A remark on the weak-star topology of $l^\infty$

by

DONALD S. ARASON (Berkeley, Calif.)

The purpose of this note is to present examples of a certain phenomenon associated with weak-star topologies. Although the phenomenon has been understood abstractly since the time of Banach, the literature contains few concrete examples.

Let $M$ be a linear manifold in the dual of a separable Banach space. Let $M^\alpha = M$, and for each countable ordinal number $\alpha$, let $M^\alpha$ be the set of all limits of weak-star convergent sequences in $\bigcup M^\alpha$. Then the set $M^- = \bigcup M^\alpha$ is the weak-star closure of $M$, and there is a least countable ordinal $\xi$, called the order of $M$, such that $M^- = M'$ (12, p. 213).

Marcinkiewicz was the first to exhibit a linear manifold of order greater than 1; his manifold is in $l^\infty (\sim c_0)$ and it has order 2 [8]. Later Banach constructed linear manifolds in $l^\infty$ of all finite orders [2], p. 209, and recently McEachin has shown that $l^1$ contains linear manifolds of all orders [7]. The present author has shown that the spaces $H^\infty$ and $l^\infty$ contain linear manifolds of all orders [8].

The examples to be presented here are of linear manifolds of all orders in the space $l^\infty$; they are much simpler than any of the examples mentioned above. A modification of the construction produces analogous examples in the space $L^\infty [0, 1]$.

The construction is based on a theorem about polynomial approximation. To prove this theorem we need the following special case of a theorem of Banach [2], p. 213):

**Theorem.** Let $B$ be a separable Banach space and $M$ a linear manifold in $B^*$. Let $M^-$ be the weak-star closure of $M$. Assume that for each $f$ in $B$,

\[
\sup \{ |\langle \Phi, f \rangle | : \Phi \in M, \| \Phi \| \leq 1 \} = \sup \{ |\langle \Phi, f \rangle | : \Phi \in M^-, \| \Phi \| \leq 1 \}.
\]

Then each $\Phi$ in $M^-$ is the weak-star limit of a sequence of elements in $M$ whose norms are uniformly bounded by $\| \Phi \|$.\(^*\)

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Short proofs of this result can be found in [3], p. 1062, and [1], Appendix I.

We now state and prove the approximation theorem. Let \( C \) be the unit circle and \( D \) the open unit disk in the complex plane.

**Theorem.** Let \( \mu \) be a finite, positive, singular Borel measure on \( C \). Let \( \varphi \) be any function in \( L^p(\mu) \) and \( \psi \) any bounded analytic function in \( D \). Then there is a sequence \( \{p_n\} \) of polynomials, uniformly bounded on \( C \) by \( \max(\|p_n\|, \|\psi\|) \), such that \( p_n \rightarrow \varphi \) in the weak-star topology of \( L^p(\mu) \) and \( p_n \rightarrow \psi \) pointwise in \( D \).

**Proof.** Let \( m \) be Lebesgue measure on \( C \) and let \( \lambda = m + \mu \). We regard \( L^p(\lambda) \) as the direct sum \( L^p(m) \oplus L^p(\mu) \) and \( L^p(\lambda) \) as the direct sum \( L^p(m) \oplus L^p(\mu) \). Let \( M \) be the set of all polynomials, regarded as a linear manifold in \( L^p(\lambda) \). If \( h \) is a function in \( L^p(\lambda) \) that annihilates \( M \), then it follows by the F. and M. Riesz theorem that the measure \( h\lambda \) is absolutely continuous with respect to \( m \), in other words, \( h \) is in \( L^q(m) \). Hence \( h \) annihilates \( H^p(m) \), and, as the latter subspace is weak-star closed, we can conclude that \( M^* = H^q(m) \oplus L^q(\mu) \).

Because of the preceding equality and Banach's theorem, we can complete the proof by showing that (1) holds for each \( f \) in \( L^q(\lambda) \). Let \( f \) be given, and let \( L \) and \( R \) denote the quantities on the left and right sides of (1). By the Hahn-Banach and Riesz Representation theorems, there is a measure \( \nu \) on \( C \) such that \( \|\nu\| = 1 \) and

\[
\int \varphi d\nu = \int f d\lambda, \quad \nu \in M.
\]

The F. and M. Riesz theorem implies that the measure \( \text{d}v = \text{d}\lambda \) is absolutely continuous with respect to \( m \), and therefore \( v \) is absolutely continuous with respect to \( \lambda \). Thus we can conclude from (2) and the weak-star density of \( M \) in \( M^* \) that

\[
\int \varphi d\nu = \int f d\lambda, \quad \Phi \in M^*.
\]

It follows that \( \text{R} < \|\nu\| = L \), and hence \( \text{R} = L \). The proof is complete.

We shall need the following special case of the approximation theorem:

**Corollary.** Let \( \varphi \) be a bounded analytic function in \( D \), \( \{z_n\} \) a sequence of distinct points on \( C \), and \( \{w_n\} \) a bounded sequence of complex numbers. Then there is a sequence \( \{p_n\} \) of polynomials, uniformly bounded on \( C \) by \( \max(\|p_n\|, \sup |w_n|) \), such that \( p_n \rightarrow \varphi \) pointwise in \( D \) and \( p_n(z_n) \rightarrow w_n \) for each \( n \).

The above proof of the approximation theorem is an adaptation of the proof in [1], Appendix II. The corollary is a special case of the theorem proved in [1], Appendix II, and also of a related theorem of Glicksberg [4].

We can now give the promised examples.

**Theorem.** There exist in \( \mathbb{F} \) weak-star dense linear manifolds of all possible orders.

**Proof.** We consider in detail the case of order 2; the construction here was used by Wermer [9] for a similar purpose. The general case is based on the same ideas and will only be sketched.

Let \( C_1 \) and \( C_2 \) be circles in the complex plane centered at the origin, with \( C_2 \) having the larger radius. Let \( S \) be a countable subset of \( C_1 \cup C_2 \) that is dense in \( C_1 \) and has at least one limit point on \( C_2 \). The space \( \mathbb{F} \) can then be identified with \( \mathbb{F}^\circ(S) \), the space of bounded complex-valued functions on \( S \). We can regard \( \mathbb{F}^\circ(S) \) as the direct sum \( \mathbb{F}^\circ(S \setminus C_1) \oplus \mathbb{F}^\circ(S \cap C_1) \).

Let \( M \) be the set of functions on \( S \) that are restrictions of polynomials. Suppose \( \Phi \) is a function in the manifold \( M \). Then there is a sequence \( \{p_n\} \) of polynomials such that \( p_n|S \rightarrow \Phi \) in the weak-star topology of \( \mathbb{F}^\circ(S) \). This means that the sequence \( \{p_n\} \) is uniformly bounded on \( S \) and converges to \( \Phi \) at each point of \( S \). Since \( S \) contains a dense subset of \( C_1 \), the polynomials \( p_n \) must be uniformly bounded on \( D_1 \), the interior of \( C_1 \). Hence \( \Phi|S \setminus C_1 \) is the restriction of a function in \( \mathbb{F}^\circ(D_1) \) (the space of bounded analytic functions on \( D_1 \)) and we have the inclusion

\[
M \subset \mathbb{F}^\circ(S \setminus C_1) \oplus \mathbb{F}^\circ(D_1)|S \cap C_1|.
\]

From the above corollary it is immediate that the inclusion is actually an equality. It is easy to see that, because \( S \) contains a limit point on \( C_1 \), the manifold \( M \) does not contain the restriction to \( S \) of the function \( \bar{z} \); thus \( M \neq \mathbb{F}^\circ(S) \).

A second application of the corollary shows that every function in \( \mathbb{F}^\circ(S) \) is the pointwise limit of a bounded sequence in \( M \), so that \( M = \mathbb{F}^\circ(S) \), as desired.

To prove the theorem in general, let \( \nu \) be a countable ordinal number. Then we can find a one-to-one order reversing map \( \alpha \rightarrow \alpha \) from the set of ordinals \( \lessdot \alpha \) into the positive real axis. For each \( \alpha \) let \( C_\alpha \) be the circle with center at the origin and radius \( \alpha \), and let \( D_\alpha \) be the interior of \( C_\alpha \). Let \( S \) be a countable subset of \( C \cup C_\alpha \) such that \( S \cap C_\alpha \) is dense in \( C_\alpha \) for every \( \alpha < \nu \), and such that \( S \) has at least one limit point on \( C_\alpha \). (If \( \nu \) is a limit ordinal the last condition can be deleted.) As before, we can identify \( \mathbb{F}^\circ(S) \). Let \( M \) be the set of functions on \( S \) that are restrictions of polynomials. By the reasoning used above for the special case \( \nu = 2 \), one can show by induction that

\[
M = \mathbb{F}^\circ(S - D_\alpha) \oplus \mathbb{F}^\circ(D_\alpha)|S \cap D_\alpha|, \quad \alpha < \nu,
\]

Thus we have

\[
M = \mathbb{F}^\circ(S).
\]
It is easy to check that $M^* \neq L^\infty(S)$ for $a < \nu$, and therefore $M$ has order $\nu$, as desired.

A similar construction gives the following result:

**Theorem.** There exist in $L^\infty([0,1])$ weak-star dense linear manifolds of all possible orders.

**Proof.** Let $\nu$ be a countable ordinal number, and let circles $C_\alpha$ and disks $D_\alpha$ be defined as in the preceding proof. Let $\mu$ be a purely nonatomic Borel probability measure on $S = \bigcup C_\alpha$ such that for each $\alpha$, the restriction of $\mu$ to $C_\alpha$ is singular with respect to Lebesgue measure on $C_\alpha$, and has support equal to all of $C_\alpha$. The measure space $(S, \mu)$ is then isomorphic to the unit interval with Lebesgue measure ([3], p. 173), so that $L^\infty([0,1])$ can be identified with $L^\infty(\mu)$. For each $\alpha$ let $\mu_\alpha$ be the restriction of $\mu$ to $S - D_\alpha$; we thus have direct sum decompositions $L^\infty(\mu) = L^\infty(\mu_\alpha) \oplus \bigoplus L^\infty(\mu - \mu_\alpha)$.

Let $M$ be the set of all polynomials, regarded as a linear manifold in $L^\infty(\mu)$. Suppose $\Phi$ is a function in $M$. Then $\Phi$ lies in the weak-star closure, and therefore in the weak $L^\infty(\mu)$ closure, of some ball in $M$. Hence $\Phi$ is in the strong $L^\infty(\mu)$ closure of the same ball in $M$, so that there is a sequence $(\Phi_n)$ of polynomials, uniformly bounded on $S$, which converges to $\Phi$ almost everywhere modulo $\mu$. The polynomials $\Phi_n$ are then uniformly bounded in $D_\alpha$, and thus $\Phi(S \cap D_\alpha)$ is the restriction of a function in $H^\infty(D_\alpha)$. We therefore have $M^* \subseteq L^\infty(\mu_\alpha) \oplus H^\infty(D_\alpha)(S \cap D_\alpha)$, and an application of the approximation theorem shows that the inclusion is actually an equality. Using the same reasoning, one can show by induction that $M^* = L^\infty(\mu_\alpha) \oplus H^\infty(D_\alpha)(S \cap D_\alpha), \quad a < \nu, \quad M^* = L^\infty(\mu)$.

It is easily seen that $M^* \neq L^\infty(\mu)$ for $a < \nu$, and thus $M$ has order $\nu$, as desired.

**References**