integer M^* we have an n_i and an m_s with $\|{}_{M^*}y - {}_{m_s}y\|_\infty \leqslant \varepsilon/4$.

Define

$$A_i = \{M: \|{}_Mx - {}_{n_i}x\|_\infty \leqslant \varepsilon/4\},$$
$$B_s = \{M: \|{}_My - {}_{m_s}y\|_\infty \leqslant \varepsilon/4\}.$$

Let $C_1, \ldots, C_p$ denote the non-void intersections $A_i \cap B_s$, as $i = 1, \ldots, k$; $s = 1, \ldots, r$. (There are some non-void interesections; e.g. M^* above belongs to $A_i \cap B_s$. Further, by the fact that both sets of translates are totally bounded, every M belongs to some such intersection). Now choose a particular $c_q \in C_q$. Let d be any integer and say $d \in A_i \cap B_s = C_q$. Then

$$\|{}_{c_q}x - {}_dx\|_\infty \leqslant \|{}_{c_q}x - {}_{n_i}x\|_\infty + \|{}_{n_i} - {}_dx\|_\infty,$$
$$\|{}_{c_q}y - {}_dy\|_\infty \leqslant \|{}_{c_q}y - {}_{m_s}y\|_\infty + \|{}_{m_s}y - {}_dy\|_\infty.$$

Since $c_q, d \in C_q$, the second terms on the right are less than $\varepsilon/4$ by definition. By statements 2) and 3) above we see that if $d > c_q$

$$\|x - {}_{d-c_q}x\|_\infty = \|{}_{c_q}x - {}_dx\|_\infty \leqslant \varepsilon/2,$$

and

$$\|y - {}_{d-c_q}y\|_\infty = \|{}_{c_q}y - {}_dy\|_\infty \leqslant \varepsilon/2.$$

Thus the set $d - c_q$, where c_q is appropriately chosen, is a set of ε-translation numbers for both (x_n) and (y_n). As d ranges over the integers, this set is seen to be relatively dense. The conclusion of the theorem is now immediate.

Note that the sequences $(\|x_n\|)$, $(\|x_n\|^2)$, $((x_n, h))$, h fixed, are AP whenever (x_n) is.

Now we can characterize AP-vector sequences in terms of their basis components.

3.2.4. Theorem. *A U-sequence is AP if and only if its components with respect to a fixed basis are AP-scalar sequences*

Proof. The necessity is obvious. For the sufficiency, we extend 3.2.3 by induction; so we can find for any finite set of almost periodic sequences (vector or scalar) a common relatively dense set of ε-translation numbers. Thus, let $\varepsilon > 0$ be given and suppose for $x = (x_n) \in U$ that (x_n, e_i) is AP for each i. Expanding $\|x_{n+p} - x_n\|^2$ as usual, we make the tail of the series small (independent of n or p) by choosing the index of summation sufficiently large. The norm of the difference is therefore approximated by

$$\sum_{i=0}^{k} |(x_{n+p}, e_i) - (x_n, e_i)|^2.$$

Then there is a relatively dense set of integers S so that $p \in S$ implies each summand in the finite sum is less than $\varepsilon/2k$. Thus $\|x_{n+p}-x_n\|^2 \leqslant k(\varepsilon/2k)+\varepsilon/2 = \varepsilon$ and (x_n) is AP.

Among U-sequences, then, the almost periodic ones are the ones whose basis components are AP-scalar sequences, and $L^0(x_n) = \Sigma_i[L(x_n, e_i)]e_i$, where $L(x_n, e_i)$ is the mean value of the sequence.

An immediate corollary to the above theorem is that a U-sequence is AP if and only if (x_n, y) is AP for each $y \in H$. Unfortunately, a complete analog of Theorem 3.1.1 cannot be proved. That is, we cannot drop the hypothesis that the sequence be in U already. Let $x_n = e_n$ whenever $k \equiv 2^n-1 \pmod{2^{n+1}}$, $n = 0, 1, 2, \ldots$ This sequence has component sequences $a_k = (x_k, e_n)$, each with period 2^{n+1}. The vector sequence is not AP however, simply because it is not in U. It is also easy to see, that for each fixed y, (x_k, y) will be AP also. Hence, neither Theorem 3.2.4 nor its corollary will be true if we drop the assumption that the sequence has range contained in a compact set.

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Reçu par la Rédaction le 17. 10. 1967

STUDIA MATHEMATICA, T. XXX. (1968)

Uniformly convex and reflexive modulared variation spaces

by

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§ 1. Introduction. In chapter 11, entitled “Modular Spaces”, of his treatise [3], Professor Hidegorô Nakano presents a theory of modulars on arbitrary (not necessarily semi-ordered) linear spaces. Namely, given any linear space X, a functional $m(x)$ defined on X with values $0 \leqslant m(x) \leqslant +\infty$ is called a *Nakano modular* if

M. 1. $m(0) = 0$,

M. 2. $(\forall x \in X)\, m(-x) = m(x)$,

M. 3. $(\forall x \in X)(\exists \lambda > 0)\, m(\lambda x) < +\infty$,

M. 4. $m(\xi x) = 0$ for all $\xi > 0 \Rightarrow x = 0$,

M. 5. $(\forall x, y \in X)(\forall \alpha, \beta \geqslant 0)\ \alpha+\beta = 1 \Rightarrow m(\alpha x+\beta y) \leqslant \alpha m(x)+\beta m(y)$,

M. 6. $(\forall x \in X)\ m(x) = \sup_{0 \leqslant \xi < 1} m(\xi x)$.

The space X associated with the functional $m(x)$ is called a *Nakano modulared space*.

It is easy to see that, for example, the p^{th} power variations (as basic papers, see [4] or [2]) are special cases of Nakano modulars on generalized variation spaces. In this paper we are concerned with a new class of spaces which include the p^{th} power variation spaces. Let x be a real function in $[a, b]$ such that $x(a) = 0$, let $p(t, s)$ be a real function of two real variables such that $t, s \in [a, b]$, $t > s$, and $1 \leqslant p(t, s) < +\infty$; let π: $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a, b]$. Define

$$B_{p(t,s)} = \Big\{x: V_{p(t,s)}(x) = \sup_{\pi} \sum_{i=1}^{n} |x(t_i)-x(t_{i-1})|^{p(t_i,t_{i-1})} < +\infty\Big\},$$

and denote by $B^*_{p(t,s)}$ the linear space generated by $B_{p(t,s)}$. Here, $V_{p(t,s)}$ is the Nakano modular on the space $B^*_{p(t,s)}$. If $p(t, s) \equiv p =$ constant $(1 \leqslant p < +\infty)$, we have the case of p^{th} variation. The spaces $B^*_{p(t,s)}$ generalize the idea of p^{th} variation in the same way as Nakano's $L_{p(t)}$-spaces generalize the classical L_p-spaces (see [3], p. 234-240). In fact, the methods employed in the present paper, although they are perhaps not widely known, are essentially due to Nakano.