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On Banach spaces containing $L_1(\mu)$

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Introduction. In the present paper we study consequences of the facts that a Banach space X or its conjugate X^* contain a subspace isomorphic (= linearly homeomorphic) to an $L_1(\mu)$ space. It is shown, in particular, that if a separable Banach space X contains a subspace isomorphic to l_1, then X^* is not only non-separable but also "sufficiently rich"; Namely X^* contains a w^*-closed subspace isomorphic to $(C[0,1])^*$ and therefore it contains a subspace isomorphic to $l_1(\Gamma)$ with card $\Gamma = 2^{\aleph_0}$. On the other hand, if a conjugate Banach space X^* contains a "nicely embedded" (precisely seminorming, cf. Definition 1.1) subspace isomorphic to $L_1(\mu)$ for some non-purely atomic measure μ, then X contains a subspace isomorphic to l_1. Combining these two facts with the easy lemma that if μ is σ-finite, then $L_1(\mu)$ does not contain any subspace isomorphic to $l_1(\Gamma)$ with Γ uncountable, we prove that if μ is σ-finite and non-purely atomic, then $L_1(\mu)$ is not isomorphic to any conjugate Banach space. This result generalizes Gelfand's theorem [8] that the space $L_1 = L_1[0,1]$ is not isomorphic to any conjugate Banach space, and gives a partial solution of the following problem raised by Dieudonné [5]: characterize those $L_1(\mu)$-spaces which are isomorphic to a conjugate Banach space. For various proofs of Gelfand's theorem the reader is reffered to the papers [2], [5], [15]-[17] and [20].

1. Preliminaries. Any unexplained notation will be that of either [3] or [7]. If A is a set, then card A denotes the cardinality of A.

Capital letters X, Y, Z denote Banach spaces. The first and the second conjugate of X are denoted by X^* and X^{**} respectively. By "subspace" we always mean a closed linear subspace. By "operator" — a bounded linear operator. If $u\colon X \to Y$ is an operator, then $u^*\colon Y^* \to X^*$ denotes the adjoint operator of u. By $\varkappa_X$ we denote the canonical embedding of X into X^{**}. An operator $u\colon X \to Y$ is called: an (*isometrically*) *isomorphic embedding* if there is $v\colon u(X) \to X$ such that for all x in X, $vu(x) = x$ (and $\|ux\| = \|x\|$); an *epimorphism* if $u(X) = Y$; an (*isometric*) *isomorphism* if u is an (isometrically) isomorphic embedding and an epimorphism. X is said to be *(isometrically) isomorphic* to Y if there is an (isometric) isomorphism from X onto Y. If X_1 is a subspace of X, then the map $x \to x+X_1$ from X onto the quotient space X/X_1 is called the *quotient map* or the *natural map*. A subspace Y of X is said to be *complemented* if there is a projection (= bounded linear idempotent) from X onto Y. A Banach space X is a $\mathscr{P}_1$-space if for each Banach space Y and for an arbitrary subspace Y_1 of Y every operator $u\colon Y_1 \to X$ has an extension $v\colon Y \to X$ (i.e. $uy = vy$ for $y \in Y_1$) with $\|v\| = \|u\|$.

In the conjugate space X^* we consider the norm topology and the X topology of X^* which is called sometimes the *w^*-topology*. This topology is obtained by taking as a base all sets of the form

$$N(x^*, V, \varepsilon) = \{y^* \in X^*\colon |x^*x - y^*x| < \varepsilon,\ x \in V\}$$

where $x^* \in X^*$, V is a finite subset of X, and $\varepsilon > 0$.

If A is a subset of X^*, then $\mathrm{cl}^*(A)$ denotes the closure of A in the w*-topology. In the sequal an important role plays the following concept.

Definition 1.1. A subspace Z of X^* is said to be *seminorming* if there is $c > 0$ such that

$$c(\mathrm{cl}^*(Z) \cap S^*) \subset \mathrm{cl}^*(Z \cap S^*),$$

where $S^* = \{x^* \in X^*\colon \|x^*\| \leqslant 1\}$ is the unit ball of X^*.

If moreover $\mathrm{cl}^*(Z) = X^*$, then Z is called *norming* (i.e. Z has positive characteristic in the sense of Dixmier [6]).

An isomorphic embedding $u\colon Y \to X^*$ is said to be *proper* (precisely proper with respect to the X topology of X^*) if $u(Y)$ is a seminorming subspace of X^*.

Proposition 1.2. *For every isomorphic embedding $u\colon Y \to X^*$ the following conditions are equivalent*:

(α) *u is proper*,

(β) *the range of the operator $v = u^*\varkappa_X\colon X \to Y^*$ is closed.*

Proof. (α) ⇒ (β). Let us set

$$X_1 = \{x \in X\colon (uy)(x) = 0 \quad \text{for } y \in Y\},$$
$$X_1^\perp = \{x^* \in X^*\colon x^*(x) = 0 \quad \text{for } x \in X_1\}.$$

Let $h\colon X \to X/X_1$ be the quotient map. Then h^* regarded as an operator from $(X/X_1)^*$ onto $X_1^\perp$ is an isometric isomorphism. Define the isomorphic embedding $u_1\colon Y \to (X/X_1)^*$ so that $h^*u_1 = u$. Since $u(Y)$ is seminorming and since $\mathrm{cl}^*[u(Y)] = X_1^\perp$, the subspace $u_1(Y)$ of $(X/X_1)^*$ is norming (because h^* is a continuous map if both spaces $(X/X_1)^*$ and $X_1^\perp$ carry the X/X_1 topology of $(X/X_1)^*$ and the X topology of X^* respectively). Hence there is $c > 0$ such that

$$\sup_{u_1y \in W} |(u_1y)(e)| \geqslant c\|e\| \quad \text{for} \quad e \in X/X_1,$$

where $W = u_1(Y) \cap \{e^* \in (X/X_1)^*\colon \|e^*\| \leqslant 1\}$. Since u_1 is an isomorphism, there is $a > 0$ such that $W \subset u_1\{y \in Y\colon \|y\| \leqslant a\}$. Thus

$$\sup_{\|y\| \leqslant 1} |(u_1y)(e)| \geqslant ca^{-1}\|e\| \quad \text{for} \quad e \in X/X_1.$$

The last inequality means nothing else but the fact that the operator $v_1 = u_1^*\varkappa_{X/X_1}$ is an isomorphic embedding. Observe that $v = v_1h$. Hence v is the composition of the epimorphism h with the isomorphic embedding v_1. Therefore $v(X) = v_1(X)$. Thus the range of v is closed.

(β) → (α). By Banach open map theorem ([7], p. 55), there exists $c > 0$ such that for each y^* in $v(X)$ there is an x_1 in X such that $vx_1 = y^*$ and $c\|x_1\| \leqslant \|y^*\|$. Suppose that $u(Y)$ is not seminorming. Then there is an element x^* in $\mathrm{cl}^*\big(u(Y)\big)$ with $\|x^*\| = 1$ such that $(2\|u\|)^{-1}cx^* \notin \mathrm{cl}^*\big(u(Y) \cap S^*\big)$. Observe that $\mathrm{cl}^*\big(u(Y) \cap S^*\big)$ is a w*-compact convex and symmetric set. Hence, by separation theorem ([7], p. 412), there is an x in X such that $\|x\| = x^*x = 1$ and $|(uy)(x)| < (2\|u\|)^{-1}c$ for $\|uy\| \leqslant 1$. Hence $\|vx\| \leqslant 2^{-1}c < c$. Choose an x_1 in X so that

$$\text{(o)} \qquad c\|x_1\| \leqslant \|vx\| < c,$$

$$\text{(oo)} \qquad vx_1 = vx.$$

It follows from (oo) that $(uy)(x) = (uy)(x_1)$ for all y in Y. Hence the condition $x^* \in \mathrm{cl}^*\big(u(Y)\big)$ implies that $x^*(x) = x^*(x_1) = 1$. Thus $1 \leqslant \|x^*\|\,\|x_1\| = \|x_1\|$. On the other hand, by (o), $\|x_1\| < 1$, a contradiction.

By $C(Q)$ we denote the space of all continuous real-valued functions on a compact Hausdorff space Q, with $\|f\| = \sup_{q \in Q} |f(q)|$.

Let μ be a measure defined on a σ-field Σ of subsets of a set T. By $L_1(\mu) = L_1(T, \Sigma, \mu)$ (resp. $L_2(\mu)$; $L_\infty(\mu)$) we denote the space of all μ-equivalence classes of μ-measurable functions such that

$$\|f\| = \int_T |f|\,d\mu < +\infty$$

$$\text{(resp. } \|f\| = \Big(\int_T |f|^2 d\mu\Big)^{1/2} < +\infty; \quad \|f\| = \operatorname{vrai\,sup} |f(s)| < +\infty).$$

If Γ is an abstract set, Σ is the field of all subsets of Γ, and μ is the unique measure on Σ such that $\mu(\{\gamma\}) = 1$ for all γ in Γ, then the space $L_1(\mu)$ is denoted by $l_1(\Gamma)$ (resp. $L_\infty(\mu)$ by $m(\Gamma)$). The space $l_1(\Gamma)$

(resp. $m(\Gamma)$) consists of all real functions $\{t(\gamma)\}_{\gamma\in\Gamma}$ such that

$$\|t(\cdot)\| = \sum_{\gamma\in\Gamma} |t(\gamma)| < +\infty \quad (\text{resp. } \|t(\cdot)\| = \sup_{\gamma\in\Gamma} |t(\gamma)| < +\infty).$$

If Γ is countable, then we shall write l_1 instead of $l_1(\Gamma)$. If $T = [0,1]$ is the unit interval, Σ — the field of all Borel subsets of $[0,1]$, and μ-the usual Lebesgue measure, then $L_1(\mu)$ is denoted by L_1 and $L_\infty(\mu)$ by L_∞. We shall write $[L_1(\mu)]^* = L_\infty(\mu)$ if the operator $\varphi \to x_\varphi^*$, where

$$x_\varphi^*(f) = \int_T \varphi f d\mu \quad \text{for} \quad f \in L_1(\mu)$$

is an isometric isomorphism from $L_\infty(\mu)$ onto $[L_1(\mu)]^*$. If μ is σ-finite, then $[L_1(\mu)]^* = L_\infty(\mu)$ ([7], p. 289). More sophisticated is the following result (cf. [7], p. 290):

PROPOSITION 1.3. *For every space $L_1(\mu)$ there is a space $L_1(\nu)$ isometrically isomorphic to $L_1(\mu)$ and such that $[L_1(\nu)]^* = L_\infty(\nu)$.*

Proof. By the Kakutani representation theorem (cf. [10], [3], p. 108), there is a collection of finite measures $\{\mu_\gamma\}_{\gamma\in\Gamma}$ such that there is an isometric isomorphism from $L_1(\mu)$ onto the l_1-product, $X = P_1(L_1(T_\gamma, \Sigma_\gamma, \mu_\gamma))$ (the elements of the space X are indexed families of functions $x = \{f_\gamma\}_{\gamma\in\Gamma}$ such that $f_\gamma \in L_1(\mu_\gamma)$ and $\|x\| = \sum_{\gamma\in\Gamma} \|f_\gamma\| < +\infty$). Moreover, this isometric isomorphism takes positive functions into families consisting of positive functions. Since all μ_γ are finite measures, we have $[L_1(\mu_\gamma)]^* = L_\infty(\mu_\gamma)$. Therefore, by general result on l_1-products (cf. [3], p. 31 (11)), we have $X^* = P_\infty(L_\infty(T_\gamma, \Sigma_\gamma, \mu_\gamma))$ (the elements of the space $P_\infty(L_\infty(\mu_\gamma))$ are indexed families of functions $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ such that $\varphi_\gamma \in L_\infty(\mu_\gamma)$ and $\|\{\varphi_\gamma\}\| = \sup_{\gamma\in\Gamma} \|\varphi_\gamma\|$). Precisely the operator $\{\varphi_\gamma\}_{\gamma\in\Gamma} \to x^*_{\{\varphi_\gamma\}}$, where

$$(1) \qquad x^*_{\{\varphi_\gamma\}}(\{f_\gamma\}) = \sum_{\gamma\in\Gamma} \int_{T_\gamma} f_\gamma \varphi_\gamma d\mu_\gamma, \qquad \{f_\gamma\}_{\gamma\in\Gamma} \in X,$$

is the isometric isomorphism from $P_\infty(L_\infty(\mu_\gamma))$ onto X^*.

Let us set (the sets T_γ are regarded to be mutually disjoint)

$$T = \bigcup_{\gamma\in\Gamma} T_\gamma,$$

$$\Sigma = \{A \subset T: A \cap T_\gamma \in \Sigma_\gamma,\ \gamma \in \Gamma\},$$

$$\nu(A) = \sum_{\gamma\in\Gamma} \mu_\gamma(A \cap T_\gamma).$$

Clearly, Σ is a σ-field of subsets of T and ν is a measure on Σ. Let $u: L_1(T, \Sigma, \nu) \to X$ be defined by

$$uf = \{f_{|T_\gamma}\} \quad \text{for} \quad f \in L_1(\nu),$$

where $f_{|T_\gamma}$ denotes the restriction of f to T_γ. By (1) for $x^* = \{\varphi_\gamma\}_{\gamma\in\Gamma} \in X^*$ we have

$$(2) \qquad (u^*x^*)(f) = x^*(uf) = \sum_{\gamma\in\Gamma} \int_{T_\gamma} \varphi_\gamma f_{|T_\gamma} d\mu_\gamma = \int_T \varphi f d\nu,$$

where $\varphi \in L_\infty(\nu)$ is the unique function such that $\varphi_{|T_\gamma} = \varphi_\gamma$ for $\gamma \in \Gamma$. Clearly, $\|x^*\| = \|\varphi\|$. Since u is an isometric isomorphism from $L_1(\nu)$ onto X, u^* is an isometric isomorphism from X^* onto $[L_1(\nu)]^*$. Hence $\|u^*x^*\| = \|x^*\| = \|\varphi\|$. Clearly, the operator $\varphi \to \{\varphi_{|T_\gamma}\}$ is an isometric isomorphism from $L_\infty(\nu)$ onto X^*. Thus, in view of (2), we conclude that $[L_1(\nu)]^* = L_\infty(\nu)$. That completes the proof.

2. Constructions of subspaces isomorphic to l_1 in norming subspaces of $[L_1(\mu)]^*$.

Definition 2.1. A double sequence $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{n=0}^\infty$ of non-empty sets is said to be a *dyadic tree* if

(i) if $k_1 \neq k_2$, then $A_{k_1}^{(n)} \cap A_{k_2}^{(n)} = \emptyset$,

(ii) $A_k^{(n)} \supset A_{2k}^{(n+1)} \cup A_{2k+1}^{(n+1)}$.

A family $\mathscr{A}$ of sets is called a *dyadic jungle* if for each A in $\mathscr{A}$ there is a dyadic tree $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{k=0}^\infty$ such that $A_0^{(0)} = A$ and each $A_k^{(n)}$ belongs to $\mathscr{A}$.

PROPOSITION 2.2. *Let Y be a subspace of $m(\Gamma)$ satisfying the following condition:*

(+) there are $K \geqslant 1$ and $\delta > 0$ and a dyadic jungle $\mathscr{A}$ of subsets of Γ such that for every $m = 1, 2, \ldots$ and for arbitrary finite collection $B_0, B_1, \ldots, B_{2^m-1}$ of mutually disjoint sets in $\mathscr{A}$ there exist $f \in Y$ with $\|f\| \leqslant K$ and $A_0, A_1, \ldots, A_{2^m-1}$ in $\mathscr{A}$ such that

$$\{\gamma \in B_r: (-1)^r f(\gamma) > \delta\} \supset A_r \qquad (r = 0, 1, \ldots, 2^m-1).$$

Then Y contains a subspace isomorphic to l_1.

Proof. We will construct a sequence $\{f^{(n)}\}_{n=0}^\infty$ in Y and a dyadic tree $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{n=0}^\infty$ in $\mathscr{A}$ with the following properties

(a) $\|f^{(n)}\| \leqslant K$,

(b) if $\gamma \in A_k^{(n)}$ and if $n \geqslant 1$, then $(-1)^k f_n(\gamma) > \delta$.

Assume that we have done this. Then for arbitrary real $t(1), t(2), \ldots, t(n)$ $(n = 1, 2, \ldots)$ we have

$$(3) \qquad \delta \sum_{r=1}^n |t(r)| \leqslant \Big\| \sum_{r=1}^n t(r) f^{(r)} \Big\| \leqslant K \sum_{r=1}^n |t(r)|.$$

Indeed, by (a), we have

$$\Big\| \sum_{r=1}^n t(r) f^{(r)} \Big\| \leqslant \sum_{r=1}^n |t(r)| \, \|f^{(r)}\| \leqslant K \sum_{r=1}^n |t(r)|.$$

To prove the left-hand side inequality we define for $r = 0, 1, \ldots, n$ the indices $p(r)$ as follows:

$$p(0) = 0;$$

$$p(r+1) = \begin{cases} 2p(r) & \text{if } t(r) \geqslant 0 \\ 2p(r)+1 & \text{if } t(r) < 0. \end{cases} \quad (r = 0, 1, \ldots, n-1)$$

Clearly $0 \leqslant p(r) < 2^r$. By (ii), $A_{p(n)}^{(n)} \subset A_{p(r)}^{(r)}$ for $r = 0, 1, \ldots, n$. Hence, by (b) and the definition of $p(r)$, if $\gamma \in A_{p(n)}^{(n)}$, then $t(r)f^{(r)}(\gamma) \geqslant \delta|t(r)|$ for $r = 1, 2, \ldots, n$. Hence

$$\Big\|\sum_{r=1}^{n} t(r)f^{(r)}\Big\| \geqslant \sup_{\gamma \in A_{p(n)}^{(n)}} \Big|\sum_{r=1}^{n} t(r)f^{(r)}(\gamma)\Big| \geqslant \delta \sum_{r=1}^{n} |t(r)|,$$

which completes the proof of inequality (3).

Let us set

$$u(t(\cdot)) = \sum_{r=1}^{\infty} t(r)f^{(r)} \quad \text{for} \quad t(\cdot) = \{t(r)\}_{r=1}^{\infty} \in l_1.$$

It follows from (3) that

$$\delta\|t(\cdot)\| \leqslant \|u(t(\cdot))\| \leqslant K\|t(\cdot)\| \quad \text{for} \quad t(\cdot) \in l_1.$$

Hence u is an isomorphism from l_1 onto the closure of the linear manifold spanned by the sequence $\{f^{(r)}\}_{r=1}^{\infty}$. Since $f^{(r)} \in Y$ $(r = 1, 2, \ldots)$, the range of u is a subspace of Y which is isomorphic to l_1.

We define the sequence $\{f^{(n)}\}_{n=0}^{\infty}$ and the dyadic tree $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{n=0}^{\infty}$ by induction. Let $f^{(0)} = 0$ and let $A_0^{(0)}$ be an arbitrary set in A. Suppose that for some $m \geqslant 1$ the functions $f^{(0)}, f^{(1)}, \ldots, f^{(m-1)}$ in Y and the sets $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{n=0}^{\infty}$ in $\mathscr{A}$ have been chosen to satisfy conditions (i), (ii), (a), and (b). Since $\mathscr{A}$ is a dyadic jungle, there are sets $B_0, B_1, \ldots, B_{2^m-1}$ in $\mathscr{A}$ such that $B_{2k} \cup B_{2k+1} \subset A_k^{(m-1)}$ and $B_{2k} \cap B_{2k+1} = \emptyset$. Let us set $f^{(m)} = f$ and $A_r^{(m)} = A_r$ $(r = 0, 1, \ldots, 2^m-1)$, where f and $\{A_r\}_{r=0}^{2^m-1}$ are chosen for $B_0, B_1, \ldots, B_{2^m-1}$ as in $(+)$. Then for $n = m$ conditions (a) and (b) are obviously satisfied. Let $k(r) = 2^{-1}r$ if r is even and $k(r) = 2^{-1}(r-1)$ if r is odd. Then $B_r \subset A_{k(r)}^{(m-1)}$ for $r = 0, 1, \ldots, 2^m-1$. Hence $A_r^{(m)} = A_r \subset B_r \subset A_{k(r)}^{(m-1)}$. Thus if $r_1 \neq r_2$, then $A_{r_1}^{(m)} \cap A_{r_2}^{(m)} = \emptyset$. Indeed, there are two possibilities.

1° $k(r_1) = k(r_2) = k$. Then (assuming that $r_1 < r_2$) $r_1 = 2k$ and $r_2 = 2k+1$. Hence $A_{r_1}^{(m)} \cap A_{r_2}^{(m)} \subset B_{2k} \cap B_{2k+1} = \emptyset$.

2° $k(r_1) \neq k(r_2)$. Then, by the inductive hypothesis, $A_{r_1}^{(m)} \cap A_{r_2}^{(m)} \subset B_{r_1} \cap B_{r_2} \subset A_{k(r_1)}^{(m-1)} \cap A_{k(r_2)}^{(m-1)} = \emptyset$. Finally we have, $A_{2k}^{(m)} \cup A_{2k+1}^{(m)} \subset B_{2k} \cup B_{2k+1} \subset A_k^{(m-1)}$. Hence the sets $\{\{A_k^{(n)}\}_{k=0}^{2^n-1}\}_{n=0}^{m'}$ satisfy conditions (i) and (ii). That completes the induction and the proof of the Proposition.

PROPOSITION 2.3. *Let μ be a non purely atomic measure. If Y is a norming subspace of the space $[L_1(\mu)]^*$, then Y contains a subspace isomorphic to l_1.*

Proof. By Proposition 1.3, we may assume that $[L_1(\mu)]^* = L_\infty(\mu)$. By Kakutani's representation theorem ([11], [3], p. 103) the space $L_\infty(\mu) = L_\infty(T, \Sigma, \mu)$ is isometrically and latically isomorphic to a space $C(Q) \subset m(Q)$. This isometric isomorphism induces via characteristic functions a Boolean isomorphism between the Boolean algebra of closed-open sets of Q and of μ — equivalence classes of sets in Σ. In the sequal ve shall identify closed-open sets in Q with suitable members of Σ and $L_\infty(\mu)$ with $C(Q)$. Let $\mathscr{A}$ be the family of those $A \in \Sigma$ that $0 < \mu(A) < +\infty$ and no subset of $\mathscr{A}$ is an atom for μ. Then $\mathscr{A}$ is a dyadic jungle. By Proposition 2.2 to complete the proof it suffices to show that Y together with $\mathscr{A}$ satisfies the condition $(+)$. Since Y is norming,

$(*)$ there is $K \geqslant 1$ such that for each $\varphi \in L_\infty(\mu)$ and for arbitrary finite set $\{g_1, g_2, \ldots, g_k\}$ in $L_1(\mu)$ there is f in Y so that $\|f\| \leqslant K\|\varphi\|$ and

$$\Big|\int_T fg_r\,d\mu - \int_T \varphi g_r\,d\mu\Big| < 2^{-1} \quad (r = 1, 2, \ldots, k).$$

Let $B_0, B_1, \ldots, B_{2^m-1}$ be a finite collection of mutually disjoint sets in $\mathscr{A}$. Let χ_r denote the characteristic function of B_r. Let us set

$$g_r = [\mu(B_r)]^{-1}\chi_r \in L_1(\mu) \quad (r = 0, 1, \ldots, 2^m-1).$$

Let $\varphi \in L_\infty(\mu)$ be defined by

$$\varphi(q) = \begin{cases} (-1)^r & \text{for } q \in B_r \quad (r = 0, 1, \ldots, 2^m-1), \\ 0 & \text{for } q \notin \bigcup_{r=0}^{2^m-1} B_r. \end{cases}$$

Clearly, $\|\varphi\| = 1$. By $(*)$, there exists in Y an f with $\|f\| \leqslant K$ so that, for $r = 0, 1, \ldots, 2^m-1$,

$$\Big|[\mu(B_r)]^{-1}\int_{B_r} f\,d\mu - (-1)^r\Big| < 2^{-1}.$$

Hence

$$(-1)^r[\mu(B_r)]^{-1}\int_{B_r} f\,d\mu > 2^{-1}.$$

Thus if $A_r = \{q \in B_r : (-1)^r f(q) > 2^{-1}\}$, then $\mu(A_r) > 0$ and $A_r \subset B_r$. Therefore $A_r \in \mathscr{A}$. Finally, if $q \in A_r$, then $(-1)^r f(q) > 2^{-1}$. Hence Y satisfies $(+)$ with $\delta = 2^{-1}$. That completes the proof.

For purely atomic measures we have

PROPOSITION 2.4. *Let Γ be an uncountable set. Let Y be a norming separable subspace of $(l_1(\Gamma))^* = m(\Gamma)$. Then Y contains a subspace isomorphic to l_1*

Proof. Let $\mathscr{A} = \{A \subset \Gamma\colon \text{card } A > \aleph_0\}$. Clearly $\mathscr{A}$ is a dyadic jungle. We will show that Y together with $\mathscr{A}$ satisfies (+).

By separability of Y, there are y_n in Y such that $\|y_n\| \leqslant 1$ and the set $\bigcup_{n=1}^{\infty}\{y_n\}$ is dense in the unit ball $\{y \in Y\colon \|y\| \leqslant 1\}$. Let us put

$$d(\gamma_1, \gamma_2) = \sum_{n=1}^{\infty} 2^{-n}|y_n(\gamma_1) - y_n(\gamma_2)| \quad \text{for} \quad (\gamma_1, \gamma_2) \in \Gamma \times \Gamma.$$

Clearly d is a pseudometric. Let $\gamma_1 \neq \gamma_2$. Choose $\psi = \{\psi(\gamma)\}_{\gamma\in\Gamma}$ in $m(\Gamma)$ and $t = \{t(\gamma)\}_{\gamma\in\Gamma}$ in $l_1(\Gamma)$ such that $\psi(\gamma_i) = (-1)^i = (-1)^i t(\gamma_i)$ and $t(\gamma) = 0$ for $\gamma \neq \gamma_i$ $(i = 1, 2)$. Since Y is dense in $m(\Gamma)$ in the $l_1(\Gamma)$ topology of $m(\Gamma)$, there is $y \in Y$ so that

$$\sum_{\gamma\in\Gamma} \psi(\gamma)t(\gamma) - \sum_{\gamma\in\Gamma} y(\gamma)t(\gamma) = 2 - [y(\gamma_2) - y(\gamma_1)] < 1.$$

Hence $\|y\|^{-1}[y(\gamma_2) - y(\gamma_1)] > \|y\|^{-1}$. Since the set $\bigcup_{n=1}^{\infty}\{y_n\}$ is dense in the unit ball of Y, there is an index n such that $\big\|y\,\|y\|^{-1} - y_n\big\| < (4\,\|y\|)^{-1}$. We have

$$y_n(\gamma_2) - y_n(\gamma_1) > \|y\|^{-1}\big(y(\gamma_2) - y(\gamma_1)\big) - 2\big\|\|y\|^{-1}y - y_n\big\| > (2\,\|y\|)^{-1} > 0.$$

Therefore $d(\gamma_2, \gamma_1) > 0$. Hence d is a metric in Γ. The metric space (Γ, d) is separable, because the map $\gamma \to \{y_n(\gamma)2^{-n}\}$ is a homeomorphic embedding of (Γ, d) into the Hilbert cube $\{s = \{s_i\}_{i=1}^{\infty}\colon |s_i| \leqslant 2^{-i}\}$. Therefore each A in $\mathscr{A}$ has a condensation point (i.e. a point with the property that every its d — neighbourhood is uncountable).

Let $B_0, B_1, \ldots, B_{2^m-1}$ be mutually disjoint sets in $\mathscr{A}$. Choose γ_r in B_r so that γ_r is a condensation point of B_r. Define t_r in $l_1(\Gamma)$ by

$$t_r(\gamma) = \begin{cases} 0 & \text{for} \quad \gamma \neq \gamma_r, \\ 1 & \text{for} \quad \gamma = \gamma_r. \end{cases}$$

Next define φ in $m(\Gamma)$ by

$$\varphi(\gamma) = \begin{cases} (-1)^r & \text{for} \quad \gamma = \gamma_r \ (r = 0, 1, \ldots, 2^m-1), \\ 0 & \text{for} \quad \gamma \notin \{\gamma_0, \gamma_1, \ldots, \gamma_r\}. \end{cases}$$

Since Y is norming, there is $K \geqslant 1$ such that the ball $\{y \in Y\colon \|y\| \leqslant K\}$ is dense in the ball $\{\varphi \in m(\Gamma)\colon \|\varphi\| \leqslant 1\}$ in the $l_1(\Gamma)$-topology of $m(\Gamma)$. Hence there is an y in Y with $\|y\| \leqslant K$ such that

$$\Big|\sum_{\gamma\in\Gamma} \varphi(\gamma)t_r(\gamma) - \sum_{\gamma\in\Gamma} y(\gamma)t_r(\gamma)\Big| < 2^{-1} \quad (r = 0, 1, \ldots, 2^m-1).$$

Computing the left-hand side of this inequality we obtain

$$|(-1)^r - y(\gamma_r)| < 2^{-1}.$$

Hence $(-1)^r y(\gamma_r) > 2^{-1}$ for $r = 0, 1, \ldots, 2^m-1$.

Since $y(\cdot)$ is a continuous function on (Γ, d) and since γ_r is a condensation point of B_r, the set A_r belongs to $\mathscr{A}$ where

$$A_r = \{\gamma \in B_r\colon (-1)^r y(\gamma) > 2^{-1}\}.$$

Clearly, we have $A_r \subset B_r$ and $(-1)^r y(\gamma) > 2^{-1}$ for $\gamma \in A_r$. Hence Y satisfies (+) with $\delta = 2^{-1}$. That completes the proof.

3. Main results. We begin with the following lemma:

LEMMA 3.1. *Let X be a Banach space and let Γ be an abstract set. Then X contains a subspace isomorphic to $l_1(\Gamma)$ if and only if there exist a Banach space Y and a linear operator $u\colon X \to Y$ such that the range of u contains a subspace isomorphic to $l_1(\Gamma)$.*

Proof. Necessity. Put $u =$ the identity operator on X.

Sufficiency. Let Y_1 be a subspace of Y which is isomorphic to $l_1(\Gamma)$ and which is contained in $u(X)$. Since Y_1 is isomorphic to $l_1(\Gamma)$, there is $K \geqslant 1$ and an indexed set $\{y_\gamma\}_{\gamma\in\Gamma}$ of elements of Y_1 such that

$$K^{-1}\sum_{\gamma\in\Gamma}|t(\gamma)| \leqslant \Big\|\sum_{\gamma\in\Gamma} t(\gamma)y_\gamma\Big\| \leqslant K\sum_{\gamma\in\Gamma}|t(\gamma)| \quad \text{for} \quad \{t(\gamma)\} \in l_1(\Gamma).$$

Let $X_1 = \{x \in X\colon ux \in Y_1\}$. Then X_1 is a closed linear manifold in X. Obviously the restriction of u to the subspace X_1 is an epimorphism. Hence (cf. [7], p. 55) there are $K_1 > 0$ and indexed set $\{x_\gamma\}_{\gamma\in\Gamma}$ of elements of X_1 such that $ux_\gamma = y_\gamma$ and $\|x_\gamma\| \leqslant K_1 (\gamma \in \Gamma)$. For $\{t(\gamma)\}_{\gamma\in\Gamma} \in l_1(\Gamma)$ we have

$$(K\|u\|)^{-1}\sum_{\gamma\in\Gamma}|t(\gamma)| \leqslant \|u\|^{-1}\Big\|\sum_{\gamma\in\Gamma} t(\gamma)y_\gamma\Big\| \leqslant \Big\|\sum_{\gamma\in\Gamma} t(\gamma)x_\gamma\Big\| \leqslant K_1\sum_{\gamma\in\Gamma}|t(\gamma)|.$$

Hence the operator $v\colon l_1(\Gamma) \to X$ defined by

$$v(\{t(\gamma)\}) = \sum_{\gamma\in\Gamma} t(\gamma)x_\gamma$$

is the required isomorphic embedding. That completes the proof.

PROPOSITION 3.2. *Let X^* contain a seminorming subspace isomorphic to $L_1(\mu)$. If either* (I) *μ is not purely atomic, or* (II) *X is separable and μ has uncountably many atoms, then X contains a subspace isomorphic to l_1.*

Proof. By the assumption, there is a proper isomorphism $u\colon L_1(\mu) \to X^*$. Let $v = u^*\varkappa_X\colon X \to [L_1(\mu)]^*$ and let $Y = v(X)$. By Proposition 1.2, Y is a subspace of $[L_1(\mu)]^*$. Let S, S^{**}, B and B^{**} denote the unit balls of X, X^{**}, Y, and $[L_1(\mu)]^*$ respectively. By Goldstine's theorem ([7], p. 424), $\mathrm{cl}^*(\varkappa_X S) = S^{**}$. By continuity of v, there is an $a > 0$ such that $B \supset av(S) = au^*(\varkappa_X S)$. Since u is an isomorphic embedding, u^* is an epimorphism. Hence ([7], p. 487] there is a $b > 0$ such that $u^*(S^{**}) \supset bB^{**}$. Finally, the adjoint operator u^* is continuous if both spaces X^{**} and $[L_1(\mu)]^*$ carry their w*-topologies. Thus

$$u^*(S^{**}) = u^*\big(\mathrm{cl}^*(\varkappa_X S)\big) \subset \mathrm{cl}^* u^*(\varkappa_X S).$$

Combining all these inclusions we get

$$\mathrm{cl}^* B \supset a\,\mathrm{cl}^* v(S) \supset au^*(S^{**}) \supset abB^{**}.$$

But this means nothing else but the fact that Y is a norming subspace of $[L_1(\mu)]^*$.

Next we will show that in both cases (I) and (II) Y contains a subspace isomorphic to l_1. Assume that we have done this. Then, since v maps X onto Y, to complete the proof it is enough to apply Lemma 3.1.

If (I), then the desired conclusion on Y is an immediate consequence of Proposition 2.3.

If (II) but (I) does not hold, then the assumption on μ implies that $L_1(\mu)$ is isometrically isomorphic to $l_1(\Gamma)$ with $\mathrm{card}\,\Gamma > \aleph_0$. Since X is separable, $Y = v(X)$ has the same property. The desired conclusion on Y follows from Proposition 2.4.

CONJECTURE. *Let X^* contain a seminorming subspace isomorphic to $L_1(\mu)$ where μ is a homogeneous* (cf. [13], [4]) *finite measure. Then X contains a subspace isomorphic to $l_1(\Gamma)$ with* $\mathrm{card}\,\Gamma = \mathrm{dens}\big(L_1(\mu)\big)$, *where* $\mathrm{dens}\big(L_1(\mu)\big)$ *is the smallest cardinal number $\mathfrak{n}$ such that there is in $L_1(\mu)$ a linearly dense set A with* $\mathrm{card}\,A = \mathfrak{n}$.

In the "opposite direction" we have the following result (by D^Γ we denote the Cartesian product of Γ copies of the discrete two point space D):

PROPOSITION 3.3. *If a Banach space X contains a subspace isomorphic to $l_1(\Gamma)$ with $\mathrm{card}\,\Gamma \geqslant \aleph_0$, then X^* contains a subspace isomorphic to $[C(D^\Gamma)]^*$. Hence X^* contains a subspace isomorphic to $l_1(\Gamma_1)$ with $\mathrm{card}\,\Gamma_1 = 2^{\mathrm{card}\,\Gamma}$.*

Proof. The space $C(D^\Gamma)$ has a dense set, say A, with $\mathrm{card}\,A = \mathrm{card}\,\Gamma$ (e.g. the set of all finite linear combinations with rational coefficients of characteristic functions of closed-open subsets of D^Γ). Hence $C(D^\Gamma)$ is a quotient space of $l_1(\Gamma)$ (cf. [12], p. 29). Let $h\colon l_1(\Gamma) \to C(D^\Gamma)$ be the quotient map. Let $u\colon l_1(\Gamma) \to X$ be an isomorphic embedding and let $j\colon u\big(l_1(\Gamma)\big) \to l_1(\Gamma)$ denote the left inverse of u. For the sake of brevity we put $\varkappa = \varkappa_{C(D^\Gamma)}$ and $\varkappa_* = \varkappa_{[C(D^\Gamma)]^*}$. Since $[C(D^\Gamma)]^{**}$ is a $\mathscr{P}_1$-space (cf. [3], p. 100; 106-107), the operator $\varkappa hj\colon u\big(l_1(\Gamma)\big) \to [C(D^\Gamma)]^{**}$ has a norm-preserving extension to an operator from X into $[C(D^\Gamma)]^{**}$, say v.

We will show that the operator $v^*\varkappa_*\colon [C(D^\Gamma)]^* \to X^*$ is the required isomorphic embedding. To this end it is enough to show that there is $c > 0$ such that

$$\|(v^*\varkappa_*)(\mu)\| \geqslant c\|\mu\| \quad \text{for} \quad \mu \in [C(D^\Gamma)]^*. \tag{4}$$

Let $W = \{x \in u\big(l_1(\Gamma)\big)\colon \|x\| \leqslant 1\}$. Clearly,

$$j(W) \supset \{t \in l_1(\Gamma)\colon \|t\| \leqslant \|u\|^{-1}\}.$$

Using this inclusion and the identity $(\varkappa^*\varkappa_*)(\mu) = \mu$ for each $\mu \in [C(D^\Gamma)]^*$ we obtain

$$\begin{aligned}\|(v^*\varkappa_*)(\mu)\| &= \sup_{\|x\|\leqslant 1}[\varkappa_*(\mu)](vx) \geqslant \sup_{x\in W}[\varkappa_*(\mu)](vx)\\ &= \sup_{x\in W}[\varkappa_*(\mu)][(\varkappa hj)(x)] = \sup_{x\in W}[(\varkappa^*\varkappa_*)(\mu)][hj(x)]\\ &= \sup_{x\in W}\mu[hj(x)] \geqslant \sup_{\|t\|\leqslant\|u\|^{-1}}\mu(ht) = \|u\|^{-1}\sup_{\|t\|\leqslant 1}(h^*\mu)(t)\\ &= \|u\|^{-1}\|h^*\mu\| = \|u\|^{-1}\|\mu\|.\end{aligned}$$

(We use the fact that if $h\colon X \to Y$ is a quotient map, then $h^*\colon Y^* \to X^*$ is an isometrically isomorphic embedding.) Hence $v^*\varkappa_*$ satisfies (4) with $c = \|u\|^{-1}$. This completes the proof of the first part of the Proposition.

The second part of the Proposition is an easy consequence of the first one, because the subspace of $[C(D^\Gamma)]^*$ generated by the family of functionals $\{\delta_q\}_{q\in D^\Gamma}$ is isometrically isomorphic to $l_1(D^\Gamma)$. Clearly, $\mathrm{card}\,D^\Gamma = 2^{\mathrm{card}\,\Gamma}$. Here δ_q denotes the "point mass" at q, i.e., δ_q is the Borel measure which equals 1 on any Borel subset of D^Γ which contains q, and equals 0 otherwise. That completes the proof.

For separable Banach spaces we have the more clear picture than that which is given by Propositions 3.2 and 3.3.

THEOREM 3.4. *If X is a separable Banach space, then the following conditions are equivalent:*

(i) *X contains a subspace isomorphic to l_1;*

(ii) *$C = C([0;1])$ is isomorphic to a quotient space of X;*

(iii) *X^* contains a (w^*-closed) subspace isomorphic to C^* such that on this subspace the X topology of X^* and the C topology of C^* coincide;*

(iv) *X^* contains a seminorming subspace isomorphic to C^*;*

(v) *X^* contains a seminorming subspace isomorphic to L_1;*

(vi) *X^* contains a seminorming subspace isomorphic to $l_1(\Gamma)$ with* $\mathrm{card}\,\Gamma = 2^{\aleph_0}$;

(vii) *X^* contains a seminorming subspace isomorphic to $l_1(\Gamma)$ with* $\mathrm{card}\,\Gamma > \aleph_0$.

Proof. We will prove the following implications:

$$\begin{array}{ccccc} & & & (\text{iv}) & \\ & & \nearrow & & \searrow \\ (\text{i}) \Rightarrow (\text{ii}) \Rightarrow (\text{iii}) & \Rightarrow & (\text{v}) & \Rightarrow & (\text{i}) \\ & \searrow & & & \Uparrow \\ & & (\text{vi}) & \Rightarrow & (\text{vii}) \end{array}$$

(i) $\Rightarrow$ (ii). We need the following result (cf. [19]):

(*) *If a separable Banach space Z contains a subspace Z_1 isomorphic to C, then there is a subspace $Z_2 \subset Z_1$ such that Z_2 is isomorphic to C and complemented in Z.*

Let Y be a subspace of X which is isomorphic to l_1. Since every separable Banach space is a quotient of l_1 (cf. [1]), there is an epimorphism u from Y onto C. Let Z be a separable Banach space containing C and such that the operator $u\colon Y \to C$ has an extension $v\colon X \to Z$ (i.e. $vy = uy$ for $y \in Y$) with $\|u\| = \|v\|$. (The space Z is constructed as follows. Regarding C as a subspace of a $\mathscr{P}_1$-space, say m, we may construct an extension $v_1\colon X \to m$ of u with $\|v_1\| = \|u\|$. Define Z as the closure of the range of v_1. Since X is separable, Z has the same property. Since $u(X) = C$, we have $Z \supset v_1(X) \supset u(X) = C$. We define $v\colon X \to Z$ by $vx = v_1 x$ for $x \in X$.) By (*), there is a subspace $Z_2 \subset C$ such that Z_2 is isomorphic to C and Z_2 is complemented in Z. Hence there is a projection p from Z onto Z_2. Put $u_1 = pv\colon X \to Z_2$. Then u_1 is the required epimorphism. (Indeed for $f \in Z_2$ there is $x \in Y$ such that $ux = vx = f$. Since $f \in Z_2$, we have $f = pf = pux = u_1 x$.) That completes the proof.

(ii) $\Rightarrow$ (iii). If $u\colon X \to C$ is an epimorphism, then $u^*\colon C^* \to X^*$ is the required isomorphic embedding.

(iii) $\Rightarrow$ (iv). This is an obvious consequence of the general fact that every w^*-closed subspace of X^* is seminorming.

(iii) $\Rightarrow$ (v). This follows from the fact that L_1, regarded as a subspace of C^* consisting of all measures absolutely continuous with respect to the usual lebesgue measure, is a norming subspace of C^*.

(iii) $\Rightarrow$ (vi). This follows from the fact that a subspace of C^* generated by all point masses is norming and is isomorphic to $l_1(\Gamma)$ with $\operatorname{card}\Gamma = 2^{\aleph_0}$.

(vi) $\Rightarrow$ (vii). This implication is trivial.

(iv) $\Rightarrow$ (i), (v) $\Rightarrow$ (i), (vii) $\Rightarrow$ (i). These implications follow immediately from Proposition 3.2.

Remark. In the statement of Theorem 3.4 the space C may be replaced by any space $C(Q)$, where Q is an arbitrary uncountable compact metric space. This clearly follows from the profound result of Miljutin [14] (cf. also [18]) that for every uncountable compact metric space Q the space $C(Q)$ is isomorphic to C.

The next examples show that in Theorem 3.4 the assumption of separability of X is in general essential.

Example 1. Let $X = c_0(\Gamma)$ with $\operatorname{card}\Gamma > \aleph_0$. Then $X^* = l_1(\Gamma)$. But X does not contain any subspace isomorphic to l_1.

Example 2. Let $X = m$. Then C is not isomorphic to any quotient space of m, because every separable quotient space of m is reflexive ([9], p. 168). Hence X does not satisfy (ii) and therefore (iii) (because for every Banach space X the conditions (ii) and (iii) are equivalent). Clearly, m contains a subspace isomorphic to l_1.

4. A characterization of σ-finite measures μ for which the spaces $L_1(\mu)$ are isomorphic to conjugate spaces.

THEOREM 4.1. *If μ is a σ-finite measure, then the space $L_1(\mu)$ is isomorphic to a conjugate space to a Banach space if and only if μ is purely atomic.*

Proof. Sufficiency. If μ is purely atomic, then $L_1(\mu)$ is isometrically isomorphic to $l_1(\Gamma) = [c_0(\Gamma)]^*$. Moreover, since μ is σ-finite, $\operatorname{card}\Gamma \leqslant \aleph_0$.

The proof of the necessity part is based on the next two lemmas. The first of them generalizes Lemma 3.1.

LEMMA 4.2. *If the closure of the range of an operator $u\colon X \to Y$ contains a subspace isomorphic to $l_1(\Gamma)$ for some uncountable set Γ, then X contains a subspace isomorphic to $l_1(\Gamma_1)$ for another uncountable set Γ_1.*

Proof. By the assumption, there are $K > 0$ and a family $\{y_\gamma\}_{\gamma\in\Gamma}$ of elements of the closure of $u(X)$ such that

$$K^{-1}\sum_{\gamma\in\Gamma}|t(\gamma)| \leqslant \Big\|\sum_{\gamma\in\Gamma} t(\gamma)y_\gamma\Big\| \leqslant K\sum_{\gamma\in\Gamma}|t(\gamma)| \quad \text{for} \quad \{t(\gamma)\}\in l_1(\Gamma).$$

For each $\gamma\in\Gamma$ choose x_γ in X so that $\|y_\gamma - ux_\gamma\| < (2K)^{-1}$. Let $\Gamma^{(n)} = \{\gamma\in\Gamma\colon \|x_\gamma\| \leqslant n\}$, $n = 1, 2, \ldots$ Since $\bigcup_{n=1}^{\infty}\Gamma^{(n)} = \Gamma$ and since Γ is uncountable, there is an index N such that $\Gamma^{(N)}$ is uncountable. Put $\Gamma_1 = \Gamma^{(N)}$. For $\{t(\gamma)\}_{\gamma\in\Gamma_1} \in l_1(\Gamma_1)$ we have

$$\begin{aligned} N\sum_{\gamma\in\Gamma_1}|t(\gamma)| &\geqslant \Big\|\sum_{\gamma\in\Gamma_1} t(\gamma)x_\gamma\Big\| \geqslant \|u\|^{-1}\Big\|\sum_{\gamma\in\Gamma_1} t(\gamma)ux_\gamma\Big\| \\ &\geqslant \|u\|^{-1}\Big(\Big\|\sum_{\gamma\in\Gamma_1} t(\gamma)x_\gamma\Big\| - \sum_{\gamma\in\Gamma_1}|t(\gamma)|\,\|ux_\gamma - y_\gamma\|\Big) \\ &\geqslant [(K\|u\|)^{-1} - (2K\|u\|)^{-1}]\sum_{\gamma\in\Gamma_1}|t(\gamma)| \\ &= (2K\|u\|)^{-1}\sum_{\gamma\in\Gamma_1}|t(\gamma)|. \end{aligned}$$

Hence the operator $v\colon l_1(\Gamma_1)\to X$ defined by

$$v\big(\{t(\gamma)\}_{\gamma\in\Gamma_1}\big)=\sum_{\gamma\in\Gamma_1}t(\gamma)x_\gamma \quad \text{for} \quad \{t(\gamma)\}_{\gamma\in\Gamma_1}\in l_1(\Gamma_1)$$

is the required isomorphic embedding. That completes the proof.

Remark. If Γ can not be represented as the countable union of sets of cardinalities less than card Γ (for instance if card $\Gamma=2^{\aleph_0}$), then we may choose Γ_1 in Lemma 4.2 so that card $\Gamma=$ card Γ_1.

LEMMA 4.3. *If μ is a σ-finite measure, then no subspace of $L_1(\mu)=L_1(T,\Sigma,\mu)$ is isomorphic to a space $l_1(\Gamma)$ with* card $\Gamma>\aleph_0$.

Proof. If card $\Gamma\geqslant\aleph_0$, then $l_1(\Gamma)$ is not isomorphic to a Hilbert space. Since every subspace of a Hilbert space is isometrically isomorphic to a Hilbert space, no subspace of the Hilbert space $L_2(\mu)$ is isomorphic to $l_1(\Gamma)$. Therefore, in view of Lemma 4.2, it is enough to show that if μ is σ-finite, then there exists a linear operator $u\colon L_2(\mu)\to L_1(\mu)$ such that the range of u is dense in $L_1(\mu)$.

Since μ is σ-finite, there is in Σ a sequence $\{T_n\}_{n=1}^{\infty}$ of mutually disjoint sets such that $\bigcup_{n=1}^{\infty}T_n=T$ and $0<\mu(T_n)<+\infty$ for $n=1,2,\ldots$ Let χ_n denote the characteristic function of T_n. Let us set

$$uf=\sum_{n=1}^{\infty}2^{-n}\big(\mu(T_n)\big)^{-1/2}\chi_n\cdot f \quad \text{for} \quad f\in L_1(\mu).$$

Then, by the Schwartz inequality, we have

$$\|uf\|_1\leqslant\sum_{n=1}^{\infty}2^{-n}[\mu(T_n)]^{-1/2}\|\chi_n\cdot f\|_1\leqslant\|f\|_2.$$

(Here by $\|\cdot\|_i$ we denote the norm in $L_i(\mu)$ for $i=1,2$, and ∞).

If $f\in L_1(\mu)$, then

$$f=\sum_{n=1}^{\infty}\chi_n f.$$

Let $\varepsilon>0$. Choose $N=N(f,\varepsilon)$ so that

$$\Big\|f-\sum_{n=1}^{N}\chi_n\cdot f\Big\|_1<2^{-1}\varepsilon.$$

Since $\mu(T_n)<+\infty$, there are functions $f_n\in L_\infty(\mu)$ such that $f_n=\chi_n\cdot f_n$, and

$$\|f_n-\chi_n\cdot f\|_1<(2N)^{-1}\varepsilon \quad (n=1,2,\ldots,N).$$

Let

$$g=\sum_{n=1}^{N}2^n[\mu(T_n)]^{1/2}f_n.$$

Then

$$\|g\|_2=\Big(\sum_{n=1}^{N}2^{2n}\mu(T_n)\int_{T_n}|f_n|^2d\mu\Big)^{1/2}\leqslant\Big(\sum_{n=1}^{N}2^{2n}[\mu(T_n)]^2\|f_n\|_\infty^2\Big)^{1/2}<+\infty.$$

Hence $g\in L_2$. Clearly, $ug=\sum_{n=1}^{N}f_n$. Therefore

$$\|ug-f\|_1\leqslant\Big\|\sum_{n=1}^{N}\chi_n\cdot f-f\Big\|_1+\sum_{n=1}^{N}\|\chi_n f-f_n\|_1<\varepsilon.$$

Thus the range of u is dense in $L_1(\mu)$. That completes the proof.

We return to the proof of Theorem 4.1.

Necessity. Suppose that for some σ-finite non-purely atomic measure μ the space $L_1(\mu)$ is isomorphic to a conjugate Banach space. Then, by Propositions 3.2 (case I), and 3.3, $L_1(\mu)$ would contain a subspace isomorphic to $l_1(\Gamma)$ with card $\Gamma=2^{\aleph_0}$. But this would contradict Lemma 4.3.

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On differentiability in an important class of locally convex spaces

by

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The most interesting facts in the theory of differentiation in Banach spaces are based on the fact that in a Banach space there exist bounded neighbourhoods of zero. In more general cases these two properties (of being open or bounded) exclude one another. So the generalizations of this theory which are known to us have followed two different ways: defining differentiability "with respect to bounded sets" or "with respect to open sets". The first way was chosen by Sebastião e Silva [12]. A very disagreeable defect of this theory is that differentiability does not imply continuity. However, this implication is valid for Fréchet spaces but it requires a non-trivial proof. Besides, the lack of the mean value theorem in Silva's theory makes it impossible to estimate the remainder by the derivative.

The second way was chosen by several authors, e. g. Marinescu [11], Bastiani [1], Binz [2], Keller [8], Frölicher and Bucher [6]. As is well known (cf. an excellent review of Keller [9]), in the case of a general locally convex space E there does not exist any locally convex topology in $\mathscr{L}(E, F)$ in which the mapping

$$E \times \mathscr{L}(E, F) \ni (h, L) \to L(h) \in F$$

is continuous. No wonder that nobody succeeded in obtaining in the general case the mean value theorem or an equivalent theorem stating that the continuously Gâteaux differentiable mapping is also Fréchet differentiable. Replacing the continuity of a Gâteaux derivative by a much stronger non-topological condition, Marinescu [11], Bastiani [1] and other authors mentioned above obtain the Fréchet differentiability.

We prove in the present paper that in an important for applications class of Fréchet spaces one can develop a theory of differentiation "with respect to open sets" without assuming this condition. Many theorems known in the classical theory of differentiation in Banach spaces are proved here. We give also (in section 3) a natural criterion for the