On Banach spaces containing $L_1(\mu)$

by

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Introduction. In the present paper we study consequences of the facts that a Banach space $X$ or its conjugate $X^*$ contain a subspace isomorphic (linearly homeomorphic) to an $L_1(\mu)$ space. It is shown, in particular, that if a separable Banach space $X$ contains a subspace isomorphic to $l_1$, then $X^*$ is not only non-separable but also "sufficiently rich"; Namely $X^*$ contains a w*-closed subspace isomorphic to $\{0, 1\}^\omega$ and therefore it contains a subspace isomorphic to $l_1(\mathcal{F})$ with card $\mathcal{F} = 2^\omega$.

On the other hand, if a conjugate Banach space $X^*$ contains a "nicely embedded" (precisely seminorming, cf. Definition 1.1) subspace isomorphic to $L_1(\mu)$ for some non-purely atomic measure $\mu$, then $X$ contains a subspace isomorphic to $l_1$. Combining these two facts with the easy lemma that if $\mu$ is $\sigma$-finite, then $L_1(\mu)$ does not contain any subspace isomorphic to $l_1(\mathcal{F})$ with $\mathcal{F}$ uncountable, we prove that if $\mu$ is $\sigma$-finite and non-purely atomic, then $L_1(\mu)$ is not isomorphic to any conjugate Banach space.

This result generalizes Gelfand's theorem [3] that the space $L_1 = L_1([0, 1])$ is not isomorphic to any conjugate Banach space, and gives a partial solution of the following problem raised by Dieudonné [5]: characterize those $L_1(\mu)$-spaces which are isomorphic to a conjugate Banach space. For various proofs of Gelfand's theorem the reader is referred to the papers [2], [5], [15];[17] and [20].

1. Preliminaries. Any unexplained notation will be that of either [3] or [7]. If $A$ is a set, then card $A$ denotes the cardinality of $A$.

Capital letters $X, Y, Z$ denote Banach spaces. The first and the second conjugate of $X$ are denoted by $X^*$ and $X^{**}$ respectively. By "subspace" we always mean a closed linear subspace. By "operator" a bounded linear operator. If $u : X \to Y$ is an operator, then $u^* : Y^* \to X^*$ denotes the adjoint operator of $u$. By $\kappa_X$ we denote the canonical embedding of $X$ into $X^{**}$. An operator $u : X \to Y$ is called: an (isometrically) isomorphic embedding if there is $v : u(X) \to X$ such that for all $x$ in $X$, $\varepsilon(u(x)) = x$ (and $[u] = [v]$); an epimorphism if $u(X) = Y$; an (isometric) isomorphism if $u$ is an (isometrically) isomorphic embedding and an
epimorphism. $X$ is said to be (isometrically) isomorphic to $Y$ if there is an (isometric) isomorphism from $X$ onto $Y$. If $X_1$ is a subspace of $X$, then the map $x \mapsto x + X_1$ from $X$ onto the quotient space $X/X_1$ is called the quotient map or the natural map. A subspace $Y$ of $X$ is said to be complemented if there is a projection (i.e., bounded linear idempotent) from $X$ onto $Y$. A Banach space $X$ is a $W^*$-space if for each Banach space $Y$ and for an arbitrary subspace $Y_1$ of $Y$ every operator $w: Y_1 \to X$ has an extension $v: Y \to X$ (i.e., $wy = vx$ for $y \in Y_1$) with $\|w\| = \|v\|$.

In the conjugate space $X^*$ we consider the norm topology and the $w^*$-topology of $X^*$ which is called sometimes the $w^*$-topology. This topology is obtained by taking as a base all sets of the form

$$X^* \setminus \overline{B^*} = \{y^* \in X^*: \|y^* - x^*\| < \varepsilon, x^* \in X^*\}$$

where $x^* \in X^*$, $\varepsilon$ is a finite subset of $X$, and $\varepsilon > 0$.

If $A$ is a subset of $X^*$, then $cl^w(A)$ denotes the closure of $A$ in the $w^*$-topology. In the sequel an important role plays the following concept.

**Definition 1.1.** A subspace $Z$ of $X^*$ is said to be seminormalizing if there is $\varepsilon > 0$ such that

$$\varepsilon = \inf\{\varepsilon > 0: \exists x \in Z, \|x\| = 1, \text{ such that } \|x - x^*\| < \varepsilon, x^* \in X^*\}$$

where $B^*$ is the unit ball of $X^*$.

If moreover $cl^w(Z) = X^*$, then $Z$ is called norming (i.e., $Z$ has positive characteristic in the sense of Dziniak [6]).

An isomorphic embedding $w: Y \to X^*$ is said to be proper (precisely proper with respect to the $X$ topology of $X^*$) if $w(Y)$ is a seminormalizing subspace of $X^*$.

**Proposition 1.2.** For every isomorphic embedding $w: Y \to X^*$, the following conditions are equivalent:

(a) $w$ is proper,
(b) the range of the operator $v = w^*|X_2: X \to Y^*$ is closed.

Proof. (a) $\Rightarrow$ (b).

Let $n$ be a unitary operator

$$X_1 = \{x \in X: (wy)(x) = 0 \text{ for all } y \in Y\},$$

$$X_1^* = \{y^* \in X^*: x^* (x) = 0 \text{ for all } x \in X_1\}.$$

Let $k: X \to X/X_1$ be the quotient map. Then $k^*$ regarded as an operator from $(X/X_1)^*$ onto $X_1^*$ is an isometric isomorphism. Define the isomorphic embedding $w_1: Y \to (X/X_1)^*$ so that $w(w_1) = w$. Since $w(Y)$ is seminormalizing and since $cl^w[w(Y)] = X_1^*$, the subspace $w_1(Y)$ of $(X/X_1)^*$ is norming (because $k^*$ is a continuous map if both spaces $(X/X_1)^*$ and $X_1^*$ carry the $X/X_1$ topology of $(X/X_1)^*$ and the $X$ topology of $X^*$ respectively). Hence there is $\varepsilon > 0$ such that

$$\sup_{x \in X} \|w_1(y)(x)\| \geq \varepsilon \|y\|$$

for $y \in Y$. Thus

$$\sup_{x \in X} \|w_1(y)(x)\| \geq \varepsilon \|y\|$$

for $x \in X$. Hence $y = v_1(x)$ is an isomorphic embedding. Observe that $v = v_1h$. Hence $v$ is the composition of the epimorphism $h$ with the isomorphic embedding $v_1$. Therefore $v(X) = v_1(X)$. Thus the range of $v$ is closed.

(b) $\Rightarrow$ (a).

By Banach open map theorem ([17], p. 55), there exists $\varepsilon > 0$ such that for each $y \in v(X)$ there is an $x_1 \in X$ such that $\varepsilon < \varepsilon < \|y\|$. Suppose that $w(Y)$ is not seminormalizing. Then there is no element $y^* = w^*(w(Y))$ with $\|y^*\| = 1$ such that $\|w(Y) - y^*\| < \varepsilon$. Now, since $cl^w[w(Y) \cap \Delta]$ is a $\psi^*$-compact convex and symmetric set. Hence, by separation theorem ([17], p. 412), there is an $x \in X$ such that $\|x\| = \|y\|$ and $\|w(y)(x)\| < \|y\| < 1$. Hence $\|x\| \leq \|\psi^*\| < \varepsilon$. Choose an $x_1 \in X$ so that

$$e = \inf\{\varepsilon > 0: \exists x \in X, \|x\| = 1, \text{ such that } \|x - x^*\| < \varepsilon, x^* \in X^*\}$$

so that

$$e = \inf\{\varepsilon > 0: \exists x \in X, \|x\| = 1, \text{ such that } \|x - x^*\| < \varepsilon, x^* \in X^*\}$$

$$\|w_1(y)(x)\| < \varepsilon$$

and

$$w_1(x) = \psi.$$
(resp. \(m(I')\)) consists of all real functions \((f)_{\nu T}\) such that
\[
||f|| = \sum_{\nu T} |(f)| < +\infty \quad \text{(resp. } |f|) = \sup_{\nu T} |(f)| < +\infty \).

If \(I'\) is countable, then we shall write \(I_1\) instead of \((I_1)_{T'}\). If \(T' = [0, 1]\) is the unit interval, \(\Sigma\) the field of all Borel subsets of \([0, 1]\), and \(\mu\) the usual Lebesgue measure, then \(L_\mu = L_\mu (\mu)\) is \(L_\mu\) by \(L_\mu\).

We shall write \([L_\mu (\mu')]\) = \(L_\mu (\mu)\) if the operator \(\varphi \rightarrow \varphi_\mu\), where
\[
\varphi_\mu = \int \varphi \mu \, d\mu
\]
is an isometric isomorphism from \(L_\mu (\mu)\) onto \([L_\mu (\mu')]\). If \(\mu\) is \(\infty\)-finite, then \([L_\mu (\mu')] = L_\mu (\mu)\) ([7], p. 289). More sophisticated is the following result (cf. [7], p. 290):

**Proposition 1.3.** For every space \(L_\mu (\mu)\) there is a space \(L_\mu (\nu)\) isometrically isomorphic to \(L_\mu (\mu)\) and such that \([L_\mu (\nu)] = [L_\mu (\mu)]\).

**Proof.** By the Kakutani representation theorem (cf. [10], [2], p. 108), there is a collection of finite measures \((\mu_\nu)\) such that there is an isometric isomorphism from \(L_\mu (\mu)\) onto the unit interval, \(X = \prod_{\nu T} [L_\mu (\nu)\mu]\) (the elements of the space \(X\) are indexed families of functions \(x_\nu = (f_\nu, \nu T)\) such that \(f_\nu \in L_\mu (\mu)\) and \(||x_\nu|| = \sum_{\nu T} |f_\nu| < +\infty\)). Moreover, this isometric isomorphism takes positive functions into families consisting of positive functions. Since all \(\mu_\nu\) are finite measures, we have \([L_\mu (\mu)] = [L_\mu (\mu)]\). Therefore, by general result on \(L_\mu (\nu)\) products (cf. [3], p. 31 (13)), we have \(X' = L_\mu (\nu)\prod_{\nu T} [L_\mu (\nu)\mu]\) (the elements of the space \(X_\nu = L_\mu (\nu)\mu\) are indexed families of functions \((f_\nu, \nu T)\) such that \(f_\nu \in L_\mu (\mu)\) and \(||f_\nu|| = \sup_{\nu T} |f_\nu|\)). Precisely the operator \((f_\nu, \nu T) \rightarrow \varphi_\mu f_\nu\), where
\[
\varphi_\mu f_\nu = \int_{\nu T} f_\nu \mu \, d\mu
\]
is the isometric isomorphism from \(L_\mu (\mu)\) onto \(X'\).

Let us set (the sets \(T_\nu\) are regarded to be mutually disjoint)
\[
T = \bigcup_{\nu T} T_\nu,
\]
\[
\Sigma = \{ A \subseteq T: A \cap T_\nu \neq \emptyset, \nu \in I'\},
\]
\[
\nu (A) = \sum_{\nu T} \mu (A \cap T_\nu).
\]

Clearly, \(\Sigma\) is a \(\sigma\)-field of subsets of \(T\) and \(\nu\) is a measure on \(\Sigma\). Let \(w: L_\mu (T, \Sigma, \nu) \rightarrow X\) be defined by
\[
w(f) = \{ f|_T \}
\]
for \(f: L_\mu (v)\),

where \(f|_T\) denotes the restriction of \(f\) to \(T_\nu\). By (1) for \(x' = \varphi_\mu x\), we have
\[
\varphi (u) = \int \varphi (u) \mu \, d\mu = \sum_{\nu T} \varphi (u_\nu) \mu \, d\mu = \int \varphi (u) \mu \, d\mu
\]
where \(v \in L_\mu (\nu)\) is the unique function such that \(\varphi (u) = \varphi (v)\). Clearly, \(||u|| = ||v||\). Since \(u\) is an isometric isomorphism from \(L_\mu (\nu)\) onto \(X\), \(u\) is an isometric isomorphism from \(X'\) onto \([L_\mu (\nu)]\). Hence \(||u|| = ||x'|| = ||v||\). Clearly, the operator \(v \rightarrow \varphi (u)\) is an isometric isomorphism from \(L_\mu (\nu)\) onto \(X'\). Thus, in view of (2), we conclude that \([L_\mu (\nu)] = [L_\mu (\mu)]\).

That completes the proof.

**2. Constructions of subspaces isomorphic to \(L_\mu (\nu)\) in norming subspaces of \([L_\mu (\mu)]\).**

**Definition 2.1.** A double sequence \(\{ (A_n^m)_{n=1}^{\infty} \}_{m=1}^{\infty}\) of non-empty sets is said to be a dyadic tree if
(i) if \(k_i \neq k_i',\) then \(A^m_{n_i} \cap A^m_{n_i'} = \emptyset\),
(ii) \(A^m_{n_i} \supseteq A^m_{n_i'} \cup A^m_{n_i+1}\).

A family \(\mathcal{A}\) of sets is called a dyadic jungle if for each \(A \in \mathcal{A}\) there is a dyadic tree \(\{ (A^m_{n_i})_{n=1}^{\infty} \}_{m=1}^{\infty}\) such that \(A^m_{n_i} = A\) and each \(A^m_{n_i}\) belongs to \(\mathcal{A}\).

**Proposition 2.2.** Let \(Y\) be a subspace of \(m(I')\) satisfying the following conditions:
\[
(+) \text{ there are } K > 1 \text{ and } \delta > 0 \text{ and a dyadic jungle } \mathcal{A}\text{ of subsets of } I\text{ such that for every } m = 1, 2, \ldots, \text{ and for arbitrary finite collection } B_1, B_2, \ldots, B_{m-1}\text{ of mutually disjoint sets in } \mathcal{A}\text{ there exist } f, g \in Y \text{ with } ||f|| \leq K \text{ and } A_1, A_2, \ldots, A_{m-1}\text{ in } \mathcal{A}\text{ such that}
\]
\[
\{ y \in B_i: |f(y) - g(y)| \geq \delta \} \supseteq A_i \quad (i = 0, 1, 2, \ldots, 2^m-1).
\]

Then \(Y\) contains a subspace isomorphic to \(L_\mu (\nu)\).

**Proof.** We will construct a sequence \(\{f^m\}_{m=1}^{\infty}\) in \(Y\) and a dyadic tree \(\{ (A^m_{n_i})_{n=1}^{\infty} \}_{m=1}^{\infty}\) in \(\mathcal{A}\) with the following properties:
(a) \(||f^m|| \leq K \text{ if } m > 0\),
(b) if \(y \in A^m_{n_i}\) and if \(n \geq 1\), then \((-1)^{n+1} f^m(y) > \delta\).

Assume that we have done this. Then for arbitrary real \(t(1), t(2), \ldots, t(n)\) \((n = 1, 2, \ldots)\) we have
\[
\delta \sum_{i=1}^{n} |t(i)| \leq \sum_{i=1}^{n} |t(i)||f^m(i)| \leq K \sum_{i=1}^{n} |t(i)|.
\]

Indeed, by (b), we have
\[
\sum_{i=1}^{n} |t(i)||f^m(i)| \leq K \sum_{i=1}^{n} |t(i)|.
\]
To prove the left-hand side inequality we define for \( r = 0, 1, \ldots, n \) the indices \( p(r) \) as follows:
\[
p(0) = 0; \\
p(r+1) = \begin{cases} 2p(r) & \text{if } t(r) \geq 0, \\
2p(r)+1 & \text{if } t(r) < 0.
\end{cases} \quad (r = 0, 1, \ldots, n-1)
\]

Clearly, \( 0 \leq p(r) \leq 2^r \). By (ii), \( A_n^{(p)} = A_{n-1}^{(p)} \) for \( r = 0, 1, \ldots, n-1 \). Hence, by (b) and the definition of \( p(r) \), if \( \mathbf{q} \in A_n^{(p)} \), then \( \mathbf{t}(\mathbf{q})f^{(p)}(\mathbf{q}) \geq 3l(\mathbf{q}) \) for \( r = 1, 2, \ldots, n \). Hence
\[
\sum_{r=1}^n l(\mathbf{q})f^{(p)}(\mathbf{q}) \geq \sup_{r \leq n} \sum_{r=1}^n l(\mathbf{q})f^{(p)}(\mathbf{q}) \geq \delta \sum_{r=1}^n l(\mathbf{q}),
\]

which completes the proof of inequality (3).

Let us set
\[
w(\mathbf{t}) = \sum_{r=1}^{n} l(\mathbf{t})f^{(p)}(\mathbf{t}) \quad \text{for} \quad \mathbf{t} = (l(\mathbf{t}))_{1}^{n} \subset l.
\]

It follows from (3) that
\[
\delta w(\mathbf{t}) \leq \|w(\mathbf{t})\| \leq K \|w(\mathbf{t})\| \quad \text{for} \quad \mathbf{t} \subset l.
\]

Hence \( w \) is an isomorphism from \( l \) onto the closure of the linear manifold spanned by the sequence \( f^{(p)}(\mathbf{t}) \). Since \( f^{(p)}, X(\mathbf{r} = 1, 2, \ldots, n) \), the range of \( w \) is a subspace of \( X \) which is isomorphic to \( l \).

We define the sequence \( f^{(p)}(\mathbf{t}) \) and the dyadic tree \( [(A_n^{(p)})_{1}^{n}]_{1}^{n} \) by induction. Let \( f^{(p)} = 0 \) and let \( A_0^{(p)} \) be an arbitrary set in \( A \). Suppose that for some \( m \geq 1 \) the functions \( f^{(p)}, f^{(p)}, \ldots, f^{(p-m)} \) in \( Y \) and the sets \( [(A_n^{(p)})_{1}^{n}]_{1}^{n} \) in \( \mathcal{A} \) have been chosen to satisfy conditions (i), (ii), (a), and (b). Since \( \mathcal{A} \) is a dyadic jungle, there are sets \( B_1, B_2, \ldots, B_{2m-1} \) in \( \mathcal{A} \) such that \( B_k \cup B_{2m-k+1} = A_{2m-1}^{(p)} \) and \( B_k \cap B_{2m-k+1} = \emptyset \). Let us set \( f^{(p)} = f \) and \( A_m^{(p)} = A_m \). Then \( f \) and \( (A_n^{(p)})_{1}^{n} \) are chosen for \( B_0, B_1, \ldots, B_{2m-1} \), as in (i). Then for \( m = n \) conditions (a) and (b) are obviously satisfied. Let \( k(\mathbf{r}) = 2^r \mathbf{r} \) if \( \mathbf{r} \) is even and \( k(\mathbf{r}) = 2^{-r}(\mathbf{r}-1) \) if \( \mathbf{r} \) is odd. Then \( B_0 \subset A_0^{(p)} \). Thus \( A_m^{(p)} = A_m \subset B_0 \subset A_0^{(p)} \). Indeed, there are two possibilities.

1. \( k(\mathbf{r}) \neq k(\mathbf{r}) \).

Then (assuming that \( r_1 < r_2 \)) \( r_1 = 2k \) and \( r_2 = 2k+1 \). Hence \( A_0^{(p)} \subset B_0 \subset A_0^{(p)} \subset A_m^{(p)} \). Finally we have \( A_0^{(p)} \subset A_m^{(p)} \subset B_k \subset B_{2m-k} = A_{2m}^{(p)} \). Hence the sets \( [(A_n^{(p)})_{1}^{n}]_{1}^{n} \) satisfy conditions (i) and (ii). That completes the induction and the proof of the Proposition.

**Proposition 2.3.** Let \( \mu \) be a non purely atomic measure. If \( Y \) is a norming subspace of the space \( [L_n(\mu)]^* \), then \( Y \) contains a subspace isomorphic to \( l \).

**Proof.** By Proposition 1.3, we may assume that \( [L_n(\mu)]^* = L_n(\mu) \). By Kakutani's representation theorem ([11], [3], p. 163) the space \( L_n(\mu) = L_n(T, \Sigma, \mu) \) is isometrically and latomically isomorphic to a space \( C(Q) \subset m(\mu) \). This isometric isomorphism induces via characteristic functions a Boolean isomorphism between the Boolean algebra of closed-open sets of \( Q \) and of \( \mu \) - equivalence classes of sets in \( \Sigma \). In the sequel we shall identify closed-open sets in \( Q \) with suitable members of \( \Sigma \) and \( L_n(\mu) \) with \( C(Q) \). Let \( \mathcal{A} \) be the family of those \( A \subset \Sigma \) that \( 0 < \mu(A) < \infty \) and no subset of \( \mathcal{A} \) is an atom for \( \mu \). Then \( \mathcal{A} \) is a dyadic jungle. By Proposition 2.2 to complete the proof it suffices to show that \( Y \) together with \( \mathcal{A} \) satisfies the condition (i). Since \( Y \) is norming,

(*) there is \( K \geq 1 \) such that for each \( \mathbf{q} \in L_n(\mu) \) and for arbitrary finite set \( \{g_1, g_2, \ldots, g_k\} \in L_n(\mu) \) there is \( \mathbf{f} \in Y \) so that \( \|\mathbf{f}\| \leq K\|\mathbf{g}\| \)

\[
\int \mathbf{f} \, d\mathbf{p} = \int g \, d\mu < 2^{-r} \quad (r = 1, 2, \ldots, k).
\]

Let \( B_1, B_2, \ldots, B_{2m-1} \) be a finite collection of mutually disjoint sets in \( \mathcal{A} \). Let \( \text{char} \) denote the characteristic function of \( B_i \). Let us set
\[
\mathbf{g} = [\mu(B_i)]^{-1} \text{char} L_n(\mu) \quad (r = 0, 1, \ldots, 2^m-1).
\]

Let \( \varphi \in L_n(\mu) \) be defined by
\[
\varphi(\mathbf{q}) = \begin{cases} (1)^r & \text{for } \mathbf{q} \in B_i \quad (r = 0, 1, \ldots, 2^m-1), \\
0 & \text{for } \mathbf{q} \notin \bigcup_{r \leq n} B_i.
\end{cases}
\]

Clearly, \( \|\mathbf{g}\| = 1 \). By (i), there exists in \( Y \) an \( f \) with \( \|\mathbf{f}\| \leq K \) so that, for \( r = 0, 1, \ldots, 2^m-1 \),
\[
[\mu(B_i)]^{-1} \int \mathbf{g} \, d\mu - (1)^r < 2^{-r}.
\]

Hence
\[
(1)^r[\mu(B_i)]^{-1} \int \mathbf{g} \, d\mu > 2^{-r}.
\]

Thus if \( A_0 = \{\mathbf{q} \in B_i : (1)^r[\mu(B_i)]^{-1} \int \mathbf{g} \, d\mu > 2^{-r}\} \) and \( A_0 > 0 \) and \( A_0 \subset B_i \). Therefore \( \text{char} \subset \mathcal{A} \). Finally, if \( \mathbf{q} \in A_0 \), then \( (1)^r[\mu(B_i)]^{-1} \int \mathbf{g} \, d\mu > 2^{-r} \). Hence \( Y \) satisfies (i) with \( \delta = 2^{-r} \). That completes the proof.

For purely atomic measures we have

**Proposition 2.4.** Let \( X \) be an uncountable set. Let \( Y \) be a norming separable subspace of \( [l(T)]^* = m(\mu) \). Then \( Y \) contains a subspace isomorphic to \( l \).

**Proof.**
Proof. Let $A = \{A \in \mathcal{F}; \text{ card } A > n\}$. Clearly $\mathcal{A}$ is a dyadic jungle. We will show that $Y$ together with $\mathcal{A}$ satisfies $(\pm)$.

By separability of $Y$, there are $y_n$ in $Y$ such that $\|y_n\| \leq 1$ and the set $\bigcup_{n=1}^{\infty} \{y_n\}$ is dense in the unit ball $\{y \in Y; \|y\| \leq 1\}$. Let us put

$$d(y_1, y_2) = \sum_{n=1}^{\infty} 2^{-n} |y_n(y_1) - y_n(y_2)| \quad \text{for} \quad (y_1, y_2) \in \mathcal{F} \times \mathcal{F}.$$ 

Clearly $d$ is a pseudometric. Let $\gamma_1 \neq \gamma_2$. Choose $\psi = \{(\psi(y))_m\}_{r \in \mathbb{N}}$ in $m(\mathcal{F})$ and $t = (t(\gamma))_{r \in \mathbb{N}}$ in $l_1(\mathcal{F})$ such that $\psi(\gamma_1) = (-1)^{\gamma_1} = (-1)^{\gamma_2}$ and $t(\gamma) = 0$ for $\gamma \neq \gamma_1$ ($\gamma = 1, 2$). Since $Y$ is dense in $m(\mathcal{F})$ in the $l_1(\mathcal{F})$ topology of $m(\mathcal{F})$, there is $y \in Y$ so that

$$\sum_{r=1}^{\infty} \psi(\gamma_1)(\gamma) - \sum_{r=1}^{\infty} \psi(\gamma_2)(\gamma) = 2 - |y(\gamma_1) - y(\gamma_2)| < 1.$$ 

Hence $|y_1(\gamma_1) - y_2(\gamma_2)| > |y(\gamma)|^{-1}$. Since the set $\bigcup_{n=1}^{\infty} \{y_n\}$ is dense in the unit ball of $Y$, there is an index $n$ such that $|y_n|^{-1} - |y_n| < (4|y(\gamma)|)^{-1}$. We have

$y_n(\gamma_2) - y_n(\gamma_1) > |y(\gamma)|^{-1} |y(\gamma_2) - y(\gamma_1)| - 2 |y_1|^{-1} |y_n - y_{n+1}| > (2|y(\gamma)|)^{-1} > 0.$

Therefore $d(\gamma_2, \gamma_1) > 0$. Hence $d$ is a metric in $\mathcal{F}$. The metric space $(\mathcal{F}, d)$ is separable, because the map $\gamma \rightarrow (\gamma_1, 2^{-\gamma_1})$ is a homeomorphic embedding of $(\mathcal{F}, d)$ into the Hilbert cube $(\mathbb{S} = (\{0\} \cup \{-1\})^\infty; \|\cdot\| \leq 2^{-1})$. Therefore each $A$ in $\mathcal{A}$ has a condensation point (i.e. a point with the property that every its $d$- neighborhood is uncountable).

Let $B_1, B_2, \ldots, B_m$ be mutually disjoint sets in $\mathcal{A}$. Choose $\gamma_1$ in $B_1$ so that $\gamma_1$ is a condensation point of $B_1$. Define $\psi$ in $l_1(\mathcal{F})$ by

$$\psi(\gamma) = \begin{cases} 0 & \text{ for } \gamma \neq \gamma_1, \\ 1 & \text{ for } \gamma = \gamma_1. \end{cases}$$

Next define $\varphi$ in $m(\mathcal{F})$ by

$$\varphi(\gamma) = \begin{cases} (-1)^{\gamma} & \text{ for } \gamma = \gamma_r (r = 0, 1, \ldots, 2^m - 1), \\ 0 & \text{ for } \gamma \neq \gamma_r, \gamma_{r+1}, \ldots, \gamma_{2^m}. \end{cases}$$

Since $Y$ is norming, there is $K > 1$ such that the ball $\{y \in Y; \|y\| \leq K\}$ is dense in the ball $\{y \in m(\mathcal{F}); \|y\| \leq 1\}$ in the $l_1(\mathcal{F})$-topology of $m(\mathcal{F})$. Hence there is an $y$ in $Y$ with $\|y\| \leq K$ such that

$$\left| \sum_{r=1}^{2^m} \varphi(\gamma)(\gamma) - \sum_{r=1}^{2^m} \psi(\gamma)(\gamma) \right| < 2^{-2} \quad (r = 0, 1, \ldots, 2^m - 1).$$

Computing the left-hand side of this inequality we obtain

$$|(-1)^{\gamma} - y(\gamma)| < 2^{-1}.$$ 

Hence $(-1)^{\gamma} y(\gamma) > 2^{-1}$ for $r = 0, 1, \ldots, 2^m - 1$.

Since $y(\gamma)$ is a continuous function on $(\mathcal{F}, d)$ and since $\gamma_1$ is a condensation point of $B_1$, the set $A_1$ belongs to $\mathcal{A}$ where

$$A_1 = \{\gamma \in B_1; (-1)^{\gamma} y(\gamma) > 2^{-1}.\}$$

Clearly, we have $A_1 \subset B_1$ and $(-1)^{\gamma} y(\gamma) > 2^{-1}$ for $\gamma \in A_1$. Hence $Y$ satisfies $(\pm)$ with $d = 2^{-1}$. That completes the proof.

3. Main results. We begin with the following lemma:

LEMMA 3.1. Let $X$ be a Banach space and let $\mathcal{L}$ be an abstract set. Then $X$ contains a subspace isomorphic to $l_1(\mathcal{L})$ if and only if there exist a Banach space $Y$ and a linear operator $u: X \rightarrow Y$ such that the range of $u$ contains a subspace isomorphic to $l_1(\mathcal{L})$.

Proof. Necessity. Put $u = \text{id}$ the identity operator on $X$.

Sufficiency. Let $X_1$ be a subspace of $X$ which is isomorphic to $l_1(\mathcal{L})$ and which is contained in $u(X)$. Since $X_1$ is isomorphic to $l_1(\mathcal{L})$, there is $K > 1$ and an indexed set $(\mathcal{L})_{\mathcal{L}}$ of elements of $X_1$ such that

$$K^{-1} \sum_{\mathcal{L}} |t(\mathcal{L})| \leq K \sum_{\mathcal{L}} |t(\mathcal{L})| \leq K \sum_{\mathcal{L}} \|t(\mathcal{L})\| \quad \text{for} \quad \{t(\mathcal{L})\}_{\mathcal{L}} \in l_1(\mathcal{L}).$$

Let $X_1 = \{y \in X; u(y) \in X_1\}$. Then $X_1$ is a closed linear manifold in $X$. Obviously the restriction of $u$ to the subspace $X_1$ is an epimorphism. Hence (cf. [1], p. 58) there are $K_2 > 0$ and indexed set $(\mathcal{L})_{\mathcal{L}}$ of elements of $X_1$ such that $\|u_n - y_n\| \leq K_2 \|x_1(\mathcal{L})\|$. For $\{t(\mathcal{L})\}_{\mathcal{L}} \in l_1(\mathcal{L})$ we have

$$K_2^{-1} \sum_{\mathcal{L}} |t(\mathcal{L})| \leq K_2 \sum_{\mathcal{L}} |t(\mathcal{L})| \leq K_2 \sum_{\mathcal{L}} \|t(\mathcal{L})\| \quad \text{for} \quad \{t(\mathcal{L})\}_{\mathcal{L}} \in l_1(\mathcal{L}).$$

Hence the required operator $u: l_1(\mathcal{L}) \rightarrow X$ defined by

$$u(t(\mathcal{L})) = \sum_{\mathcal{L}} t(\mathcal{L}) \cdot x_1(\mathcal{L})$$

is the required isomorphic embedding. That completes the proof.

PROPOSITION 3.2. Let $X$ contain a seminorming subspace isomorphic to $l_1(\mu)$. If either (1) $\mu$ is not purely atomic, or (2) $X$ is separable and $\mu$ has uncountably many atoms, then $X$ contains a subspace isomorphic to $l_1$.

Proof. By the assumption, there is a proper isomorphism $u: l_1(\mu) \rightarrow X$. Let $u = u_0 \circ \alpha: X \rightarrow [l_1(\mu)]^p$ and let $X = v(X)$. By Proposition 1.2, $Y$ is a subspace of $[l_1(\mu)]^p$. Let $S, S^* \in B$ denote the unit
balls of \(X, X^**, Y, \) and \([L_0(\mu)]^\star\) respectively. By Goldstine’s theorem \((17), \text{ p. 424}\), \(cl^*(\kappa_X S) = S^**\). By continuity of \(\varepsilon\), there is an \(e > 0\) such that \(B = av(S) = av(\kappa_X S)\). Since \(u\) is an isomorphic embedding, \(u^*\) is an isomorphism. Hence \((17), \text{ p. 487}\) there is a \(b > 0\) such that \(u^*(S^*) \supseteq B^*\). Finally, the adjoint operator \(u^*\) is continuous if both spaces \(X^*\) and \([L_0(\mu)]^\star\) carry their \(\tau^\star\)-topologies. Thus
\[
u^*(S^*) = u^*cl^*(\kappa_X S) = cl^*\kappa_X S.
\]
Combining all these inclusions we get
\[
u^*B = av\kappa_X S = av(S^*) = abB^*.
\]
But this means nothing else but the fact that \(Y\) is a norming subspace of \([L_0(\mu)]^\star\).

Next we will show that in both cases (I) and (II) \(Y\) contains a subspace isomorphic to \(l_1\). Assume that we have done this, then, since \(\varepsilon\) maps \(X\) onto \(Y\), to complete the proof it is enough to apply Lemma 3.1.

If (I) then the desired conclusion on \(Y\) is an immediate consequence of Proposition 2.3.

If (II) but (I) does not hold, then the assumption on \(\mu\) implies that \(L_0(\mu)\) is isometrically isomorphic to \(l_1(\Gamma)\) with \(\text{card} \Gamma \geq \aleph_0\). Since \(X\) is separable, \(Y = \varepsilon(X)\) has the same property. The desired conclusion on \(Y\) follows from Proposition 2.4.

**Conjecture.** Let \(X^*\) contain a seminorming subspace isomorphic to \(l_1(\mu)\) where \(\mu\) is a homogenous \((\text{cf. } [13], \text{ [4]}\)) finite measure. Then \(X\) contains a subspace isomorphic to \(l_1(\Gamma)\) with \(\text{card} \Gamma = \text{den}(L_0(\mu))\), where \(\text{den}(L_0(\mu))\) is the smallest cardinal number \(n\) such that there is in \(L_0(\mu)\) a linearly dense set \(A\) with \(\text{card} A = n\).

In the “opposite direction” we have the following result (by \(D^*\) we denote the Cartesian product of \(D\) copies of the discrete two point space \(D\)):

**Proposition 3.3.** If a Banach space \(X\) contains a subspace isomorphic to \(l_1(\Gamma)\) with \(\text{card} \Gamma \geq \aleph_0\), then \(X^*\) contains a subspace isomorphic to \(l_1(\Gamma)\) with \(\text{card} \Gamma = 2^{\aleph_0}\).

Proof. The space \(O(D^*)\) has a dense set, say \(A\), with \(\text{card} A = \text{card} D\) (e.g. the set of all finite linear combinations with rational coefficients of characteristic functions of closed-open subsets of \(D^*)\). Hence \(O(D^*)\) is a quotient space of \(l_1(\Gamma)\) (cf. [12], p. 29). Let \(h: l_1(\Gamma) \to O(D^*)\) be the quotient map. Let \(u: l_1(\Gamma) \to X\) be an isomorphic embedding and let \(j: u(l_1(\Gamma)) \to l_1(\Gamma)\) denote the left inverse of \(u\). For the sake of brevity we put \(\kappa = \kappa_{op} \rho\) and \(\kappa = \kappa_{rop} \rho\). Since \((O(D^*))^\star\) is a \(\mu\)-space (cf. [3], p. 100; 106-107), the operator \(\kappa h: u(l_1(\Gamma)) \to (O(D^*))^\star\) has a norm-preserving extension to an operator from \(X\) into \((O(D^*))^\star\), say \(v\).

We will show that the operator \(\kappa v = (O(D^*))^\star \to X^*\) is the required isomorphic embedding. To this end it is enough to show that there is \(c > 0\) such that
\[
||v^*\kappa u(\mu)|| \geq c||u(\mu)|| \quad \text{for} \quad \mu \in (O(D^*))^\star.
\]

Let \(W = \{u(l_1(\Gamma)): ||u|| \leq 1\}\). Clearly,
\[
j(W) = \{t \Sigma l_1(\Gamma): ||t|| \leq ||u||^{-1}\}.
\]

Using this inclusion and the identity \((v^*\kappa u(\mu))(\mu) = \mu\) for each \(\mu \in (O(D^*))^\star\) we obtain
\[
||v^*\kappa u(\mu)|| = \sup_{||u||=1} ||uv(\mu)||(\mu) = \sup_{||u||=1} ||uv(\mu)|| = \sup_{||u||=1} \sup_{\mu \in \Omega} ||uv(\mu)|| = \sup_{||u||=1} \sup_{\mu \in \Omega} \sup_{||\mu||=1} ||uv(\mu)|| = \sup_{||u||=1} \sup_{||\mu||=1} ||uv(\mu)|| = ||u||^{-1}||v^*\kappa u(\mu)|| = ||\mu||^{-1}||v^*\kappa u(\mu)||
\]

(We use the fact that if \(h: X \to Y\) is a quotient map, then \(h^*: Y^* \to X^*\) is an isometrically isomorphic embedding.) Hence \(v^*\kappa u\) satisfies (4) with \(c = ||u||^{-1}\). This completes the proof of the first part of the Proposition.

The second part of the Proposition is a consequence of the first one, because the subspace of \((O(D^*))^\star\) generated by the family of functionals \((\delta_0)^{op}\) is isometrically isomorphic to \(l_1(\Gamma)\). Clearly, \(D^* = 2^{\aleph_0}\). Here \(\delta_0\) denotes the “point mass” at \(q\), i.e., \(\delta_0\) is the Borel measure which equals \(1\) on any Borel subset of \(D^*\) which contains \(q\), and equals \(0\) otherwise. That completes the proof.

For separable Banach spaces we have the more clear picture than that which is given by Propositions 3.2 and 3.3.

**Theorem 3.4.** If \(X\) is a separable Banach space, then the following conditions are equivalent:

1. \(X\) contains a subspace isomorphic to \(l_1\);
2. \(C = C([0; 1])\) is isomorphic to a quotient space of \(X\);
3. \(X^*\) contains a \((w^*\text{-closed})\) subspace isomorphic to \(C^*\) such that on this subspace the \(X\) topology of \(X^*\) and the \(C\) topology of \(C^*\) coincide;
4. \(X^*\) contains a seminorming subspace isomorphic to \(C^*\);
5. \(X^*\) contains a seminorming subspace isomorphic to \(l_1\);
6. \(X^*\) contains a seminorming subspace isomorphic to \(l_0\) with \(\text{card} \Gamma = 2^{\aleph_0}\);
7. \(X^*\) contains a seminorming subspace isomorphic to \(l_0(\Gamma)\) with \(\text{card} \Gamma \geq \aleph_0\).

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The next examples show that in Theorem 3.4 the assumption of separability of X is in general essential.

Example 1. Let \( X = l_q(l') \) with \( \text{card} l' > \mathfrak{c} \). Then \( X^* = l_1(l') \).
But \( X \) does not contain any subspace isomorphic to \( l_1 \).

Example 2. Let \( X = m \). Then \( G \) is not isomorphic to any quotient space of \( m \), because every separable quotient space of \( m \) is reflexive ([9], p. 168). Hence \( X \) does not satisfy (ii) and therefore (iii) (because for every Banach space \( X \) the conditions (ii) and (iii) are equivalent). Clearly, \( m \) contains a subspace isomorphic to \( l_1 \).

4. A characterization of \( \sigma \)-finite measures \( \mu \) for which the spaces \( L_1(\mu) \) are isomorphic to conjugate spaces.

**Theorem 4.1.** If \( \mu \) is a \( \sigma \)-finite measure, then the space \( L_1(\mu) \) is isomorphic to a conjugate space to a Banach space if and only if \( \mu \) is purely atomic.

**Proof.** Sufficiency. If \( \mu \) is purely atomic, then \( L_1(\mu) \) is isometrically isomorphic to \( l_1(l') \) for some uncountable set \( l' \), then \( X \) contains a subspace isomorphic to \( l_1(l') \) for another uncountable set \( l' \).

The proof of the necessity part is based on the next two lemmas. The first of them generalizes Lemma 3.1.

**Lemma 4.2.** If the closure of the operator \( u : X \to Y \) contains a subspace isomorphic to \( l_1(l') \) for some uncountable set \( l' \), then \( X \) contains a subspace isomorphic to \( l_1(l_1) \) for another uncountable set \( l' \).

**Proof.** By the assumption, there are \( K > 0 \) and a family \( \{y_i\}_{i \in I} \) of elements of the closure of \( u(X) \) such that

\[
K^{-1} \sum_{\gamma \notin I} |t(\gamma)| \leq \sum_{\gamma \in I} |t(\gamma)y_i| \leq K \sum_{\gamma \in I} |t(\gamma)| \quad \text{for} \quad \{t(\gamma)\}_{\gamma \in I} \in l_1(l_1).
\]

For each \( \gamma \in I \), choose \( z_\gamma \) in \( X \) so that \( \|y_\gamma - wz_\gamma\| < (2K)^{-1} \). Let

\( F^{(n)} = \{\gamma \in I : |x_\gamma| \leq n\}, \quad n = 1, 2, \ldots \). 

Since \( \bigcup_{n=1}^{\infty} F^{(n)} = I \) and since \( I \) is uncountable, there is an index \( N \) such that \( F^{(N)} \) is uncountable. Put

\( l' \equiv F^{(N)} \). For \( \{t(\gamma)\}_{\gamma \notin I} \in l_1(l_1) \) we have

\[
N \sum_{\gamma \in I} |t(\gamma)| \leq \sum_{\gamma \in I} |t(\gamma)y_i| \leq \sum_{\gamma \in I} |t(\gamma)| \|y_i\| \leq (K\|w\|^{-1} - (2K\|w\|^{-1})) \sum_{\gamma \in I} |t(\gamma)|.
\]
Hence the operator \( v: L_1(\Gamma') \to X \) defined by
\[
v((t(y))_{m=1}^\infty) = \sum_{m=1}^\infty t(y)_{m} x_m \quad \text{for} \quad ((y))_{m=1}^\infty e_1\Lambda(\Gamma')
\]
is the required isomorphic embedding. That completes the proof.

**Remark.** If \( \Gamma \) cannot be represented as the countable union of sets of cardinalities less than \( \text{card} \Gamma \) (for instance if \( \text{card} \Gamma = 2^{\aleph_0} \)), then we may choose \( \Gamma_1 \) in Lemma 4.2 so that \( \text{card} \Gamma_1 = \text{card} \Gamma \).

**Lemma 4.3.** If \( \mu \) is a \( \sigma \)-finite measure, then no subspace of \( L_1(\mu) = L_1(\Gamma, \Sigma, \mu) \) is isomorphic to a space \( l_1(\ell_2) \) with \( \text{card} \Gamma = \aleph_0 \).

**Proof.** If \( \text{card} \Gamma = \aleph_0 \), then \( L_1(\Gamma) \) is not isomorphic to a Hilbert space. Since every subspace of a Hilbert space is isometrically isomorphic to a Hilbert space, no subspace of the Hilbert space \( L_1(\mu) \) is isomorphic to \( l_1(\ell_2) \). Therefore, in view of Lemma 4.2, it is enough to show that if \( \mu \) is \( \sigma \)-finite, then there exists a linear operator \( u: L_1(\mu) \to L_1(\mu) \) such that the range of \( u \) is dense in \( L_1(\mu) \).

Since \( \mu \) is \( \sigma \)-finite, there is in \( \Sigma \) a sequence \( (T_n)_{n=1}^\infty \) of mutually disjoint sets such that \( \bigcup_{n=1}^\infty T_n = \Gamma \) and \( 0 < \mu(T_n) < \infty \) for \( n = 1, 2, \ldots \). Let \( \chi_n \) denote the characteristic function of \( T_n \). Let us set
\[
u f = \sum_{n=1}^\infty 2^{-n}[\mu(T_n)]^{-1/2} \chi_n f \quad \text{for} \quad f \in L_1(\mu).
\]

Then, by the Schwartz inequality, we have
\[
\|\nu f\|_1 \leq \sum_{n=1}^\infty 2^{-n}[\mu(T_n)]^{-1/2} \|\chi_n f\|_1 \leq \|f\|_2.
\]
(Here by \( \|f\| \) we denote the norm in \( L_1(\mu) \) for \( f = 1, 2, \) and \( \infty \)).

If \( f \in L_1(\mu) \), then
\[
f = \sum_{n=1}^\infty \chi_n f.
\]

Let \( \varepsilon > 0 \). Choose \( N = N(f, \varepsilon) \) so that
\[
\left\|f - \sum_{n=1}^N \chi_n f\right\|_1 < 2^{-1} \varepsilon.
\]

Since \( \mu(T_n) < \infty \), there are functions \( f_n \in L_1(\mu) \) such that \( f_n = \chi_n f_n \), and
\[
\|f_n - \chi_n f_n\|_1 < (2N)^{-1} \varepsilon \quad (n = 1, 2, \ldots, N).
\]

**References**

On differentiability in an important class of locally convex spaces

by

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The most interesting facts in the theory of differentiation in Banach spaces are based on the fact that in a Banach space there exist bounded neighbourhoods of zero. In more general cases these two properties (of being open or bounded) exclude one another. So the generalizations of this theory which are known to us have followed two different ways: defining differentiability "with respect to bounded sets" or "with respect to open sets". The first way was chosen by Sebastião e Silva [12]. A very disagreeable defect of this theory is that differentiability does not imply continuity. However, this implication is valid for Fréchet spaces but it requires a non-trivial proof. Besides, the lack of the mean value theorem in Silva's theory makes it impossible to estimate the remainder by the derivative.

The second way was chosen by several authors, e.g. Marinescu [11], Bastiani [1], Binz [2], Keller [8], Frülicher and Bucher [9]. As is well known (cf. an excellent review of Keller [9]), in the case of a general locally convex space $E$ there does not exist any locally convex topology in $\mathcal{L}(E, F)$ in which the mapping

$$E \times \mathcal{L}(E, F) \times \{h\} \rightarrow L(h) \ast F$$

is continuous. No wonder that nobody succeeded in obtaining in the general case the mean value theorem or an equivalent theorem stating that the continuously Gâteaux differentiable mapping is also Fréchet differentiable. Replacing the continuity of a Gâteaux derivative by a much stronger non-topological condition, Marinescu [11], Bastiani [1] and other authors mentioned above obtain the Fréchet differentiability.

We prove in the present paper that in an important for applications class of Fréchet spaces one can develop a theory of differentation "with respect to open sets" without assuming this condition. Many theorems known in the classical theory of differentiation in Banach spaces are proved here. We give also (in section 3) a natural criterion for the