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Measures in independent sets

by

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In this note we construct an independent compact set E of real numbers, and a positive measure μ in E whose Fourier-Stieltjes coefficient

$$\hat{\mu}(n) = \int e^{-int}\mu(dt) \qquad (n = 0, \ \pm 1, \ \pm 2, \ldots..)$$

is $o(1)$ as $|n| \to \infty$. This has been accomplished by Rudin [2], using the sets of multiplicity of Salem [4]. Inasmuch as Salem's method is probabilistic it is of interest to perform as much of the work as possible in the same spirit. Kahane has previously achieved this by a method different from ours, involving the Brownian motion process [1]. We wish to thank Professor Kahane for his aid in preparing this note.

1. Let X be the infinite-dimensional cube, its elements being denoted by $x = (x_1, x_2, \ldots, x_k, \ldots)$, where $0 \leqslant x_k \leqslant 1$ $(1 \leqslant k < \infty)$; X is provided always with the product Lebesgue measure P. Similarly, Y is the space of sequences $y = (y_1, y_2, \ldots, y_k, \ldots)$, where $y_k = \pm 1$ $(1 \leqslant k < \infty)$; Y carries the usual product measure λ. We suppose given a sequence $\{b_k\}$ of positive numbers with sum $\sum b_k < \infty$, and a continuous real-valued function h on Y. Setting

$$F(x, y) = \sum_{k=1}^{\infty} b_k x_k y_k + h(y) \qquad (x \in X, \ y \in Y),$$

we note that for each fixed x the range of the function $F(x, \cdot)$ supports a measure with Fourier-Stieltjes coefficient

$$\hat{\mu}(n) = \int \exp\big(-inF(x, y)\big)\lambda(dy).$$

In the next paragraph we shall prove that under certain conditions on h and $\{b_k\}$, $F(x, Y)$ is an independent set for almost all $x \in X$, and in the last paragraph we shall show that under certain conditions on $\{b_k\}$ alone, $\hat{\mu}(n) = o(1)$ for almost all x.

2. Concerning $\{b_k\}$ we assume that

$$\lim_{k\to\infty}\frac{1}{k}\log b_k = -\infty,$$

or equivalently that $\lim\limits_{k\to\infty} c^k b_k = 0$ for every $c > 0$. To obtain a suitable function h, we imitate closely the method of Rudin ([3], p. 101), whereby a homeomorphism of Y onto an independent set of real numbers is constructed. The additional requirement is, that if $y_j = y'_j$ for $1 \leqslant j \leqslant k$, then $|h(y)-h(y')| \leqslant b_k$ $(1 \leqslant k < \infty)$. This can be done with only small adjustments to the method cited, again using Baire's theorem.

THEOREM 1. *$F(x, Y)$ is an independent set for almost all x in X.*

Proof. Let $s_1, s_2, \ldots, s_m$ be integers $\neq 0$, and A the subset of X such that for each $x \in A$ one can choose distinct $y^1, \ldots, y^m$ so that

$$\sum_{l=1}^{m} s_l F(x, y^l) = 0. \tag{1}$$

We shall prove that $P(A) = 0$. Equation (1) can be expressed

$$\sum_{k=1}^{\infty} b_k x_k \Big(\sum_{l=1}^{m} s_l y_k^l\Big) = -\sum_{l=1}^{m} h(y^l) s_l. \tag{2}$$

But, from the independence properties of h, equation (2) implies, for some $J \geqslant 1$,

$$\sum_{l=1}^{m} s_l y_J^l \neq 0. \tag{3}$$

Thus A is a countable union of sets A_J such that (1) and (3) hold for some choice of $y^l (1 \leqslant l \leqslant m)$.

For each integer $r > J$ and each choice of $y^1, \ldots, y^m$ occurring in (1) and (3), let $z^1, \ldots, z^m$ be the vectors obtained by writing 0 for each coordinate beyond the r-th. Then by the continuity conditions on h,

$$\sum_{k=1}^{\infty} b_k x_k \Big(\sum_{l=1}^{m} s_l z_k^l\Big) = -\sum_{l=1}^{m} s_l h(z^l) + R_r, \tag{4}$$

where

$$R_r \leqslant 3 \sum_{l=1}^{m} |s_l| \sum_{k=r}^{\infty} b_k.$$

The number of choices of $z^1, \ldots, z^m$ which must be counted in (4) s $\leqslant 2^{mr}$ and, as will appear, the probability that (4) hold for any fixed choice is $o(2^{-rm})$ as $r \to \infty$. For if all coordinates except x_J are held constant, the left member of (4) is linear in x_J with derivative

$$b_J \sum_{l=1}^{m} s_l z_J^l = b_J \sum_{l=1}^{m} s_l y_J^l \neq 0.$$

Then (4) holds on a linear x_J-set of measure $\leqslant 2b_J^{-1}R_r$, and Fubini's theorem yields the same estimate for the product space X. Finally, $R_r = o(2^{-rm})$ as $r \to \infty$, by the hypothesis on $\{b_k\}$. Hence $P(A_J) = 0$ for each J and so $P(A) = 0$. The theorem now follows from an enumeration of all the choices $s_1, \ldots, s_m$.

3. In this paragraph h need only be measurable, but we assume that $\log b_k = -k \log\log k$ for $k \geqslant 3$. Since this is agreeable with the hypotheses of Theorem 1, the main purpose of this article is attained with the next statement.

THEOREM 2. *For almost all x, the coefficient*

$$\hat{\mu}(n) = \int_Y \exp\big(-inF(x, y)\big)\lambda(dy) = o(1), \qquad |n| \to \infty.$$

Proof. Let n and s be positive integers so that

$$|\hat{\mu}(n)|^{2s} = \int \ldots \int \exp\big(-in[F(x, y^1)+\ldots+F(x, y^s)-\ldots-F(x, y^{2s})]\big)\lambda(dy^1)\ldots$$
$$\ldots\lambda(dy^{2s}) \equiv \int \exp\big(-inH(x, \vec{y})\big)\lambda(d\vec{y}).$$

Thus

$$\int |\hat{\mu}(n)|^{2s} P(dx) = \iint \exp\big(-inH(x, \vec{y})\big)\lambda(d\vec{y})P(dx).$$

When $\vec{y}$ is fixed,

$$\frac{\partial H}{\partial x_k} = \sum_{j=1}^{s} b_k y_k^j - \sum_{j=s+1}^{2s} b_k y_k^j$$

and H is linear in $x_1, x_2, \ldots$ For any real number u

$$\Big|\int_0^1 \exp(iux)\,dx\Big| \leqslant \theta(|u|),$$

where $\theta(t) = 1$, $0 \leqslant t \leqslant 2$, and $\theta(t) = 2/t$, $2 \leqslant t < \infty$.

Writing

$$T_k = \sum_{j=1}^{s} b_k y_k^j - \sum_{j=s+1}^{2s} b_k y_k^j \qquad (1 \leqslant k < \infty),$$

we find

$$\int |\hat{\mu}(n)|^{2s} P(dx) \leqslant \prod_{k=1}^{\infty} E(\theta|nT_k|)$$

(E denotes the mean over Y).

Now $\{T_k = 0\}$ occurs with probability (in Y)

$$2^{-2s}\binom{2s}{s} \leqslant As^{-1/2},$$

and in the complementary case $|T_k| \geqslant 2$. Thus

$$(nb_k)E(\theta\,|nT_k|) \leqslant As^{-1/2} + As^{-1/2}\sum_{p=1}^{2s} p^{-1} \leqslant Bs^{-1/2}\log s.$$

Here we used the fact that for any p, $\{T_k = p\}$ occurs with a smaller probability than $\{T_k = 0\}$.

Define $s = s(n)$ for $n > 3$ by the inequality

$$s \leqslant \log n/\log\log n < s+1,$$

so that for large n,

$$\log\left(Bs(n)^{-1/2}\log s(n)\right) \leqslant -\frac{1}{3}\log\log n.$$

Also

$$1 \leqslant nb_k \quad \text{for} \quad 1 \leqslant k \leqslant 5\log n/\log\log n,$$

whence

$$\log\int |\hat{\mu}(n)|^{2s(n)}P(dx) \leqslant -\frac{5}{3}\log n, \qquad n > n_0.$$

Still following Salem [4], we use the fact that for every M

$$\sum_{n=1}^{\infty} M^{2s(n)}\int |\hat{\mu}(n)|^{2s(n)}P(dx) < \infty,$$

so

$$\limsup_{|n|\to\infty} |M\hat{\mu}(n)| \leqslant 1$$

for almost all x, and the proof is complete.

References

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Modular spaces of generalized variation

by

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In this paper the author continues investigations by J. Musielak and W. Orlicz ([2] and, especially, [3]) about modular functionals of generalized variation. Some fundamental lemmas are first established. Ratio characterizations of inclusions among variation spaces, convex and concave variation, and modular spaces of generalized variation are then treated in succession. The last topic includes the study of locally bounded and locally convex linear topological spaces of generalized variation having the Musielak-Orlicz F-norm topology. Finally, some examples are listed. It is my pleasure here to express my thanks to Dr. Takashi Itô for his excellent advice.

1. M^{th} variation and some fundamental lemmas. Given a real, even, right-continuous function $M(u)$, non-decreasing for $u \geqslant 0$, with $M(0) = 0$ and $M(u) > 0$ for $u > 0$ (such a function will be referred to as a *variation function*) and a real function $x(t)$ defined in a finite closed interval $[a, b]$, the value

$$V_M(x) = \sup_{\pi}\sum_{i=1}^{m} M[x(t_i) - x(t_{i-1})],$$

where $\pi\colon a = t_0 < t_1 < \ldots < t_m = b$ is an arbitrary partition of the interval $[a, b]$, is called the M^{th} *variation* of $x(t)$ in $[a, b]$. It can be shown that

$$M(ax+\beta y) \leqslant aM(x)+\beta M(y) \quad \text{iff} \quad V_M(ax+\beta y) \leqslant aV_M(x)+\beta V_M(y),$$

while

$$M(ax+\beta y) \geqslant aM(x)+\beta M(y) \quad \text{iff} \quad V_M(ax+\beta y) \geqslant aV_M(x)+\beta V_M(y),$$

for $a, \beta \geqslant 0$ and $a+\beta = 1$; that is, M is convex (concave) iff V_M is convex (concave). For a more detailed discussion, see [2]. Let X be the class of all real functions defined on $[a, b]$ which vanish at a. For $x, y \in X$, it is easy to verify that $V_M(x) = 0$ iff $x = 0$, $V_M(-x) = V_M(x)$, and if $a, \beta \geqslant 0$ and $a+\beta = 1$, then $V_M(ax+\beta y) \leqslant V_M(x)+V_M(y)$. Define

$$B_M = \{x \in X\colon V_M(x) < +\infty\};$$