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The non-existence of L^p estimates for certain translation-invariant operators*

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0. **Introduction.** There is a close relationship between L^p-estimates for solution to partial differential equations with constant coefficients and multipliers in $L^p(R^n)$. Denoting the Fourier transform and its inverse by $\wedge$ and $\vee$ respectively, a complex-valued function $\varphi(\xi)$, $\xi \in R^n$, is said to be a *multiplier* in $L^p(R^n)$ if there exists an estimate

$$|(\varphi\hat{f})^{\vee}(x)|_{L^p(R^n)} \leqslant C|f(x)|_{L^p(R^n)}$$

for all $f \in C_0^\infty(R^n)$ (infinitely differentiable with compact support.) We refer the reader to Hörmander's paper [3] for the basic properties of multipliers and translation invariant operators.

By means of the Fourier transform one can relate the existence of L^p-estimates for solutions and their derivatives of non-homogeneous linear partial differential equations *in all of* R^n in terms of the L^p-norm of the right-hand side. However, in the theory of partial differential equations it is very often of interest to study local L^p-estimates (in the absence of global estimates). For that purpose we introduce the notion of a local multiplier. We say that φ is a *local multiplier* in L^p if for $f \in C_0^\infty(S)$ (infinitely differentiable functions with compact support on S) we have the estimate

$$|(\varphi\hat{f})^{\vee}|_{L^p(S)} \leqslant C|f|_{L^p},$$

C independent of f, and S being a fixed finite cube in R^n. All (global) multipliers are also local multipliers, but not conversely. Namely, we know from well known properties of the wave operator that $\xi_3/(\xi_1^2+\xi_2^2-\xi_3^2)$ is a local multiplier in $L^2(R^3)$. However, it is not a global multiplier.

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The results of this paper are mainly negative. It is known that bounded rational functions of one real variable are multipliers in L^p for all p (1). That this fact has no counterpart for several variables is seen from a number of the examples presented here.

In an earlier paper [5] (see also [6]) one of the authors showed that for solution of the n-(space) dimensional wave equation $\Box u(x,t) = 0$ with smooth initial data having support contained in a fixed bounded set there exists no estimate of the type

$$\int_{R_x^n} |u_t(x,1)|^p dx \leqslant C \int_{R_x^n} (|u_t(x,0)|^p + |\mathrm{grad}_x u(x,0)|^p)\, dx,$$

except for $n=1$ or $p=2$. This can be seen to imply that $\cos|x|$ is not a multiplier in $L^p(R^n)$ except when $n=1$ or $p=2$; a fact also arrived at independently by Wainger [9] (see also Brenner [1]).

It was also shown in [9] that for $p \geqslant 2n/(n-1)$ no estimate of the type

$$\int_{R_t^1}\int_{R_x^n} |u_t(x,t)|^p dx dt \leqslant C \int_{R_t^1}\int_{R_x^n} |\Box u|^p dx dt$$

exists for functions in C_0^∞ (unit cube in $R_t^1 \times R_x^n$). The corresponding question for estimating the L^p-norm of u itself (instead of u_t) is dealt with in Section 1. Using these two results it is shown in Section 2 that for example

$$(\xi_3 + i)(\xi_1^2 + \xi_2^2 - \xi_3^2 + 1 - 2i\xi_3)^{-1},$$

a bounded rational function is not even a local multiplier in $L^p(R^3)$ for $p \geqslant 4$, and that

$$(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_5^2 + 1 - 2\xi_5 i)^{-1},$$

a bounded reciprocal of a polynomial, is not a local multiplier in $L^p(R^5)$ for $p > 8$.

In Section 3 we investigate the non-homogeneous Schrödinger equation

$$\frac{1}{i} u_t + u_{xx} - iu = f$$

(modified by addition of the term $-iu$) and conclude that $(\xi_2 - \xi_1^2 - i)^{-1}$, a bounded reciprocal of a polynomial, is not a multiplier in R^2 for $1 \leqslant p < \frac{4}{3}$. This is done by studying the convolution kernel $e^{-ix^2/4t}e^{-t}$.

Section 4 is devoted to the study of the 1-dimensional convolution kernel $|t|^{-\frac{3}{2}+a}\cos t$, which arises in the study of surface waves. It is shown that convolution with this kernel does not take $L^p \to L^p_{\mathrm{loc}}$ if $|1/p - \frac{1}{2}| > a$.

(1) A. P. Calderón, *Notes on singular integrals*, M. I. T. See also [7].

In Section 5 convolution with the kernel $t^{-\beta}\cos(|x|^2/t)$ $(x \in R^n)$ is studied, and by this means it is shown that the solution to the non-homogeneous Schrödinger equation with zero initial values satisfies no local L^p-estimates for $p \leqslant (2n+2)/(n+4)$, n being the number of space dimensions.

The range of p for which a function is a multiplier in $L^p(R^n)$ is such that the values of $1/p$ form a symmetric interval about $\frac{1}{2}$ (see for example Hörmander's article [3]). For local multipliers and local estimates it can also be easily shown by a duality argument that the range of $1/p$ for which a given function is a multiplier is symmetric about $\frac{1}{2}$. Hence we have content ourselves in stating each result either for $p \geqslant 2$ or $p \leqslant 2$ and have left it for the reader to make analogous conclusions for the conjugate range of p.

1. The wave equation. We consider the following

QUESTION 1. *Does there exist an estimate of the type*

$$\iint_{x,t\,\mathrm{space}} |\vee|^p dx dt \leqslant C \iint_{x,t\,\mathrm{space}} |\Box \vee|^p dx dt \tag{1}$$

for functions $\vee(x,t)$ which are C^∞ and vanish outside a fixed cube in x, t space $(x = (x_1, \ldots, x_n))$?

Answer: *not if* $p > 2n/(n-3)$.

The main tool in treating this question is, as in [5], the explicit solution of the following "Radiation problem": Let $f(t)$ be a C^∞-function of one variable vanishing outside the interval $(0, \varepsilon)$ and positive inside. Find a function $u(x,t)$ vanishing for $t \leqslant 0$ and satisfying

$$u = f(t)\,\delta(x). \tag{2}$$

The solution is given explicity in Courant and Hilbert [2]. For n odd it is

$$u = r^{2-n} \sum_{\nu=0}^{\frac{1}{2}(n-3)} A_\nu r^\nu f^{(\nu)}(t-r), \tag{3}$$

where the A_ν are non-zero constants whose value need not concern us here; for even n there is an analogous, silghtly more complicated formula. For simiplicity we restrict ourselves to the case of n odd. The case of even n can be treated by a combination of methods of this section and [5]. Let $u(r,t) \equiv u_\varepsilon(r,t)$ be the solution to the radiation problem for the function $f(t) = g(t/\varepsilon)$, where g is a C^∞-function vanishing outside the interval $(0, 1)$ and positive inside. We consider the function

$$v(r,t) = \varphi(t)\, u(r,t), \tag{4}$$

where φ is a C^∞-function such that

$$\varphi(t)=1 \quad \text{for} \quad t\leqslant \tfrac{1}{2},$$
$$0<\varphi(t)<1 \quad \text{for} \quad \tfrac12<t<\tfrac34,$$
$$\varphi(t)=0 \quad \text{for} \quad t\geqslant \tfrac34.$$

We then have

(6) $$\Box(\varphi u)=u\Box\varphi+\varphi\Box u-\varphi_t u_t.$$

Now, if $u=0$, which holds for $t>0$, we have

(7) $$|\Box v|^p\leqslant C(|u|^p+|u_t|^p)$$

and therefore

(8) $$\iint\limits_{\frac12<t<1}|\Box v|^p\,dx\,dt\leqslant \iint\limits_{\frac12<t<1}|u|^p\,dx\,dt+\iint\limits_{\frac12<t<1}|u_t|^p\,dx\,dt,$$

where C denotes a generic constant independent of ε (remember that $u\equiv u_\varepsilon$).

The essential step consists in showing that for an appropriate constant $\lambda>2$, and $p>2n/(n-3)$

(9) $$\frac{\lim\limits_{\varepsilon\to 0}\iint\limits_{\lambda\varepsilon<t<\frac12}|u|^p\,dx\,dt}{\iint\limits_{\frac12<t<1}(|u|^p+|u_t|^p)\,dx\,dt}=\infty.$$

Suppose that the above relation holds, and assume that the function $g(y)$ used in the definition of u is symmetric about $y=\frac12$. Consider the function

(10) $$V\equiv V_\varepsilon(r,t)=v_\varepsilon(r,t-\tfrac12\varepsilon)-v_\varepsilon(r,\tfrac12\varepsilon-t).$$

From the way the function u_ε was defined it follows that the support of V is contained in the region $|t|<1$, $|r|<1$. Furthermore,

(11) $$\Box V=\Box v_\varepsilon(r,t-\tfrac12\varepsilon)-\Box v_\varepsilon(r,\tfrac12-t)$$
$$=\delta(x)\left[g\left(\frac{t}{\varepsilon}-\frac12\right)-g\left(\frac12-\frac{t}{\varepsilon}\right)\right]\quad\text{for}\quad |t|\leqslant\tfrac12$$

($\delta(x)=$ Dirac measure). Since g is assumed even about $\frac12$, the bracketed expression vanishes and we have

(12) $$\Box V=0\quad\text{for}\quad |t|\leqslant\tfrac14,$$

at least in the sense of distributions.

Now (8) and (9) imply that

(13) $$\lim_{\varepsilon\to0}\frac{\iint\limits_{t>\lambda\varepsilon}|v|^p\,dx\,dt}{\iint\limits_{t>\lambda\varepsilon}|\Box v|^p\,dx\,dt}=\infty.$$

Hence it follows that

(14) $$\lim_{\varepsilon\to0}\frac{\iint\limits_{|t|>(\lambda-\frac12)}|V|^p\,dx\,dt}{\iint\limits_{x,t\text{ space}}|\Box V|^p\,dx\,dt}=\infty.$$

Now it is not clear whether V is a C^∞-function. Let us merely remark that if not, we can, by modifying V_ε slightly, obtain another one-parameter family of C^∞-functions depending on ε for which (14) holds.

It thus remains to prove (9) for $p>2n/(n-3)$.

We let

(15) $$S_\nu=r^{2-n+\nu}\varepsilon^{-\nu}\left(\frac{t-r}{\varepsilon}\right)\quad\text{and}\quad T_\nu=r^{2-n+\nu}\varepsilon^{-(\nu+1)}g^{(\nu+1)}\left(\frac{t-r}{\varepsilon}\right),$$

so that, letting $n'\equiv\frac12(n-3)$,

(16) $$u=\sum_0^{n'}A_\nu S_\nu\quad\text{and}\quad u_t=\sum_0^{n'}A_\nu T_\nu.$$

Keeping $t>\lambda\varepsilon$, $\lambda>2$,

(17) $$\int|S_\nu|^p\,dx=\int_0^\infty r^{(2-n+\nu)p+n-1}\varepsilon^{-p\nu}\left|g^{(\nu)}\left(\frac{t-r}{\varepsilon}\right)\right|^p dx,$$

and letting $t-r=\varepsilon r'$, $r=t-\varepsilon r'$,

$$\int|S_\nu|^p\,dx=\int_0^1(t-\varepsilon r')^{(2-n+\nu)p+n-1}\varepsilon^{-p\nu}|g^{(\nu)}(r')|^p\,dr'.$$

Now letting $-q=(2-n+\nu)p+n-1$ we see that $q>0$ and for $0\leqslant r'\leqslant1$

(18) $$1\leqslant(t-\varepsilon r')^{-q}/t^{-q}=(1-\varepsilon r'/t)^{-q}\leqslant(1-\lambda^{-1})^{-q}\leqslant2^q,$$

and hence for $t>\lambda\varepsilon$, $\lambda>2$,

(19) $$\int|S_\nu|^p\,dx=t^{(2-n+\nu)p+n-1}\varepsilon^{1-\nu p}.$$

Here the symbol $\cong$ is taken to mean the following: the ratio of the two quantities is bounded from above and below by positive constants independent ε and λ.

Similarly

(20) $$\int|T_\nu|^p\,dx\cong t^{(2-n+\nu)p+n-1}\varepsilon^{1-(\nu+1)p}.$$

Thus

(21) $$\iint\limits_{t>\lambda\varepsilon}|S_\nu|^p\,dx\,dt\cong\lambda^{(2-n+\nu)p+n}\varepsilon^{1+n+(2-n)p},$$

(22) $$\iint\limits_{\frac12<t<1}|S_\nu|^p\,dx\,dt\cong\varepsilon^{1-\nu p}\quad\text{and}\quad\iint\limits_{\frac12<t<1}|T_\nu|^p\,dx\,dt\cong\varepsilon^{1-(\nu+1)p}.$$

Now taking into account (16) we see that for λ sufficiently large

(23)
$$\iint\limits_{\lambda\varepsilon<t<\frac{1}{2}} |u|^p\,dx\,dt \simeq \iint\limits_{\lambda\varepsilon<t<\frac{1}{2}} \sum_{\nu=0}^{n'} |A_\nu|^p |S_\nu|^p\,dx\,dt \simeq \iint\limits_{\lambda\varepsilon<t<\frac{1}{2}} |S_{n'}|^p\,dx\,dt = \varepsilon^{1+n+(2-n)p}.$$

Here $n' = (n-3)/2$ and $\simeq$ means that the two quantities are bounded from above and below by positive constants which are independent of ε, but which may depend on λ and all other parameters. Similarly, we see that in the denominator of (9), u_t being the dominant term,

(24)
$$\iint\limits_{\frac{1}{2}<t<1} (|u|^p+|u_t|^p)\,dx\,dt \simeq \varepsilon^{1-(n'+1)p}.$$

Thus the ratio in (9) $\simeq \varepsilon^{n+(2-n+n'+1)p}$. This ratio will tend to infinity, if $2n+(4-2n+2n'+2)p = 2n+(3-n)p < 0$ or $p > 2n/(n-3)$.

2. Bounded rational functions which are not local multipliers.

LEMMA. *Let $P(\xi)$ and $Q(\xi)$ be polynomials in $\xi = (\xi_1, \ldots, \xi_n)$ and let $D = \frac{1}{i}\frac{\partial}{\partial x}$, in the usual notation. Suppose there exists an estimate*
$$|Q(D)u|_p \leqslant C\,|P(D)u|_p$$
for all C^∞-functions with compact support in a fixed cube K. Then there also exists the estimate
$$|Q(D+\eta)u|_p \leqslant C'\,|P(D+\eta)u|_p,$$
where η is an arbitrary, but fixed complex n-vector.

Proof. The shift formula
$$P(D)u = e^{i\eta}P(D+\eta)e^{-i\eta}u$$
tells us that
$$|e^{i\eta}Q(D+\eta)(e^{-i\eta}u)|_p \leqslant C\,|e^{i\eta}P(D+\eta)(e^{-i\eta}u)|_p \quad \text{for} \quad u \in C_0^\infty(K).$$

Letting $v = e^{-i\eta}u$ we see that
$$|Q(D+\eta)v|_p \leqslant C\,|P(D+\eta)v|_p \quad \text{for all } v \in C_0^\infty(K).$$

In [5] it was proved that there exists no estimate of the type
$$|u_t|_p \leqslant C\,|u_{xx}+u_{yy}-u_{tt}|_p, \quad |D_t u|_p \leqslant C\,|(D_x^2+D_y^2-D_t^2)u|_p$$
for $u \in C_0^\infty(K)$, K being a fixed cube in x, y, t space if $p \geqslant 4$. Hence, by the above lemma, for $p \geqslant 4$ there cannot be an estimate of the type
$$|(D_t+i)u|_p \leqslant C\,|\big(D_x^2+D_y^2-(D_t+i)^2\big)u|_p.$$

Now let $\big(D_x^2+D_y^2-(D_t+i)^2\big)u = v$ and $(D_t+i)u = g$. Then
$$\hat{g} = \Phi\hat{v}, \quad \text{where} \quad \Phi = \frac{\tau+i}{\xi^2+\eta^2-(\tau+i)^2} = \frac{\tau+i}{\xi^2+\eta^2-\tau^2+1-2i\tau}.$$

Thus if Φ is a multiplier in L^p, then the last inequality must hold. Since it is violated, Φ is not a multiplier for $p \geqslant 4$. It is easily seen that Φ is bounded.

THEOREM. *There exist bounded rational functions in R^3 (and hence in R^n, $n \geqslant 3$) which are not even local multipliers for all p.*

Applying the preceding lemma to the result proved in Section 1, we obtain, setting $n = N-1$,

THEOREM. $(\xi_1^2+\ldots+\xi_{N-1}^2-\xi_N^2+1-2\xi_N i)^{-1}$, *a bounded reciprocal of a polynomial, is not a local multiplier in $L^p(R^N)$ for $p > (2N-2)/(N-4)$.*

3. The equation $\frac{1}{i}u_t+u_{xx}-iu = f$.

THEOREM. *$\varphi(\xi, \tau) \equiv (\tau-\xi^2-i)^{-1}$ is not a multiplier in $L^p(R^2_{\xi,\tau})$ for $1 < p < 4/3$ (and the conjugate range).*

Proof. Since the function is bounded, it clearly is a multiplier in $L^2(R^2)$. Letting
$$L \equiv \frac{1}{i}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} - i,$$
a fundamental solution to L is given by $E_1 = e^{-t}E(x, t)$, where
$$E(x, t) = \begin{cases} ie^{i\frac{\pi}{4}}e^{-i\frac{x^2}{4t}}/|t|^{\frac{1}{2}} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0. \end{cases}$$

This follows from the fact that $E(x, t)$ is a fundamental solution of the Schrödinger operator
$$\frac{1}{i}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$$
(see Trèves [8], p. 72). Now let f be an L^∞-function with compact support. Then $f \in L^2 \cap L^p$ and $f_* E_1 = u$ satisfies the equation $Lu = f$. If φ is a multiplier in L^p, since $\hat{u} = \varphi\hat{f}$, we must have
$$|u|_p \leqslant C_p |f|_p.$$

We shall show that this inequality is violated. To that effect we take
$$f_h(x, t) = \begin{cases} \dfrac{1}{hk} & \text{for} \quad 0 \leqslant x \leqslant h,\ 0 \leqslant t \leqslant k, \\ 0 & \text{otherwise}, \end{cases}$$
where $k \equiv h^2/a$ and $h = 1$.

Taking

$$E_2(x,t) = \begin{cases} e^{-t}t^{-1/2}\cos\dfrac{x^2}{t} & \text{for} \quad t > 0, \\ 0 & \text{otherwise}, \end{cases}$$

we see that

$$g_h(x,t) \equiv E_2 * f_h(x,t) = \frac{1}{hk}\int_0^k\int_0^h \cos\left(\frac{(x+h')^2}{t+k'}\right)\frac{e^{-(t+k')}}{\sqrt{t+k'}}\,dh'dk'.$$

We proceed to estimate g_h. First we notice that for $1 \leqslant x \leqslant a/h$, $1 \leqslant t \leqslant 2$, $0 < h' \leqslant 1$,

$$\left|\frac{\cos(x+h')^2}{t+k'} - \frac{\cos x^2}{t}\right| \leqslant C(xh' + x^2k') \leqslant Ca.$$

We note that for $1 \leqslant t \leqslant 2$

$$\frac{1}{hk}\int_0^k\int_0^h \left|\frac{\cos(x+h')^2}{t+k'}\cdot\frac{e^{-(t+k')}}{\sqrt{t+k'}} - \frac{e^{-t}}{\sqrt{t}}\right| dh'dk' \leqslant Ck.$$

Now $|g_h - g_0| \leqslant Ck + Ca$, where $g_0 = \lim_{h\to 0} g_h = E_2$. Furthermore, letting $R = \{x, t\colon 1 \leqslant t \leqslant 2,\ 1 \leqslant x \leqslant a/h\}$, a short calculation shows that

$$\int_R |g_0|^p\,dxdt \geqslant C_p a/h \quad \text{for} \quad h \leqslant h_0.$$

Thus

$$\int_R |g_h|^p\,dxdt \geqslant [C_p - C_p' k^p - C_p'' a^p]a/h.$$

First picking a sufficiently small and then h_0 sufficiently small, and remembering that $k = h^2/a$, it follows that

$$\int_R |g_h|^p\,dxdt \geqslant Ca/h \quad \text{for} \quad h \geqslant h_0.$$

On the other hand,

$$\int_R |f_h|^p\,dxdt = O(h^{3(1-p)}).$$

Thus for the L^p-estimate in question to hold it is necessary that $-1 \leqslant 3(1-p)$ or $p \geqslant \frac{4}{3}$.

A corresponding argument in n-space and one t dimension shows that for the estimate to hold one needs that $p \geqslant (2n+2)/n+2$.

4. Convolution with the kernel $|t|^{-\frac{3}{2}+a}\cos\frac{1}{t}$**.** We next investigate a kernel arising in the theory of surface waves. See for example Lamb [4] Chapter IX.

THEOREM. *Convolution with the kernel*

$$K(t) = |t|^{-\frac{3}{2}+a}\cos\frac{1}{t}$$

(in R^1) does not take L^p into L^p_{loc} if $|1/p - \frac{1}{2}| > a$.

Note. Thus, if $a < 0$, no estimate holds for *any* p $(1 < p < \infty)$. If $a = 0$, the estimate $|K*f|_{L_2} \leqslant C|f|_{L_2}$ has been shown to hold by Frank Jones (unpublished), and is false for values of $p \neq 2$. If $a > 0$, the transformation does not take $L^p \to L^p_{\mathrm{loc}}$ for $|1/p - \frac{1}{2}| > a$. For $a > \frac{1}{2}$ the transformation is bounded $L^p \to L^p_{\mathrm{loc}}$ for all $1 \leqslant p \leqslant \infty$. By complex interpolation it can then be shown that the transformation $L^p \to L^p_{\mathrm{loc}}$ is bounded or $|1/p - \frac{1}{2}| < a$ for $0 < a < \frac{1}{2}$.

Proof of theorem. We treat only the case $a = 0$, the other cases being only slightly more complicated. We shall convolve the kernel $K(t) = t^{-3/2}\cos 1/t$ with the function $f_h(t) = 1/h$ for $0 \leqslant t \leqslant h$, 0 otherwise, to get the function $g_h(t)$. We note that for $t > 0$, $h > 0$,

$$\left|(t+h)^{-3/2}\cos\frac{1}{t+h} - t^{-3/2}\cos\frac{1}{t}\right| \leqslant Cht^{-2}t^{-3/2},$$

where C is a constant independent of h, t. Thus for $Ch/a < t^2$ the above left-hand side does not exceed $a/t^{3/2}$. From this it follows that

$$|g_h(t) - K(t)| \leqslant a/t^{3/2}.$$

Hence,

$$\int_{\sqrt{Ch/a}}^1 |g_h(t)|^p\,dt \geqslant \int_{\sqrt{Ch/a}} |K(t)|^p\,dt - a^p\int_{\sqrt{Ch/a}}^1 t^{-3p/2}\,dt.$$

Estimating as in the previous example, we see that the left-hand side

$$\geqslant \left(\frac{a}{h}\right)^{(3p/2-1)/2}[C_p - a^p C_p'] \approx C_{p,a}'' h^a, \quad \text{where} \quad a = \frac{1-\frac{3}{2}p}{2},$$

for a chosen sufficiently small. Now $\int f h^p\,dt \sim h^{1-p}$, hence for the estimate in question to hold we must have $1 - \frac{3}{2}p \geqslant 2(1-p)$ or $p \geqslant 2$.

By duality it follows that $p \leqslant 2$ hence $p = 2$.

Note. It can be seen from the proof that for $|1/p - \frac{1}{2}| > a$ the transformation does note take $L^p \to L^q_{\mathrm{loc}}$, for q sufficiently close to p, depending on a and p.

5. Convolution by the kernel $K(x,t)=t^{-\beta}\cos(|x|^2/t)$.

THEOREM. *Convolution by the kernel $k(x,t)$ given by $t^{-\beta}\cos|x|^2/t$ for $t>0$ and vanishing for $t<0$ does not satisfy the estimate*

$$|K*f|_{L^p(S)}\leqslant C_p|f|_{L^p}\quad \text{for}\quad p<\frac{n+1}{n+2-\beta},$$

S being a compact set in $R_x^n\times R_t^1$, $f\in C_0^\infty(R_x^n\times R_t^1)$.

Note. For values of β for which $K(x,t)$ is not integrable we may interpret the convolution by analytic continuation with respect to β.

Proof. We shall convolve the kernel with the function

$$f_h(x,t)=\varphi\left(\frac{x}{h},\frac{t}{h^2}\right)h^{-(n+1)},$$

where φ is C^∞, vanishes outside the cube $0<x_i<1$, $0<t<1$, and equals a constant in the cube $\frac14<x_i<\frac34$, $\frac14<t<\frac34$. Such that $\int\varphi\,dxdt=1$. Let

$$h'=(h_1',h_2',\ldots,h_n'),\quad 0\leqslant h_j'\leqslant h<1 \text{ and } 0\leqslant k'\leqslant k<1,\quad \text{where}\quad k=h^2.$$

Then if we restrict the variables x,t by the inequalities $1\leqslant x_i\leqslant 2$, $h/a\leqslant t<1$, $0<a<1$, we have

$$\left|\cos\frac{|x+h'|^2}{t+k'}-\cos\frac{|x|^2}{t}\right|\leqslant C\left[\frac{h}{t}+\frac{h^2}{t^2}\right]\leqslant Ca,$$

where C denotes a generic constant independent of a. This implies

$$\left|(t+k')^{-\beta}\cos\frac{|x+h'|^2}{t+k'}-t^{-\beta}\cos\frac{|x|^2}{t}\right|\leqslant C\frac{a}{t^\beta}+\frac{C'h^2}{t^{\beta+1}}<C\frac{a}{t^\beta}.$$

Denoting by $g_h(x,t)$ the convolution (with respect to x and t) $K*f_h$, it is easily seen that

$$|g_h(x,t)-K(x,t)|\leqslant C\frac{a}{t^\beta}\quad\text{for}\quad 0<h<1,\ \frac{h}{a}<t<1,\ 1\leqslant x_j\leqslant 2.$$

As in the previous two examples, a straight-forward computation shows that to estimate the L^p-norm of g_h (for small h) in the above named region, it suffices to do the same for the kernel $K(x,t)$ instead, provided the parameter a is chosen appropriately small and then kept fixed. Under these conditions we have

$$\int_{\mathfrak{D}_a}|K(x,t)|^p\,dxdt\geqslant C_{a,p}h^{1-p\beta},\quad h<h_0,$$

where $\mathfrak{D}_a$ is the set $1\leqslant x_i\leqslant 2$, $h/a<t<1$.

Meanwhile we have

$$\iint|f_h|^p\,dxdt\leqslant C_p h^{(1-p)(n+2)}.$$

Thus for the estimate $|K*f_h|_{L^p_{\rm loc}}\leqslant C_p|f_h|_{L^p}$ to hold, it is necessary that as we let $h\to 0$, $h^{1-p\beta}\leqslant C_p h^{(1-p)(n+2)}$, i.e., that $1-p\beta\geqslant(1-p)(n+2)$, hence that $p\geqslant(n+1)/(n+2-\beta)$.

COROLLARY. *Consider the initial value problem for the Schrödinger equation*

$$Lu=\frac{1}{i}u_t+\Delta u=f,\qquad u(x,t)\equiv 0\quad\text{for } t<0,$$

where $f\in C^\infty$ and vanishes outside the set S: $0<t<1$, $|x|<1$. Then there is no estimate of the type

$$|u|_{L^p(S)}\leqslant|f|_{L^p},\quad\text{for}\quad p<\frac{n+1}{n/2+2}.$$

The proof consists in observing that the solution is given by convolving f with the kernel of the theorem with $\beta=n/2$.

This corollary, in turn implies our final statement:

$(\xi_1^2+\xi_2^2+\ldots+\xi_{N-1}^2-\xi_N)^{-1}$ *is not a local multiplier in $L^p(R^n)$ for $p<2N/(N+3)$.*

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