Summability of vector sequences

by

JOSEPH B. DREDS (Baton Rouge, La.)

1.1. Introduction. In his paper [6] G. G. Lorentz defined the notion of almost convergence for scalar sequences, and proved the theorem characterizing them in terms of their averages. It is the purpose of this paper to extend Lorentz's results to sequences of vectors in a Hilbert space and to obtain component-wise criteria for sequences to be periodic and almost periodic. It turns out that the results for scalar sequences cannot be extended to all types of vector sequences, and it is convenient to classify the "good" and "bad" sequences in an enlarged Hilbert space. A further justification for employing this structure (although it is not heavily relied on in the paper), is that there are several methods of extending a generalized limit from scalar to vector sequences. In the enlarged space, the formal distinctions are identified with a single projection operator. Details of the brief description of $H_L$ in 1.3 may be found in [4].

1.2 Definitions and notation. The symbol $H$ will denote a fixed separable Hilbert space over the complex numbers. Let $w$ denote the space of bounded complex sequences. A positive linear functional $L : w \to \mathbb{C}$ is called a generalized limit provided it preserves ordinary limits. That is, if $\lim (a_n) = a$, $L((a_n)) = a$ also. If $L((a_{n+1})) = L((a_n))$ for all $(a_n) \in w$, $L$ is said to be translation invariant (T. I.) The existence of T. I. generalized limits was proved by Banach [1].

In order to avoid nested parentheses, we will drop them whenever possible. Thus, if $(x_n)$ and $(y_n)$ are two sequences of vectors in $H$, then $(x_n, y_n)$ is their "inner product" sequence. The value of $L$ on this sequence (provided it is bounded) is $L((x_n, y_n))$, which we write as $L(x_n, y_n)$.

1.3. The space $H_L$. Let $\Sigma(H)$ be the set of all norm bounded sequences of vectors in $H$. If operations are defined point-wise, $\Sigma(H)$ becomes a vector space. A semi-definite bi-linear form may be defined on $\Sigma(H)$ as follows: For $x = (x_n)$, $y = (y_n)$ let $[x, y] = L(x_n, y_n)$. If we let $K = \{(x_n) : L([x_n, x_n]) = 0\}$,
then $\Sigma(\mathcal{H})/K$ is a pre-Hilbert space. Denote by $H_L$ the completion of $\Sigma(\mathcal{H})/K$. We will usually view the elements of $\Sigma(\mathcal{H})/K$ as though they were sequences, rather than equivalence classes, and we persist in this viewpoint when $\Sigma(\mathcal{H})/K$ is regarded as a subset of $H_L$. This construction is due to Berberian [2].

Clearly, the map which sends each $h \in \mathcal{H}$ into $(h)$, the constant sequence whose every term is $h$, is an isometric embedding of $\mathcal{H}$ in $H_L$. The sequences having range contained in a strongly compact set in $H$ form a linear manifold, $U$, in $H_L$. The closure, $(U')$, of $U$ in $H_L$ is a subspace, and we can write $H_L = (U') \oplus U'$. The algebra $\mathcal{A}$ is isomorphic to the algebra $O(\beta \mathcal{N})$, where $\beta \mathcal{N}$ is the Stone-Čech compactification of the integers. If the isomorphism is denoted by $\beta$, we can view each sequence $(a_n)$ as being associated with an extension $a^*(\beta \mathcal{N})$ so that $a^*(\beta \mathcal{N})(n) = a_n$ for all $n \in \beta \mathcal{N}$. Likewise, each element $L_m$ of $\mathcal{A}$ is associated with a measure on $\beta \mathcal{N}$, say $a$, so that

$$L_m(a_n) = \int_{\beta \mathcal{N}} a^*(\beta \mathcal{N})(t) \, da(t)$$

for each $(a_n) \in \mathcal{A}$.

It follows from the fact that $L$ is a generalized limit that it is supported on $\beta \mathcal{N} - \mathcal{N}$. It is possible to define a similar extension for vector sequences $x = (a_n) \in \Sigma(\mathcal{H})$, and it is seen [4] that the extension $a^*(\beta \mathcal{N})$ induces a partial isomorphism on $H_L$. Under it, $(U')$ is isometric to an $L_1\mathcal{N}$-space of vector functions on $\beta \mathcal{N}$ and $U'$ goes into zero.

1.4. Extensions of $L_m$. (a) Let $L$ be a given generalized limit. For each element $(a_n) \in \Sigma(\mathcal{H})$, the equation $F(y) = L(y, a_n)$ defines a continuous linear functional on $H$. Thus there is a unique $h \in H_L$ so that $F(y) = (y, h)$. Define $L_m(a_n) = h$. The operation of $L_m$ on $\Sigma(\mathcal{H})$ is evidently linear. If $(a_n) \rightarrow k \in H_L$ in the weak topology,

$$L_m(a_n) = (y, k) = (y, L_m[a_n])$$

for every $y \in H$. Thus $L_m(a_n) = k$. In this sense, $L_m$ is still a generalized limit. Also, for $y \in H$, $x = (a_n) \in \Sigma(\mathcal{H})$

$$|y, L_m(a_n)| = |L_m[(y, a_n)]| = |L_m[|y^*| |a_n||]| \leq L_m[|y^*||a_n|].$$

Thus $L_m$ may be extended (by linearity and continuity) to $H_L$.

(b) Bourbaki [3] develops a theory of integration for vector functions $f$ with respect to a scalar measure $\alpha$. This theory is especially simple when applied to Hilbert space valued functions continuous on and supported in a compact set. Then

$$\left( \int f \, da, h \right) = \int (f(t), h) \, da(t)$$

for all $h \in H$. Applying the Stone-Čech extension to $x = (a_n)$, we define

$$L_m(a_n) = \int_{\beta \mathcal{N}} a^*(\beta \mathcal{N})(t) \, da(t),$$

where $a$ is the measure on $\beta \mathcal{N}$ associated with the generalized limit $L$ in the sense of the previous section. Again, if $(a_n) \rightarrow k$ weakly, then $(a_n) \rightarrow (h, k)$ for each $h \in H$. On the other hand, $a^*$ is at least weakly continuous on $\beta \mathcal{N}$, and therefore the scalar sequence $(a_n, k)$ has the extension $(a^*(\beta \mathcal{N}), k)$. It is not difficult to see that a scalar sequence which converges to zero must have an extension on $\beta \mathcal{N}$ which vanishes on $\beta \mathcal{N} - \mathcal{N}$. Thus $(a^*(\beta \mathcal{N}), k) = (h, k)$ for all $k \in \beta \mathcal{N} - \mathcal{N}$. Since $a$ must be supported on $\beta \mathcal{N} - \mathcal{N}$, $a^* = h$ almost everywhere and so $L_m(a_n) = h$. As before, we may extend $L_m$ to $H_L$.

(c) Let us consider only $U$-sequences for a moment. If we write $x_n = \sum (a_n, \varepsilon_k) \varepsilon_k$ (where $(\varepsilon_k)$ is a basis sequence), then the series must converge uniformly in $a$. It is reasonable to expect that an extension $L_m$ should satisfy

$$L_m(a_n) = \sum_{k} (\sum_{l} (a_n, \varepsilon_l) \varepsilon_l) \varepsilon_k.$$

Thus, for finite rank sequences

$$x_n = \sum_{k} (a_n, \varepsilon_k) \varepsilon_k$$

in $U$, we define

$$L_m(a_n) = \sum_{k} (\sum_{l} (a_n, \varepsilon_l) \varepsilon_l) \varepsilon_k.$$

$L_m$ is evaluated on the variable $a$, not $i$.) This map from a subset of $U$ into $H$ is bounded because

$$|L_m(a_n), y| = |\sum_{l} (\sum_{k} (a_n, \varepsilon_k) \varepsilon_k, y)| \leq \left( \sum_{k} (a_n, \varepsilon_k) \varepsilon_k, y \right) \leq (|\sum_{k} (a_n, \varepsilon_k) \varepsilon_k| ||y||) \leq |\sum_{k} (a_n, \varepsilon_k) \varepsilon_k| ||y||.$$
Formulas and text from the image are readable and clear. The text discusses mathematical concepts and theorems, likely from a field such as functional analysis. The notation and symbols are consistent with advanced mathematical literature. The content involves sequences and operators, with references to properties and convergence. The text is structured in paragraphs, with mathematical expressions and theorems presented in a logical sequence.
From this, it is evident that none of the usual notions of proximity apply. If \( T \) is a bounded linear map on \( H \), the relation \( L(Tx_n, y) = L(x_n, Ty) \) shows that \( T \) will preserve almost convergence. The following theorem shows that a sort of converse is true.

2.1.2. Theorem. Let \( T \) be a map from \( H \) into \( H \), continuous in the norm topology, and suppose \( T(0) = 0 \). Then a necessary and sufficient condition that \( T \) preserve almost convergence is that \( T \) be additive and real homogeneous.

Proof. Sufficiency. Write \( M(x) = (T(x) + T(2x)) - \langle x, T(2x) \rangle \). Then \( M \) and \( N \) are additive; \( M(ax) = aM(x) \) and \( N(ax) = aN(x) \) for all complex \( a \). Clearly, \( M \), being an ordinary operator, will preserve almost convergence. On the other hand, associated with the conjugate homogenous transformation \( N \) there is a well defined conjugate homogenous transformation \( N^- \) which satisfies \( \langle N(x), y \rangle = \langle N^- y, x \rangle \) for every \( x, y \in H \). Thus for any \( L \) and \( y \in H \)

\[
L(Nx_n, y) = L([a, N^- y]) = L(a, N^- y) = \langle x, N^- y \rangle = (N, y)
\]

whence \( \langle x, N^- y \rangle = \langle x, y \rangle \) almost converges also. Since \( T(x) = (1/2)[M(x) + N(ax)] \), \( T \) will preserve almost convergence.

Necessity. The sequence

\[
x_n = \begin{cases} 0 & \text{for } n \neq kp, \\
q & \text{for } n = kp, h \neq 0 \text{ in } H
\end{cases}
\]

is a \( U \)-sequence of period \( p \). We have \( (x_n) \gg \|p\| \), and so \( T(x_n) \gg T(k/p) \).

However,

\[
T(x_n) = \begin{cases} 0 & \text{for } n \neq kp, \\
q & \text{for } n = kp
\end{cases}
\]

so \( T(x_n) \gg T(k/p) \). We have thus shown that \( T(k/p) = T(k/p) \) when \( p \) is a positive integer. The sequence \( x, y, z, y, z, \ldots \) where \( x, y, z \in H \) is almost convergent to \( \langle 1/2 \rangle \{x + y\} \). But then \( T(x), T(y), \ldots \) is almost convergent to \( T(\langle 1/2 \rangle \{x + y\}) \) by the preservation assumption. We always have \( T(x), T(y), \ldots \gg \langle 1/2 \rangle \{T(x) + T(y)\} \), and so \( \langle 1/2 \rangle \{T(x) + T(y)\} = \langle 1/2 \rangle \{T(x + y)\} \), whence \( T \) is additive. Now the additivity and continuity combine to give real homogeneity.

2.2. Averages of non-\( U \) sequences. We have observed that the generalized limit of every weakly convergent to zero sequence is zero. Part 3 of Theorem 2.1.1 falls in the case of such sequences. For instance, if \( \{\} \) denotes the greatest integer function, the sequence \( y_n = e_{y_n} \) converges weakly to zero and the averages converge in norm to zero also — but not in a translation invariant fashion. An even stronger example is provided by the following sequence whose norm averages do not converge to zero. We define a sequence of integers \( n_i \) as follows. Let \( n_1 = 1 \), \( n_i \) chosen so that

\[
\left\| \frac{n_1 e_1 + \ldots + n_i e_i}{n_1 + \ldots + n_i} \right\| > \frac{1}{4}
\]

In general, let \( S_i = \sum_{j=1}^{i} n_j \). Then if we have determined \( n_1, \ldots, n_{i-1} \), we choose \( n_i \) so that

\[
\left\| \frac{n_1 e_1 + \ldots + n_i e_i}{S_{i-1} + n_i} \right\| > \frac{1}{4}
\]

This can be done because the norm term is at least as large as

\[
\left\| \frac{n_1 e_1}{S_{i-1} + n_i} \right\| > \frac{(n_i)^2}{(S_{i-1} + n_i)^2}
\]

and the fraction approaches \( 1 \) as \( n_i \) increases without bound. It is thus eventually greater than \( 1/2 \), and the sequence \( (n_i) \) is defined. Now let \( x_1 = e_1 \), \( x_n = e_{i+1} \) when \( n_i < n \leq n_{i+1} \).

Then

\[
\left\| \frac{x_1 + \ldots + x_{n_i}}{S_i} \right\| = \left\| \frac{n_1 e_1 + \ldots + n_i e_i}{S_i} \right\| > \frac{1}{4}
\]

and so the averages cannot converge to zero in norm.

3.1. Periodic sequences. Any periodic vector sequence is in \( U \) because its range is finite-dimensional. It will have periodic basis component sequences; \( (x_{i+p}, e_i) = (x_i, e_i) \) for \( i = 1, 2, \ldots \). Thus the generalized limits of such are computable from the averages of their components by the formula

\[
\{x_i\} = \lim_{i \to \infty} L(x_i, e_i) e_i
\]

Conversely, if we are given that \( (x_i, e_i) \) is periodic with period \( p_i \) for \( i = 1, 2, \ldots \), we might expect that the generalized limit is computable according to \((*)\). We now consider conditions under which this is possible.

3.1.1. Theorem. A vector sequence \( (x_n) \) is periodic if and only if \( (x_n, y) \) is periodic for every \( y \in H \).
Proof. We need to establish only the sufficiency. Let

\[ E_k = \{ h \in H : (x_{n+k} - x_n, h) = 0 \text{ for all } n \}. \]

Then \( E_k \) is a closed set and every \( h \in H \) belongs to some \( E_k \). Thus \( H = \bigcup_k E_k \) and by the Baire Category Theorem, at least one \( E_k \) contains a non-void open set, i.e. a sphere \( S \) of radius \( r > 0 \). When \( h \in S \subset E_k \), \( (x_{n+k} - x_n, h) = 0 \). Let \( g \) be interior to \( S \) so that \( 0 \) is interior to \( S - g \). If \( k' \) is chosen so that \( (x_{n+k} - x_n, g) = 0 \) for all \( n \), then \( k' = k \) has the property that \( (x_{n+k} - x_n, f - g) = 0 \) for every \( f \in S \). Thus \( (x_{n+k} - x_n, z) = 0 \) for all \( z \) in a neighborhood of zero. It now follows that \( (x_{n+k} - x_n, z) = 0 \) for all \( z \in H \), and so \( k \) is a period for the sequence \( (x_n) \).

The hypothesis of this theorem cannot be weakened to assert that \( (x_n) \) is periodic whenever \( (x_n, e) \) is periodic for each \( e \) in a basis set. In the following computation, \( L \) always operates on the variable \( n \).

Let \( C^*_n \) be the characteristic function of multiples of \( i \), that is \( C^*_n = 1 \) if \( i \) divides \( n \), and zero otherwise. Then \( L^{-1} C^*_n \) has period \( i \) (as a function of \( n \)). We may define

\[ a_n = \frac{1}{i} \sum_{i} C^*_n \epsilon_i = L^{-1}(\sum_{i} \epsilon_i) \epsilon_i. \]

The straightforward verification that this is a \( U \)-sequence depends on the fact that \( \sum^{1}_{i=1} \epsilon_i \) converges. For \( x = (a_n) \), we have

\[ L([a_n]) = \sum_i L((a_n, \epsilon_i)) = \sum_i \left( \frac{1}{i} C^*_n \right) = \sum_i \frac{1}{i}, \]

so the sequence is non-trivial. It is not periodic, however. If we view it in the spatial context of \( U \subset H \), even more is true. Let \( T \) be the shift operator defined in 2.1. Then there is no \( p \) so that \( T^p x = x \) in \( H_0 \) (regardless of which \( L \) is chosen). Thus, we assert \( L([a_{n+p} - a_n]) \neq 0 \) for every \( p \).

We have

\[ L([a_{n+p} - a_n]) = \sum_i \frac{1}{i} [C^*_{n+p} - C^*_n], \]

if and only if every term vanishes.

\[ |C^*_{n+p} - C^*_n| = \begin{cases} 1 & \text{if } i | n+p \text{ and } i \nmid n, \\ 1 & \text{if } i | n \text{ and } i \nmid n+p, \\ 0 & \text{otherwise}. \end{cases} \]

If \( p \) is given, we choose \( i \) so that \( i \nmid p \). Then for \( n = ki, i | n \), but \( i \nmid n+p \). Thus, for \( n = ki \), the value of the sequence is 1, so

\[ L([C^*_{n+p} - C^*_n]) \geq L(C^*_n) = 1/i > 0. \]

It is reasonable to expect, however, that this sequence inherits some sort of periodicity, and it does turn out to be an almost periodic sequence. We establish this in the next section.

3.2. Almost periodicity. The generalized limits of almost periodic scalar sequences [6] may be obtained from their mean value:

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_{n+i}. \]

Our object in extending this notion to vector sequences is to characterize the latter in terms of their components and to show (by virtue of the fact that such sequences are in \( U \)) that the component-wise computation of their generalized limits is valid.

Definition. A subset \( S \) of the positive integers is said to be relatively dense in \( N \) if there is a number \( m \) so that each interval \([k, k+m]\) there is at least one member of \( S \).

Definition. Let \( (a_n) \) be a sequence in a metric space whose metric is denoted by \( d(\cdot, \cdot) \). The integer \( p \) is called an \( \varepsilon \)-translation number of \( (a_n) \) provided

\[ \sup_{n} d(a_{n+p}, a_n) \leq \varepsilon. \]

We denote the set of \( \varepsilon \)-translation numbers of \( (a_n) \) by \( E(\varepsilon, a_n) \).

Definition. A sequence \( (a_n) \) in a metric space will be called almost periodic (AP) provided for each \( \varepsilon > 0 \) the set \( E(\varepsilon, a_n) \) is relatively dense in \( N \).

This definition is different from the one usually given for functions on semi-groups in that it excludes sequences which converge to zero. In more general circumstances, if almost periodicity is defined in terms of compactness of translates, the set of AP semi-group functions is the direct sum of functions almost periodic on a group and functions which vanish at infinity. This is, in particular, true for the semi-group of positive reals and the positive integers [5]. For our purposes, the sequences which converge to zero pose no question as to summability — they are merely superfluous. It should be noted also that our definitions are quite similar to H. Bohr's classic formulation of an AP real function. Some of his proofs will apply with minor modifications. Thus it is almost immediate that an almost periodic sequence is bounded. On the other hand the discrete topology of the integers eliminates the possibility of employing uniform continuity to produce an "interval" of translation numbers about zero. For this reason, it is not immediately apparent that two almost periodic sequences will have any \( \varepsilon \)-translation numbers in com-
non—hence their sum is not clearly AP. The converse of the following lemma is the definition of almost periodicity frequently employed.

We will denote by $\Sigma(H)_\infty$ the space $\Sigma(H)$ with supreme norm:

$$||x||_\infty = \sup_n |x_n|.$$ 

If $M$ is a positive integer (or zero) $x^M$ is the $M$-th translate of the sequence $x$, so $x^M = x + M \cdot r$.

3.2.2. LEMMA. If $x = (x_n)$ is AP, then the translates $\{x_n^M\}_{M=0}^\infty$ form a totally bounded set in $\Sigma(H)_\infty$.

Proof. For $\epsilon > 0$ given, let $m(\epsilon)$ be the least length of an interval in which some member of $E[x, \epsilon]$ must lie. Then the sequences $x, x^2, \ldots, x^{m(\epsilon)}$ are an $\epsilon$-mesh for the set $\{x_n^M\}_{M=0}^\infty$.

Proof. We need to show that $(x_n)$ is contained in a totally bounded set in $H$. But if $j$ and $k$ are greater than one,

$$||x_n - x_j||_M = \sup_{i \leq j} |(x_{n+i} - x_{j+i})_M| = 0.$$ 

The assertion follows from the previous lemma.

Now we can complete the example of 3.1 by showing the non-periodic sequence defined therein is almost periodic. Write

$$||x_{n+p} - x_n||^2 = \sum_{i=0}^{k-1} (x_{n+i} - x_n)^2 + \sum_{i=k}^{n+p-1} (x_{n+i} - x_n)^2.$$ 

Since $(x_n)$ and $(x_{n+p})$ are both $U$-sequences, the second term can be made smaller than $\epsilon$ for large enough $k$, independent of $p$ or $n$. Having thus determined $k$, we consider $(x_n, \epsilon)$ for $i < k$. These sequences have period $i$ and so the first $k$ summands above will vanish if $p$ is any number divisible by all $1 \leq k$. It follows that $j - k$, where $j = 1, 2, \ldots$ is a relatively dense set of $\epsilon$-translation numbers for $(x_n)$.

In order to prove the following theorem, we need three additional statements, the verification of which we leave to the reader:

1) If $(x_n)$ is AP, then so is $(x_n^k)$ for each $K$, and for each $\epsilon > 0$, $E[x, \epsilon] = E[x^k, \epsilon]$. 

2) If $x = (x_n)$ is AP, then $||x||_M = ||x^M||_M$.

3) If $x$ is AP, then $x - x^M$ is also.

3.2.3. THEOREM. Let $(x_n)$ and $(y_n)$ be AP. Then for $\epsilon > 0$, there is a relatively dense set of translation numbers which is contained in $E(x, \epsilon) \cap E(y, \epsilon)$. Thus $(x_n) + (y_n)$ is AP also.

Proof. Because the sets $\{x_n^M\}_{M=0}^\infty$ and $\{y_n^M\}_{M=0}^\infty$ are both totally bounded in $\Sigma(H)_\infty$, for $\epsilon > 0$ there is a set $\{n_1, \ldots, n_k\}$ and a set $\{m_1, \ldots, m_l\}$ so that for each integer $M^*$ we have an $n_i$ and an $m_j$ with $||x_{n_i} - y_{m_j}||_\infty \leq \epsilon/4$.

Define

$$A_i = \{M : ||x_n^M - y_{m_j}||_\infty \leq \epsilon/4\},$$

$$B_i = \{M : ||y_n^M - x_{n_i}||_\infty \leq \epsilon/4\}.$$ 

Let $C_i, \ldots, C_r$ denote the non-void intersections $A_i \cap B_i$, as $i = 1, \ldots, r$. (There are some non-void intersections; e.g. $M^*$ above belongs to $A_1 \cap B_1$. Further, by the fact that both sets of translates are totally bounded, every $M$ belongs to some such intersection.) Now choose a particular $\epsilon_i \in C_i$. Let $d$ be any integer and say $d \in A_1 \cap B_2 = C_2$. Then

$$\|x_n^d - y_{m_j}^d\|_\infty \leq \|x_n^d - y_{m_j}^d\|_\infty + \|y_{m_j}^d - y_{m_j}^d\|_\infty,$$

$$\|y_{m_j}^d - x_{n_i}^d\|_\infty \leq \|y_{m_j}^d - y_{m_j}^d\|_\infty + \|y_{m_j}^d - x_{n_i}^d\|_\infty.$$ 

Since $\epsilon_i, d \in C_2$, the second terms on the right are less than $\epsilon/4$ by definition. By statements 2) and 3) above we see that if $d > \epsilon_i^2$

$$\|x_n^{d-\epsilon_i^2}\|_\infty \leq \|x_n^{d-\epsilon_i^2}\|_\infty \leq \epsilon/2,$$

and

$$\|y_{m_j}^{d-\epsilon_i^2}\|_\infty \leq \|y_{m_j}^{d-\epsilon_i^2}\|_\infty \leq \epsilon/2.$$ 

Thus the set $d - \epsilon_i^2$, where $\epsilon_i$ is appropriately chosen, is a set of $\epsilon$-translation numbers for both $(x_n)$ and $(y_n)$. As $\epsilon_i$ ranges over the integers, this set is seen to be relatively dense. The conclusion of the theorem is now immediate.

Note that the sequences $(||x_n||), (||y_n||), ((x_n, \epsilon), (y_n, h))$, $k$ fixed, are AP whenever $(x_n)$ is.

Now we can characterize AP-vector sequences in terms of their basis components.

3.2.4. THEOREM. A $U$-sequence is AP if and only if its components with respect to a fixed basis are AP-vector scalar sequences.

Proof. The necessity is obvious. For the sufficiency, we extend 3.2.3 by induction; so we can find for any finite set of almost period sequences (vector or scalar) a common relatively dense set of $\epsilon$-translation numbers. Thus, let $\epsilon > 0$ be given and suppose for $x = (x_n) \in U$ that $(x_n, \epsilon)$ is AP for each $i$. Expanding $||x_n^M - x_n||_\infty$ as usual, we make the tail of the series small (independent of $n$ or $p$) by choosing the index of summation sufficiently large. The norm of the difference is therefore approximated by

$$\sum_{i=0}^{k-1} (||x_{n+p}|| - ||x_n||)^k.$$
Then there is a relatively dense set of integers $S$ so that $p \epsilon S$ implies each summand in the finite sum is less than $\varepsilon/2k$. Thus $\|\xi_{m_0} - x_0\| < k/\varepsilon + \varepsilon/2 = \varepsilon$ and $(\xi_n)$ is AP.

Among U-sequences, then, the almost periodic ones are the ones whose basis components are AP-scalar sequences, and $U^1(x_n) = \xi_n [L(x_n, e_n)]e_n$, where $L(x_n, e_n)$ is the mean value of the sequence.

An immediate corollary to the above theorem is that a U-sequence is AP if and only if $(x_n, y)$ is AP for each $y \in H$. Unfortunately, a complete analog of Theorem 3.2.1 cannot be proved. That is, we cannot drop the hypothesis that the sequence be in U already. Let $x_n = e_n$ whenever $k \equiv 2^n - 1 \ (\text{mod } 2^{n+1})$, $n = 0, 1, 2, \ldots$ This sequence has component sequences $a_n = (e_n, e_n)$, each with period $2^{n+1}$. The vector sequence is not AP however, simply because it is not in U. It is also easy to see, that for each fixed $y$, $(a_n, y)$ will be AP also. Hence, neither Theorem 3.2.4 nor its corollary will be true if we drop the assumption that the sequence has range contained in a compact set.

References


LOUISIANA STATE UNIVERSITY

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Uniformly convex and reflexive modulated variation spaces

HANS-HEINRICH HERDA (Salem, Massachusetts)

§ 1. Introduction. In chapter III, entitled "Modular Spaces", of his treatise [3], Professor Hidegorô Nakano presents a theory of modulars on arbitrary (not necessarily semi-ordered) linear spaces. Namely, given any linear space $X$, a functional $m(x)$ defined on $X$ with values $0 \leq m(x) \leq + \infty$ is called a Nakano modular if

- $m(0) = 0$,
- $m(yx) = m(x)$,
- $m(x + y) \leq m(x) + m(y)$,
- $m(x) = 0$ whenever $x = 0$,
- $m(\alpha x) = \alpha m(x)$,
- $m(x + y) \leq \sup_{s \in [a,b]} m(x)$.

The space $X$ associated with the functional $m(x)$ is called a Nakano modular space.

It is easy to see that, for example, the $p$th power variations (as basic papers, see [4] or [2]) are special cases of Nakano modulars on generalized variation spaces. In this paper we are concerned with a new class of spaces which include the $p$th power variation spaces. Let $x$ be a real function in $[a,b]$ such that $x(a) = 0$, let $p(t, s, k)$ be a reall function of two real variables such that $t, s \in [a,b], t > s, k, 1 \leq p(t, s) < + \infty$; let $n: a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a,b]$. Define

$$B_{p(t)} = \{x: V_{p(t)}(x) = \sup_{s \in [a,b]} \left[ \sum_{k=1}^{n} \frac{p(t_k, s, k)}{p(t_k, t_{k-1})} \right] \leq 1, [p(t) < + \infty] \},$$

and denote by $B_{p(t), k}$ the linear space generated by $B_{p(t)}$. Here, $V_{p(t)}$ is the Nakano modular on the space $B_{p(t), k}$. If $p(t, s) = p = \text{constant}$ $(1 \leq p < + \infty)$, we have the case of $p$th variation. The spaces $B_{p(t), k}$ generalize the classical $L_p$-spaces (see [3], p. 234-240). In fact, the methods employed in the present paper, although they are perhaps not widely known, are essentially due to Nakano.