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A remark on the weak-star topology of l^∞

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The purpose of this note* is to present examples of a certain phenomenon associated with weak-star topologies. Although the phenomenon has been understood abstractly since the time of Banach, the literature contains few concrete examples.

Let M be a linear manifold in the dual of a separable Banach space. Let $M^0 = M$, and for each countable ordinal number α, let M^α be the set of all limits of weak-star convergent sequences in $\bigcup_{\beta<\alpha} M^\beta$. Then the set $M^- = \bigcup_\alpha M^\alpha$ is the weak-star closure of M, and there is a least countable ordinal ν, called the *order* of M, such that $M^- = M^\nu$ ([2], p. 213).

Mazurkiewicz was the first to exhibit a linear manifold of order greater than 1; his manifold is in l^1 $(= c_0^*)$ and it has order 2 [6]. Later Banach constructed linear manifolds in l^1 of all finite orders ([2], p. 209), and recently McGehee has shown that l^1 contains linear manifolds of all orders [7]. The present author has shown that the spaces H^∞ and l^∞ contain linear manifolds of all orders [8].

The examples to be presented here are of linear manifolds of all orders in the space l^∞; they are much simpler than any of the examples mentioned above. A modification of the construction produces analogous examples in the space L^∞ $[0,1]$.

The construction is based on a theorem about polynomial approximation. To prove this theorem we need the following special case of a theorem of Banach ([2], p. 213):

THEOREM. *Let B be a separable Banach space and M a linear manifold in B^*. Let M^- be the weak-star closure of M. Assume that for each f in B,*

$$(1)\qquad \sup\{|\langle\Phi, f\rangle| : \Phi \in M,\ \|\Phi\| \leqslant 1\} = \sup\{|\langle\Phi, f\rangle| : \Phi \in M^-,\ \|\Phi\| \leqslant 1\}.$$

Then each Φ in M^- is the weak-star limit of a sequence of elements in M whose norms are uniformly bounded by $\|\Phi\|$.

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Short proofs of this result can be found in [3], p. 1062, and [1] Appendix I.

We now state and prove the approximation theorem. Let C be the unit circle and D the open unit disk in the complex plane.

THEOREM. *Let μ be a finite, positive, singular Borel measure on C. Let φ be any function in $L^\infty(\mu)$ and ψ any bounded analytic function in D. Then there is a sequence $\{p_n\}$ of polynomials, uniformly bounded on C by* $\max(\|\varphi\|_\infty, \|\psi\|_\infty)$, *such that $p_n \to \varphi$ in the weak-star topology of $L^\infty(\mu)$ and $p_n \to \psi$ pointwise in D.*

Proof. Let m be Lebesgue measure on C and let $\lambda = m+\mu$. We regard $L^1(\lambda)$ as the direct sum $L^1(m)\oplus L^1(\mu)$ and $L^\infty(\lambda)$ as the direct sum $L^\infty(m)\oplus L^\infty(\mu)$. Let M be the set of all polynomials, regarded as a linear manifold in $L^\infty(\lambda)$. If h is a function in $L^1(\lambda)$ that annihilates M, then it follows by the F. and M. Riesz theorem that the measure $hd\lambda$ is absolutely continuous with respect to m, in other words, h is in $L^1(m)$. Hence h annihilates $H^\infty(m)\oplus L^\infty(\mu)$, and, as the latter subspace is weak-star closed, we can conclude that $M^- = H^\infty(m)\oplus L^\infty(\mu)$.

Because of the preceding equality and Banach's theorem, we can complete the proof by showing that (1) holds for each f in $L^1(\lambda)$. Let f be given, and let L and R denote the quantities on the left and right sides of (1). By the Hahn-Banach and Riesz representation theorems, there is a measure ν on C such that $\|\nu\| = L$ and

$$\int p\,d\nu = \int fp\,d\lambda, \qquad p\in M. \tag{2}$$

The F. and M. Riesz theorem implies that the measure $d\nu - fd\lambda$ is absolutely continuous with respect to m, and therefore ν is absolutely continuous with respect to λ. Thus we can conclude from (2) and the weak-star density of M in M^- that

$$\int \Phi\,d\nu = \int f\Phi\,d\lambda, \qquad \Phi\in M^-.$$

It follows that $R \leqslant \|\nu\| = L$, and hence $R = L$. The proof is complete.

We shall need the following special case of the approximation theorem:

COROLLARY. *Let ψ be a bounded analytic function in D, $\{z_k\}$ a sequence of distinct points on C, and $\{w_k\}$ a bounded sequence of complex numbers. Then there is a sequence $\{p_n\}$ of polynomials, uniformly bounded on C by*

$$\max(\|\psi\|_\infty, \sup_n |w_n|),$$

such that $p_n \to \psi$ pointwise in D and $p_n(z_k) \to w_k$ for each k.

The above proof of the approximation theorem is an adaptation of the proof in [1], Appendix II. The corollary is a special case of the theorem proved in [1], Appendix II, and also of a related theorem of Glicksberg [4].

We can now give the promised examples.

THEOREM. *There exist in l^∞ weak-star dense linear manifolds of all possible orders.*

Proof. We consider in detail the case of order 2; the construction here was used by Wermer [9] for a similar purpose. The general case is based on the same ideas and will only be sketched.

Let C_1 and C_2 be circles in the complex plane centered at the origin, with C_1 having the larger radius. Let S be a countable subset of $C_1\cup C_2$ that is dense in C_1 and has at least one limit point on C_2. The space l^∞ can then be identified with $l^\infty(S)$, the space of bounded complex-valued functions on S. We can regard $l^\infty(S)$ as the direct sum $l^\infty(S\cap C_1)\oplus \oplus l^\infty(S\cap C_2)$.

Let M be the set of functions on S that are restrictions of polynomials. Suppose Φ is a function in the manifold M^1. Then there is a sequence $\{p_n\}$ of polynomials such that $p_n|S\to\Phi$ in the weak-star topology of $l^\infty(S)$. This means that the sequence $\{p_n\}$ is uniformly bounded on S and converges to Φ at each point of S. Since S contains a dense subset of C_1, the polynomials p_n must be uniformly bounded on D_1, the interior of C_1. Hence $\Phi|(S\cap C_2)$ is the restriction of a function in $H^\infty(D_1)$ (the space of bounded analytic functions on D_1), and we have the inclusion

$$M^1 \subset l^\infty(S\cap C_1)\oplus H^\infty(D_1)|(S\cap C_2).$$

From the above corollary it is immediate that the inclusion is actually an equality. It is easy to see that, because S contains a limit point on C_2, the manifold M^1 does not contain the restriction to S of the function $\bar z$; thus $M^1\neq l^\infty(S)$.

A second application of the corollary shows that every function in $l^\infty(S)$ is the pointwise limit of a bounded sequence in M^1, so that $M^2 = l^\infty(S)$, as desired.

To prove the theorem in general, let ν be a countable ordinal number. Then we can find a one-to-one order reversing map $\alpha\to r_\alpha$ from the set of ordinals $\leqslant\nu$ into the positive real axis. For each α let C_α be the circle with center at the origin and radius r_α, and let D_α be the interior of C_α. Let S be a countable subset of $\bigcup C_\alpha$ such that $S\cap C_\alpha$ is dense in C_α for every $\alpha<\nu$, and such that S has at least one limit point on C_ν. (If ν is a limit ordinal the last condition can be deleted.) As before, we can identify l^∞ with $l^\infty(S)$. Let M be the set of functions on S that are restrictions of polynomials. By the reasoning used above for the special case $\nu=2$, one can show by induction that

$$M^\alpha = l^\infty(S-D_\alpha)\oplus H^\infty(D_\alpha)|(S\cap D_\alpha), \qquad \alpha<\nu,$$
$$M^\nu = l^\infty(S).$$

It is easy to check that $M^{\alpha} \neq l^{\infty}(S)$ for $\alpha < \nu$, and therefore M has order ν, as desired.

A similar construction gives the following result:

THEOREM. *There exist in $L^{\infty}[0,1]$ weak-star dense linear manifolds of all possible orders.*

Proof. Let ν be a countable ordinal number, and let circles C_{α} and disks D_{α} be defined as in the preceding proof. Let μ be a purely nonatomic Borel probability measure on $S = \bigcup C_{\alpha}$ such that for each α, the restriction of μ to C_{α} is singular with respect to Lebesgue measure on C_{α} and has support equal to all of C_{α}. The measure space (S, μ) is then isomorphic to the unit interval with Lebesgue measure ([5], p. 173), so that $L^{\infty}[0,1]$ can be identified with $L^{\infty}(\mu)$. For each α let μ_{α} be the restriction of μ to $S-D_{\alpha}$; we thus have direct sum decompositions $L^{\infty}(\mu) = L^{\infty}(\mu_{\alpha}) \oplus \oplus L^{\infty}(\mu-\mu_{\alpha})$.

Let M be the set of all polynomials, regarded as a linear manifold in $L^{\infty}(\mu)$. Suppose Φ is a function in M^1. Then Φ lies in the weak-star closure, and therefore in the weak $L^2(\mu)$ — closure, of some ball in M. Hence Φ is in the strong $L^2(\mu)$ — closure of the same ball in M, so that there is a sequence $\{p_n\}$ of polynomials, uniformly bounded on S, which converges to Φ almost everywhere modulo μ. The polynomials p_n are then uniformly bounded in D_1, and thus $\Phi|(S \cap D_1)$ is the restriction of a function in $H^{\infty}(D_1)$. We therefore have

$$M^1 \subset L^{\infty}(\mu_1) \oplus H^{\infty}(D_1)|(S \cap D_1),$$

and an application of the approximation theorem shows that the inclusion is actually an equality. Using the same reasoning, one can show by induction that

$$M^{\alpha} = L^{\infty}(\mu_{\alpha}) \oplus H^{\infty}(D_{\alpha})|(S \cap D_{\alpha}), \qquad \alpha < \nu,$$
$$M^{\nu} = L^{\infty}(\mu).$$

It is easily seen that $M^{\alpha} \neq L^{\infty}(\mu)$ for $\alpha < \nu$, and thus M has order ν, as desired.

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