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L^p-multiplier theorems*

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Introduction. In 1939, J. Marcinkiewicz [12] published a multiplier theorem for Fourier series. It gives a sufficient condition for a sequence of numbers λ_n to have the property that multiplication of the Fourier coefficients of a periodic function f by λ_n will yield a periodic function g such that the mapping $f \to g$ is bounded in L^p. Similar results are obtained in higher dimensions.

Analogous and related theorems for Fourier transforms have been obtained by a number of authors, among them Mihlin [15, 16], Hörmander [8], Lizorkin [11], and J. Schwartz [17]. Of these Mihlin and Lizorkin make use of the theorem of Marcinkiewicz. Schwartz obtains a direct analogue of Marcinkiewicz's theorem for functions with values in a Hilbert space. His multipliers are functions taking their values as linear transformations in the Hilbert space. An advantage of his approach is that vector-valued analogues of the Marcinkiewicz theorem as well as of the Littlewood-Paley inequality are obtained as by-products of his vector-valued extension of an inequality of Calderón and Zygmund [3]. However, his results are obtained only in one space-dimension.

We obtain a generalization of Schwartz's results to higher dimensions by introducing an appropriate notion of variation. An important tool in this is the Littlewood-Paley inequality for higher dimensional Euclidean spaces.

We briefly summarize the contents of this paper. Section 1 is devoted to notation. Section 2 deals with multiplier theorems which are Fourier transform analogues of results of Marcinkiewicz [12] and Zygmund [19, 20] (cf. our theorems 2.2 and 2.1 respectively). Section 3 contains the Littlewood-Paley inequality and a resulting stronger multiplier theorem. In Section 4 we undertake a more detailed study of the properties of the variation introduced in Section 2 and give some applications. Section 5

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gives a counter-example to a very reasonable-sounding stronger multiplier theorem.

Section 6 gives an axiomatic treatment of inequalities of Paley-Littlewood type and includes an alternative method of obtaining theorems for functions of many variables as concatenations of one-variable results.

1. Notation. R^d, C^d will denote the d-dimensional real and complex Euclidean spaces respectively. For $x, y \in R^d$ or C^d we define

$$x\cdot y = \sum_{j=1}^{d} x_j\bar{y}_j; \qquad |x|^2 = x\cdot x.$$

$l^2(S)$ will denote the Hilbert space with elements $a = \{a_s\}_{s\in S}$, $a_s \in C$, such that

$$\|a\|_{l^2(S)} = \sum_{s\in S} |a_s|^2 < \infty$$

(this of course requires that $a_s = 0$ for all but countable $s \in S$). For S countable we shall write $l^2(S) = l^2$. Let H be a Hilbert space; $L^p(H)(R^d)$ is the space of H-valued functions $f(x)$ defined in R^d with norm

$$\|f\|_{L^p(H)} = \| \, \|f(x)\|_H \|_{L^p(R^d)}.$$

For $f \in L^1(H)(R^d)$ we define the Fourier transform and the anti-Fourier transform of $f(x)$ by

$$\hat{f}(\xi) = \int_{R^d} f(x)\, e^{-2i\pi x\cdot\xi}\, dx \quad \text{and} \quad \check{f}(\xi) = \int_{R^d} f(x) e^{2i\pi x\cdot \xi}\, dx$$

respectively.

$C_0^\infty(l_0^2)(R^d)$ is the space of functions $f = \{f_n(x)\}$, where all components belong to $C^\infty(R^d)$ and vanish outside a compact set, and where only a finite number of components are non-zero.

$\chi_E(x)$ shall denote the characteristic function of the set E.

2. Multiplier theorems. Let $\Omega_N = \{x;\ |x| = 1,\ x \in C^N\}$ and $d\omega$ be an element of surface on Ω_N.

LEMMA 2.1. *For $a \in C^N$, we have*

$$|a|^p = c_{p,N} \int_{\Omega_N} |a\cdot\omega|^p\, d\omega,$$

where $c_{p,N}$ depends only on p and N.

Proof. Write $a = |a|\,\alpha$; $\alpha \in \Omega_N$ and notice that

$$c_{p,N}^{-1} = \int_{\Omega_N} |\alpha\cdot\omega|^p\, d\omega$$

is independent of a.

LEMMA 2.2 (¹). *Let S be a bounded linear operation on $L^p(R^d)$ and let T be the operator on $L^p(l^2)(R^d)$ given by $T(\{f_n\}) = \{S(f_n)\}$; then $\|T\| = \|S\|$.*

Proof. $\|T\| \geqslant \|S\|$ follows immediately. To show the opposite inequality, note that elements for which all but a finite number of f_n are zero are dense in $L^p(l^2)(R^d)$, so it suffices to prove that

$$\|T(\{f_n\})\|_{L^p(l^2)} \leqslant \|S\|\, \|\{f_n\}\|_{L^p(l^2)}$$

assuming $f_n = 0$ for $n > N$. Using lemma 2.1 and interchanging the order of integration, we have

$$\|T(\{f_n\})\|^p_{L^p(l^2)} = \int_{R^d} \Big\{\sum_{n=1}^{N} |S(f_n)(x)|^2\Big\}^{p/2} dx$$

$$= \int_{R^d} \Big\{c_{p,N} \int_{\Omega_N} \Big|\sum_{n=1}^{N} \bar{\omega}_n S(f_n)(x)\Big|^p d\omega\Big\} dx$$

$$= c_{p,N} \int_{\Omega_N} \Big\{\int_{R^d} \Big| S\Big(\sum_{n=1}^{N} \bar{\omega}_n f_n(x)\Big)\Big|^p dx\Big\} d\omega$$

$$\leqslant \|S\|^p c_{p,N} \int_{\Omega_N} \Big\{\int_{R^d} \Big|\sum \bar{\omega}_n f_n(x)\Big|^p dx\Big\} d\omega$$

$$= \|S\|^p \|\{f_n\}\|^p_{L^p(l^2)}$$

as desired. Note that the above is valid for S defined on a subspace of an arbitrary L^p-space $L^p(\Lambda, d\mu)$, $p > 0$.

Remark. Lemma 2.2 extends to the case where the role of $L^p(R^d)$ is replaced by $L^p(H)(R^d)$, and that of $L^p(l^2)(R^d)$ by $L^p(l^2(H))(R^d)$, H being a Hilbert space. If $H = l^2$ and S is a diagonal operator, i.e. acts as the same scalar operator on each component, the same proof applies with minor modifications (one first works with l_0^2). In the case where S is a more general operator, lemma 1.1 in the proof should be replaced by lemma 6.1 of Section 6. Here it also suffices to take $H = l^2$. Theorem 2.1 stated below also holds for functions $f \in C_0^\infty(l_0^2)(H)(R^d)$. In the proof it is only necessary to apply the vector-valued form of lemma 2.2 in the case where S is diagonal.

Let $\mathscr{L} = \{l_1, l_2, \ldots, l_r\}$ be a finite family of real-valued affine functionals on R^d. Set

$$I_n = \{x;\ l_j(x) \leqslant a_j^{(n)},\ j = 1, \ldots, r\}.$$

THEOREM 2.1. *Let $f \in C_0^\infty(l_0^2)(R^d)$; define*

$$T(f) = \{T_n(f_n)\}, \quad \text{where} \quad T\ (f_n)(x) = (\chi_{I_n}\hat{f}_n)^\vee(x).$$

(¹) cf. Zygmund [18], p. 224.

Then

$$\|T(f)\|_{L^p(l^2)} \leqslant c_p^r \|f\|_{L^p(l^2)},$$

where c_p will denote the norm of the Hilbert transform on $L^p(R^1)$, for the remainder of this section.

Proof. Observe that

$$\chi_{I_n} = \prod_{j=1}^{r} \frac{1}{2}\{1-\operatorname{sgn}(l_j(x)-a_j^{(n)})\}.$$

Pick $y^{(n)} \in R^d$ such that $l_r(y_{(n)}) = a_r^{(n)}$ and denote with " ′ " the operations and sets defined above with $\mathscr{L}$ replaced by $\mathscr{L}' = \{l_1, l_2, \ldots, l_{r-1}\}$. Then

$$T_n(f_n)(x) = \frac{1}{2}\{I-e^{iy(n).x}H\}(T_n'(f_n))(x),$$

where I is the identity and

$$H(f)(x) = (\operatorname{sgn} l_r(\xi)\hat{f}(\xi))^{\vee}(x)$$

is a 1-dimensional Hilbert transform in the direction of grad l_r. We have

$$\|T(f)\|_{L^p(l^2)} \leqslant \frac{1}{2}(\|T'(f)\|_{L^p(l^2)} + \|H^{\#}(T'(f))\|_{L^p(l^2)},$$

where $H^{\#}(f) = \{e^{iy(n).x}H(f_n)\}$. Using lemma 1.2 and the fact that H is a 1-dimensional Hilbert transform (see Zygmund [18]), we obtain

$$\|H^{\#}(f)\|_{L^p(l^2)} = \|H(f)\|_{L^p(l^2)} \leqslant c_p\|f\|_{L^p(l^2)}.$$

The theorem now follows by induction on the number r of linear functionals.

To state the next multiplier theorem we shall need the following definition of $\mathscr{L}$-variation:

Definition 1. Let $\mathscr{L} = \{l_1, \ldots, l_r\}$ be as above. Then for any measurable function $g(x)$ set $V_{\mathscr{L}}(g) = \inf\{M\}$, such that there exists a sequence of functions

$$h^{(m)}(x) = \sum_{k=-\infty}^{\infty} c_{km}\chi_{I_{km}}(x),$$

where $I_{km} = \{x;\ l_j(x) \leqslant a_j^{(km)},\ j = 1, \ldots, r\}$, satisfying:

1. $\sum_k |c_{km}| \leqslant M$,
2. $h^{(m)}(x) \to g(x)$ a.e.

Clearly $|g(x)| \leqslant V_{\mathscr{L}}(g)$ a.e. We postpone a more detailed discussion of the $\mathscr{L}$-variation until Section 4.

Definition 1′. For $g(x)$ taking values in a Banach space B, we simply pick $c_{km} \in B$, replace absolute values by norms and convergence by convergence in the Banach space.

THEOREM 2.2. *For $f \in C_0^\infty(l^2)(R^d)$ define $T(f) = \{T_n(f_n)\}$, where $T_n(f_n) = (g_n\hat{f}_n)^{\vee}$. If $V_{\mathscr{L}}(g_n) \leqslant M$ for all n, then*

$$\|T(f)\|_{L^p(l^2)} \leqslant c_p^r M\|f\|_{L^p(l^2)}, \quad 1 < p < \infty.$$

Proof. Let

$$h_n(x) = \sum_k c_k^{(n)}\chi_{I_k}(n)(x)$$

be an element of the sequence approximating $g_n(x)$ according to definition 1. Let

$$S_n(f_n) = (h_n\hat{f}_n)^{\vee} \quad \text{and} \quad S(f) = \{S_n(f_n)\}.$$

We first show that

$$(*) \qquad \|S(f)\|_{L^p(l^2)} \leqslant c_p^r M\|f\|_{L^p(l^2)}.$$

Namely, using Schwarz's inequality,

$$|S_n(f_n)(x)|^2 = \Big|\sum_k c_k^{(n)}(\chi_{I_k}(n)\hat{f}_n)^{\vee}\Big|^2 \leqslant M\Big\{\sum_k |c_k^{(n)}|\,|\chi_{I_k}(n)\hat{f}_n)|^2\Big\}.$$

Therefore, by Theorem 2.1,

$$\begin{aligned}\|S(f)\|_{L^p(l^2)}^p &= \int_{R^d}\Big[\sum_n |S_n(f_n)(x)|^2\Big]^{p/2}dx \\ &\leqslant M^{p/2}\int_{R^d}\Big[\sum_{n,k}|c_k^{(n)}|\,|(\chi_{I_k}(n)\hat{f}_n)^{\vee}(x)|^2\Big]^{p/2}dx \\ &= M^{p/2}\int_{R^d}\Big[\sum_{n,k}|(\chi_{I_k}(n)(|c_k^{(n)}|^{1/2}f_n)^{\wedge})^{\vee}(x)|^2\Big]^{p/2}dx \\ &\leqslant c_p^{rp}M^{p/2}\int_{R^d}\Big[\sum_{n,k}|c_k^{(n)}|\,|f_n(x)|^2\Big]^{p/2}dx \leqslant c_p^{rp}M^p\|f\|_{L^p(l^2)}^p,\end{aligned}$$

thus establishing $(*)$. That S may be replaced by T in $(*)$ can be seen direct by using the Lebesgue dominated convergence theorem and Fatou's lemma.

THEOREM 2.2′. *The above theorem remains true if the functions $f_n(x)$ take values in a Hilbert space H, and $g_n(x)$ take values in $\mathscr{L}(H, H)$.*

3. Inequalities of the Littlewood-Paley type. Combining theorem 2.2 (or 2.2′) with a Littlewood-Paley type inequality will result in a strengthened multiplier theorem. We begin by quoting a 1-dimensional version of this inequality:

THEOREM 3.1 (J. Schwartz [17]). *Let $F\colon R\to H$ (H — a Hilbert space), and $f\in L^p(H)(R)$. Define*

$$Q_n=\{t\in R;\ 2^{n-1}\leqslant |t|<2^n\},\ n=0,\pm1,\pm2,\ldots,\quad \textit{and}\quad \hat f_n=\chi_{Q_n}\hat f$$

(where the Fourier transform is taken in R with values in H). Then

$$C_p^{-1}\int_R\Big(\sum_n\|f_n(t)\|_H^2\Big)^{p/2}dt\leqslant\int_R\|f(t)\|_H^p\,dt\leqslant C_p\int_R\Big(\sum_n\|f_n(t)\|_H^2\Big)^{p/2}dt$$

where C_p is a constant depending only on p, $1<p<\infty$.

THEOREM 3.2. *Let $\{l_1,\ldots,l_r\}$ be as before. For the multi-index $N=(n_1,\ldots,n_r)$, $n_j=0,\pm1,\pm2,\ldots$, define*

$$Q_N=\{x\in R^d;\ 2^{n_j-1}\leqslant|l_j(x)|\leqslant 2^{n_j},\ j=1,\ldots,r\}.$$

Consider $f\in L^p(H)(R^d)$ and set $\hat f_N=\chi_{Q_N}\hat f$. Then

$$C_p^{-r}\int_{R^d}\Big(\sum_N\|f_N(x)\|_H^2\Big)^{p/2}dx\leqslant\int_{R^d}\|f(x)\|_H^p\,dx\leqslant C_p^r\int_{R^d}\Big(\sum_N\|f_N(x)\|_H^2\Big)^{p/2}dx.$$

Proof. The proof easily follows by induction on r with the application of theorem 3.1, and we include it.

When $r=1$, identifying $R=\{t;\ l_1(x)=t\}$ and integrating the inequality of theorem 3.1 in the orthogonal $(d-1)$-dimensional subspace of R^d ($\{x,\ l_1(x)=0\}$) the result easily follows.

For $r>1$, let $\mathscr H$ be the Hilbert space of $r-1$ tuples of elements in H. Denote with "$'$" sets, functions and operations as defined above for the family of affine functionals $\{l_1,\ldots,l_{r-1}\}$. Set $F\colon R^d\to\mathscr H$ $F=\{f_{N'}\}$; identify again $R=\{t,\ l_r(x)=t\}$ and $R^{d-1}=\{x,\ l_r(x)=0\}$. If $y\in R^{d-1}$, using theorem 3.1. we get

$$C_p^{-1}\int_{R^d}\Big(\sum_N\|f_N(x)\|_H^2\Big)^{p/2}dx=C_p^{-1}\int_{R^{d-1}}\Big\{\int_R\Big(\sum_n\|F_n(y,t)\|_{\mathscr H}^2\Big)^{p/2}dt\Big\}dy$$

$$\leqslant\int_{R^{d-1}}\Big\{\int_R\|F(y,t)\|_{\mathscr H}^p\,dt\Big\}dy=\int_{R^d}\|F(x)\|_{\mathscr H}^p\,dx$$

$$\leqslant C_p\int_{R^{d-1}}\Big\{\int_R\Big(\sum_n\|F_n(y,t)\|_{\mathscr H}^2\Big)^{p/2}dt\Big\}dy=C_p\int_{R^d}\Big(\sum_N\|f_N(x)\|_H^2\Big)^{p/2}dx.$$

But using our inductive hypothesis we have

$$C_p^{-(r-1)}\int_{R^d}\|F(x)\|_{\mathscr H}^p\,dx\leqslant\int_{R^d}\|f(x)\|_H^p\,dx\leqslant C_p^{r-1}\int_{R^d}\|F(x)\|_{\mathscr H}^p\,dx$$

and the theorem follows.

As a corollary of theorem 3.2 we obtain a strengthened version of theorem 2.2 (or 2.2′) which bears a close relation to a theorem of Marcinkiewicz [12] on Fourier series (see also Schwartz [17]).

In the next theorem the $\mathscr L$ and $\{Q_N\}$ are defined as before, using the same set of affine functionals for both.

THEOREM 3.3. *Let $G(x)$ be a scalar-valued function defined in R^d; assume that $V_{\mathscr L}(\chi_{Q_N}G)\leqslant B$ (independent of N). Then setting, for $f\in C_0^\infty(R^d)$, $[T(f)]^\wedge(x)=G(x)\hat f(x)$, we have*

$$\|T(f)\|_{L^p}\leqslant B_p^d B\|g\|_{L^p},$$

B_p depending on p only, $1<p<\infty$.

Proof. Recalling the definition $[T(f)]_N=\{(T(f))^\wedge\chi_{Q_N}\}^\vee$ and applying theorem 2.2 and 3.2, we get

$$\|T(f)\|_{L^p}^p\leqslant C_p^d\int_{R^d}\Big(\sum_N|[T(f)]_N(x)|^2\Big)^{p/2}dx$$

$$\leqslant c_p^dC_p^dB^p\int_{R^d}\Big(\sum_N|f_N(x)|^2\Big)^{p/2}dx\leqslant c_p^dC_p^{2d}B^d\|f\|_{L^p}$$

and the theorem follows.

4. The $\mathscr L$-variation. We first note that if $\bar{\mathscr L}=\mathscr L\cup\{\bar l\}$, $\bar l(x)=\lambda l(x)+a$ ($\lambda, a\in R$; $l\in\mathscr L$), then $V_{\bar{\mathscr L}}(g)=V_{\mathscr L}(g)$ for any g. Hence without loss of generality assume that the l's are linear and $\lambda l\notin\mathscr L$ for $\lambda\neq1$, $l\in\mathscr L$.

Next let us compare, for functions of a single variable, $V_{\mathscr L}$ with the classical notion of variation. In that case it is easily seen that if $P=\{x_n\}_{-\infty}^{\infty}$ is a partition of R^1 and $g(x)$ is a function of bounded variation, then, setting

$$c_{j,P}=g(x_j)-g(x_{j-1})\quad\text{and}\quad \chi_{I_j}=(-\infty,x_j),$$

the functions $g_P(x)=\sum_j c_{j,P}\chi_{I_j}(x)$ satisfy the conditions for the approximating functions of definition 1.1. As P becomes more refined, $g_P(x)-g(x)\to$ constant a.e.

Define

$$\mathfrak v_{\mathscr L}(R^d)=\{f;\ f\colon R^d\to C;\ V_{\mathscr L}(f)<\infty\}.$$

THEOREM 4.1. *$\mathfrak v_{\mathscr L}(R^d)$ is a Banach algebra with respect to the norm $V_{\mathscr L}(\cdot)$ and pointwise multiplication. If $\mathscr L_1\subsetneqq\mathscr L_2$ (possible only if $d>1$), then*

$$\mathfrak v_{\mathscr L_1}(R^d)\subsetneqq\mathfrak v_{\mathscr L_2}(R^d)\quad\textit{and}\quad V_{\mathscr L_2}(g)\leqslant V_{\mathscr L_1}(g).$$

Proof. It is easily checked that $V_{\mathscr L}(\cdot)$ enjoys all the properties of a norm. To prove that $\mathfrak v_{\mathscr L}(R^d)$ is an algebra, we note first that if $f,g\in\mathfrak v_{\mathscr L}(R^d)$ and φ_n and ψ_n are two sequences of functions converging a.e. and boundedly to f and g respectively (as in definition 1), then $\varphi_n(x)\psi_n(x)$ converges to $f\cdot g$ in the same sense. Hence $f\cdot g\in\mathfrak{v}_{\mathscr{L}}(R^d)$ and $V_{\mathscr{L}}(fg)\leqslant V_{\mathscr{L}}(f)V_{\mathscr{L}}(g)$. Next, let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{v}_{\mathscr{L}}(R^d)$. As has been observed,

$$\|f_{n_1}-f_{n_2}\|_{L^\infty}\leqslant V_{\mathscr{L}}(f_{n_1}-f_{n_2})$$

and therefore $\{f_n\}$ is a Cauchy sequence in $L^\infty(R^d)$, and converges to a function $f(x)\in L^\infty(R^d)$ a.e. and boundedly. It remains to show that $f\in\mathfrak{v}_{\mathscr{L}}(R^d)$ and the convergence is in $V_{\mathscr{L}}(\cdot)$. To that effect, let $\{\varphi_{n,k}\}$ be a sequence of function converging to f_n a.e. such that $V_{\mathscr{L}}(\varphi_{n,k}-f_n)<2^{-n}$ (as in definition 1); then the double index sequence $\{\varphi_{n,k}\}$ converges to f a.e. as $n\to\infty$. Hence $f\in\mathfrak{v}_{\mathscr{L}}(R^d)$; moreover, $\varphi_{n,k}-\varphi_{n_1,k_1}$ converges a.e. to f_n-f, as $k, k_1, n_1\to\infty$ and n remains fixed; and since $V_{\mathscr{L}}(\varphi_{n,k}-\varphi_{n_1,k_1})\leqslant V_{\mathscr{L}}(f_n-f_{n_1})+o(1)$, the first of the theorem is established.

To prove the second part, observe from definition 1 that when $l\in\mathscr{L}_2$ and $l\notin\mathscr{L}_1$, then $V_{\mathscr{L}_1}(\operatorname{sgn}(l))=\infty$. We leave the rest to the reader.

Now we temporarily restrict ourselves to the case where $r=d$ and $l_j(x)=x_j$, and in that special case denote $\mathfrak{v}_{\mathscr{L}}(R^d)$ by $\mathfrak{v}(R^d)$ and $V_{\mathscr{L}}(\cdot)$ by $V(\cdot)$. Our immediate aim is to obtain an equivalent characterization of the space $\mathfrak{v}(R^d)$. In introducing the next few definitions, we follow Cramér [4], p. 76-83. If $F(x_1,\ldots,x_d)$ is an arbitrary complex-valued function defined in R^d, and I is the half-open interval $a_j<x_j\leqslant b_j$, we define

$$\Delta_I F=\sum_{j=0}^{d}(-1)^{d-j}F(b_1,\ldots\ldots,b_j,a_{j+1},\ldots,a_d)$$

and

$$J=J(I)=\{x;\,x_j\leqslant b_j,\ j=1,\ldots,d\}.$$

Let $\pi=\{I\}$ be the finite collection of intervals of the previous type induced by a partition $\ldots<x_j^{(k-1)}<x_j^{(k)}<\ldots$ $(j=1,\ldots,d)$ and write

$$U_\pi(F)=\sum_{I\in\pi}|\Delta_I F|\quad\text{and}\quad U(F)=\sup_\pi U_\pi(F)\ \text{for all partitions }\pi.$$

Let $\mathscr{U}(R^d)$ be the space of all F such that $U(F)<\infty$.

Helly's Theorem. *Let $\{F_n\}$ be a sequence of uniformly bounded functions belonging to $\mathscr{U}(R^d)$ such that $U(F_n)\leqslant M$. Then there exists a subsequence converging pointwise everywhere to a function $F\in\mathscr{U}(R^d)$, and $U(F)\leqslant M$.*

Proof. It is shown in Cramér [3] that there exists an $F^*\in\mathscr{U}(R^d)$ such that a subsequence converges to F^* at all points of continuity F^*, which are contained in the complement of the union, S, of a denumerable number of $(d-1)$-dimensional hyperplanes each perpendicular to a coordinate axis. Assume that the theorem is true for $\mathscr{U}(R^{d-1})$. Then by a diagonal process we can extract a further subsequence converging to a function $F\in\mathscr{U}(R^d)$ agreeing with F^* in the complement of S, since

$$\lim_{n_k\to\infty}U_\pi(F_{n_k})=U_\pi(F)\leqslant M.$$

Theorem 4.2. *Given $g\in\mathfrak{v}(R^d)$ with compact support, there exists $F\in\mathscr{U}(R^d)$ with compact support, such that $g=F$ a.e. and conversely if $F\in\mathscr{U}(R^d)$, then $F\in\mathfrak{v}(R^d)$. Moreover,*

$$V(g)=U(F)=\int_{R^d}d|\mu|,\quad\text{where}\quad\mu=\frac{\partial^d F}{\partial x_1\ldots\partial x_d},$$

in the sense of distributions, is measure of bounded variation.

Proof. Suppose $g\in\mathfrak{v}(R^d)$. Then there exists a sequence

$$h^{(m)}(\tau)=\sum_k c_{km}\chi_{I_{km}}(x),$$

where $I_{km}(x)=\{x;\,x_j\leqslant d_j^{(km)}\}$, and $\Sigma|c_{km}|\leqslant V(g)+\varepsilon$.

Since $U(\cdot)$ is a semi-norm, we have

$$U(h^{(m)})\leqslant\sum_k|c_{km}|\,U(\chi_{I_{km}})=\sum_k|c_{km}|$$

due to the fact that $U(\chi_{I_{km}})=1$. Hence $U(h^{(m)})\leqslant V(g)+\varepsilon$, and applying Helly's theorem, there exists a subsequence of $\{h^{(m)}\}$ converging everywhere to $F\in\mathscr{U}(R^d)$, with $U(F)\leqslant V(g)+\varepsilon$, thus proving the first part of the theorem.

To prove the second part of the theorem we first observe that if $F\in\mathscr{U}(R^d)$ and has compact support, then

$$\mu=\frac{\partial^d F}{\partial x_1\ldots\partial x_d}$$

is a finite measure of bounded variation (where differentiation is taken in the sense of distributions), and

$$\int_{R^d}d|\mu|=U(F)$$

(see Cramér [4]). Let $\pi=\{I\}$ be a partition of R^d (as above) where $I=\{x;\,a_j<x_j\leqslant b_j\}$ and $a=\{a_j\}$. Let

$$F_\pi=\sum_{I\in\pi}F(a)\chi_I(x).$$

Then $F_\pi \to F$ a.e., and pointwise boundedly, as the maximal diameter of the partition π tends to zero. By a simple summation by parts argument we have

$$F_\pi(x) = \sum_{I\in\pi} (\Delta_I F)\,\chi_J(x),$$

where we continue to use the notation introduced earlier; then

$$V(F) \leqslant \sup_j \sum_j |\Delta_I F| \leqslant U(F),$$

and the theorem follows.

Our next aim is to establish a correspondence between the spaces $\mathfrak{v}_{\mathscr{L}}(R^d)$ and $\mathscr{U}(R^r)$ for $r \geqslant d$. Let $\mathscr{L}$, as before, denote the real affine functionals $l_1(x), \ldots, l_r(x)$ and introduce the affine transformation $P\colon R_x^d \to R_\xi^r$ given by $\xi_j = l_i(x_1, \ldots, x_d)$, $j = 1, \ldots, r$. Each function $f(\xi)$ on R_ξ^r gives rise to a function $(\bar{P}f)(x) = f\big(P(x)\big)$ in $x \in R_x^d$.

THEOREM 4.3. *Given $g \in \mathfrak{v}_{\mathscr{L}}(R^d)$ with compact support there exists $F \in \mathscr{U}(R^r)$ having compact support such that $\bar{P}(F) = g$ a.e. in R^d. Conversely, if $F \in \mathscr{U}(R^r)$, then $\bar{P}(F) \in \mathfrak{v}_{\mathscr{L}}(R^d)$. Moreover,*

$$U(F) = V_{\mathscr{L}}(g).$$

Proof. Observing that if $I = \{x;\, l_j(x) \leqslant a_j,\, j = 1, \ldots, r\}$ and $\bar{I} = \{\xi;\, \xi \leqslant a_j,\, j = 1, \ldots, r\}$, then $\bar{P}(\chi_{\bar{I}})(x) = \chi_I(x)$, the proof proceeds along the same lines as in theorem 4.2.

Definition. Let $Q_N = \{x;\, 2^{n_j - 1} \leqslant |x| \leqslant 2^{n_j},\ j = 1, \ldots, d\}$, $n_j = 0, \pm 1, \pm 2, \ldots$ $F(x)$ defined in R^d is said to belong to $\mathscr{M}(R^d)$ if

$$\sup_N U(\chi_{Q_N} F) = M(F) < \infty.$$

This definition applied in conjunction with theorem 4.3 yields the following alternate version of theorem 3.3:

THEOREM 4.4. *Let G be scalar-valued function in R^d such that $G = \bar{P}(F)$ a.e. in R^d, where $F \in \mathscr{M}(R^r)$; then, for $f \in C_0^\infty(R^d)$,*

$$\|(G\hat{f})^\vee\|_{L^p(R^d)} \leqslant B_p^r M(F)\, \|f\|_{L^p(R^d)},$$

B^p depending only on p, $1 < p < \infty$.

Our next aim is to obtain a more explicit formula for $M(F)$ when $F \in C^d(R^d)$. If we denote by $J(a_1, \ldots, a_d)$ the set $\{x;\, x_j \leqslant a_j\}$, then the following identity is valid:

$$\chi_I = \sum_{k=0}^{d} (-1)^{d-k} \chi_{J(a_1, \ldots, a_k, b_{k+1}, \ldots, b_d)},$$

where $I = \{x;\, a_j < x_j \leqslant b_j\}$. We observe that

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d} (\chi_{J(a)}) = \delta_a(x) = \delta(x-a).$$

In what follows α, β will denote d-tuples of zeros and ones. Denote by R_α the subspace of R^d of dimension $|\alpha|$ given by $R_\alpha = \{x;\, x_j = 0$ for $\alpha_j = 0\}$; let I_α by the projection of I on R_α, and x_α be the variable in R_α. We observe that

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d}(\chi_I F) = \sum_{|\alpha+\beta|=d} \left(\frac{\partial}{\partial x}\right)^\alpha (\chi_I)(x) \left(\frac{\partial}{\partial x}\right)^\beta (F)(x)$$
$$= \sum_{|\alpha+\beta|=d} \frac{\partial^{|\alpha|}}{\partial x^\alpha}(\chi_{I_\alpha})(x_\alpha) \cdot \chi_{I_\beta}(x_\beta) \left(\frac{\partial}{\partial x}\right)^\beta (F)(x),$$

where the equality is understood in the sense of measures. Now since

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(\chi_{I_\alpha})(x_\alpha) = \sum_{k=0}^{|\alpha|} (-1)^{|\alpha|-k} \delta_{a_{j_1} a_{j_2} \ldots a_{j_k} b_{j_{k+1}} \ldots b_{j_{|\alpha|}}}(x_\alpha)$$

if we write

$$\frac{\partial^d}{\partial x_1 \ldots \partial x_d}(\chi_I F) = \mu,$$

we have

$$\int_{R^d} d|\mu| = \sum_{|\alpha+\beta|=d} \left(\sum_{k=0}^{|\alpha|} \int_{I_\beta} \left| \left(\frac{\partial}{\partial x}\right)^\beta (F)(x_\beta, a_{j_1}, \ldots, a_{j_k}, b_{j_{k+1}}, \ldots, b_{j_{|\alpha|}}) \right| dx_\beta \right).$$

With this in mind, we can write for $F \in C^d(R^d - \bigcup_j \{x;\, x_j = 0\}$,

$$M(F) = \sup_{n=(n_1,\ldots n_d)} \left\{ \sum_{|\alpha+\beta|=d} \left(\sum_{k=0}^{|\alpha|} \int_{Q_{N,\beta}} \left| \left(\frac{\partial}{\partial x}\right)^\beta F(x_\beta, 2^{n_{j_1}-1}, \ldots, 2^{n_{j_k}-1}, 2^{n_{j_k}}) \right| dx_\beta \right) \right\},$$

where $Q_{N,\beta}$ is the projection of Q_N on R_β.

THEOREM 4.5. *If $F \in C^d(R^d - \bigcup_j \{x, x_j = 0\})$ and $|x^\alpha (\partial/\partial x)^\alpha (f)(x)| \leqslant B$, where α is an arbitrary d-tuple of zeros and ones with $|\alpha| \leqslant d$, then $F \in \mathscr{M}(R^d)$ and $M(F) \leqslant 2^{2d} dB$.*

This theorem in conjunction with theorem 4.4 gives a generalization of a result of Mihlin [15], [16] and Lizorkin [11] who considered only the case $r = d$. As an illustration we mention

THEOREM 4.6. *Let $h(t)$ be a complex-valued function of the real variable $t \in R^1$. Assume that*

$$\left| t^k \frac{d^k h}{dt}(t) \right| \leqslant C_k \quad \text{for} \quad t \neq 0,\ k = 0, \ldots, r.$$

Let $\{l_k(x)\}_{k=1}^r = \mathscr{L}$ *be a family of affine functionals from* R^d *into* R^1. *Then*

$$G(x) = h\Big(\prod_{k=1}^r l_k(x)\Big)$$

is a multiplier in $L^p(R^d)$. *More explicitly, for* $f \in C_0^\infty(R^d)$

$$\|(G\hat{f})^\vee\|_{L^p} \leqslant B_p \max_k \{C_k\} \|f\|_{L^p},$$

$1 < p < \infty$. B_p *depending on* p, r *and* d *only.*

Proof. Consider the affine transformation P from R_x^d into R_ξ^r given by $\xi_k = l_k(x)$, and the corresponding linear mapping $\bar{P}$ from functions in R^r to functions in R^d,

$$\bar{P}(F)(x) = F\big(P(x)\big).$$

Set $F(\xi) = h(\xi_1, \ldots, \xi_r)$. Then $\bar{P}(F)(x) = G(x)$. Hence by theorems 4.4 and 4.5 it suffices to show that

$$\left|\xi^a\left(\frac{\partial}{\partial\xi}\right)^a F(\xi)\right| \leqslant B_p \max_k \{C_k\},$$

where a is any r-tuple of zeros and ones. It easily follows by induction that for such a's

$$\xi^a\left(\frac{\partial}{\partial\xi}\right)^a F(\xi) = \sum_{j=0}^{|a|} C_j(\xi_1, \xi_2, \ldots, \xi_r)^j h^{(j)}(\xi_1, \xi_2, \ldots, \xi_j),$$

where the C_j are constants depending on $|a|$ and r only, and $h^{(j)}$ denotes the j-th derivative of h. From the assumption on $h(t)$, the theorem follows.

Since the bounded ratio of two polynomials satisfies the conditions of the previous theorem, we obtain the following

COROLLARY. *Let* $h(t) = P(t)/Q(t)$ *be uniformly bounded for all* $t \in R^1$, *where* P *and* Q *are polynomials, and let* $\mathscr{L} = \{l_k(x)\}_{k=1}^r$ *be a family of affine functionals from* R^d *into* R^1. *Then*

$$G(x) = h\Big(\prod_{k=1}^r l_k(x)\Big)$$

is a multiplier in $L^p(R^d)$, $1 < p < \infty$.

We proceed to give some alternate forms of theorem 4.5 which may be easier to apply in some concrete situations.

Let

$$\theta_i(\cdot) = x_i \frac{\partial}{\partial x_i}(\cdot) \quad \text{and} \quad \theta^a = \theta_1^{a_1}, \theta_2^{a_2}, \ldots, \theta_d^{a_d}.$$

Then a straight forvard argument shows that the conditions in the hypothesis of theorem 4.5 can be replaced by the conditions $|\theta^a F(x)| \leqslant B$, $0 \leqslant |a| \leqslant d$. Making the substitution $x_i = e^{t_i}$ for $t_i > 0$, and $x_i = -e^{t_i}$ for $x_i < 0$, we have

$$\theta_i\big(F(x)\big) = \pm\frac{\partial}{\partial t} F(\pm e^{t_1}, \pm e^{t_2}, \ldots, \pm e^{t_d}).$$

Thus we have the

COROLLARY TO THEOREM 4.5. *Suppose*

$$F(x_1, \ldots, x_d) \in C^d\big(R^d - \bigcup_j \{x; x_j = 0\}\big),$$

and for every d-tuple of zeros and ones a, $|a| \leqslant d$, *we have*

$$\left|\frac{\partial^a}{\partial t^a} F(\pm e^{t_1}, \pm e^{t_2}, \ldots, \pm e^{t_d})\right| \leqslant B,$$

for all possible combinations of plus and minus signs; then $F \in \mathscr{M}(R^d)$ *and* $M(F) \leqslant 2^{2d} dB$.

As an illustration, we note that the function of $n+5$ variables

$$\frac{x_i x_j}{iy_1 y_2 y_3 y_4 y_5 + x_1^2 + x_2^2 + \ldots + x_r^2}$$

can be seen (almost by inspection) to satisfy the conditions of the above corollary. (For example take $x_j = \pm e^{t_j}$, $0 \leqslant j \leqslant n$, $y_j = \pm e^{t_{j+5}}$, $0 \leqslant j \leqslant 5$). Thus we obtain the estimate for the function $\mu(x, y) = \mu(x_1, x_2, \ldots, x_n, y_1, \ldots, y_5) \in C_0^\infty(R_x^n \times R_y^5)$:

$$\|\mu_{x_i x_j}\|_{L^p(R_x^n \times R_y^5)} \leqslant C \|\mu_{y_1 y_2 y_3 y_4 y_5} - (\mu_{x_1 x_1} + \mu_{x_2 x_2} + \ldots + \mu_{x_n x_n})\|_{L^p(R_x^n \times R_y^5)}.$$

Applying theorem 4.4 we obtain the following result: Let $\partial_1, \partial_2, \ldots, \partial_5$ denote differentiation in five directions (respectively in R_y^5). Then the estimate

$$\|\mu_{x_i x_j}\|_{L^p(R_x^n \times R_y^5)} \leqslant C \|\partial_1\partial_2\partial_3\partial_4\partial_5 \mu - \Delta_x \mu\|_{L^p(R_x^n \times R_y^5)}$$

holds for all functions $\mu \in C_0^\infty(R_x^n \times R_y^5)$. Obviously the number 5 can be replaced by any odd number and the Laplacian by a more general elliptic operator.

5. Counter-example to a strong multiplier theorem. Theorem 3.3 naturally leads to the following question:

Let $g_n(\xi)$ be a function on R^1 with support in $[2^n, 2^{n+1}]$, $n = 1, 2, \ldots$ Suppose the norm of g_n as a multiplier on $L^p(E^1)$ is bounded uniformly in n. Then is $g = \sum_n g_n$ necessarily a multiplier?

The answer turns out to be negative. A counter-example to this conjecture can be obtained by taking

$$g_n(\xi) = e^{i\pi 2^n \xi} \chi_{[2^n, 2^n+1]}(\xi).$$

g_n acts as the product of a translation in χ (an isometry) and an idempotent which has bound uniform in n. However, we will show g is not a multiplier. Our computations are carried out in L^4 because of the relative ease of explicit calculation, although the example is valid for $p \neq 2$.

Let

$$f(x) = \sum_{n=1}^{\infty} n^{-1/2} f_n(x), \quad \text{where} \quad \hat{f}_n = \frac{1}{2}\chi_{[2^n, 2^n+2]},$$

so that

$$f_n(x) = \frac{1}{2}\int_{2^n}^{2^n+2} e^{i\pi x\xi}\, d\xi = e^{(2^n+1)\pi i x} \frac{\sin \pi x}{\pi x}.$$

$\hat{f} = g\hat{h}_n$, where

$$h_n(x) = e^{(2^n+1)\pi i(x-2^n)} \frac{\sin \pi(x-2^n)}{\pi(x-2^n)}$$

so that $h = \sum_n n^{-1/2} h_n$ is carried by g into f. We estimate the L^4 norms of h and f by menans of the Paley-Littlewood-Schwartz theorem:

$$\|h\|_4^4 \sim \int_{-\infty}^{\infty} \Big(\sum_{n=1}^{\infty} |n^{-1/2} h_n(x)|^2\Big)^2 dx$$

$$= \sum_{m,n} n^{-1} m^{-1} \int_{-\infty}^{\infty} \frac{\sin^2\pi(x-2^n)}{\pi^2(x-2^n)^2}\, \frac{\sin^2\pi(x-2^m)}{\pi^2(x-2^m)^2}\, dx.$$

Recalling the formula

$$\int_{-\infty}^{\infty} \frac{\sin^2\pi(x-\alpha)}{\pi^2(x-\alpha)^2}\, \frac{\sin^2\pi(x-\beta)}{\pi^2(x-\beta)^2}\, dx = \frac{1}{4\pi^2(\alpha-\beta)^2}\Big\{1 - \frac{\sin 4\pi(\alpha-\beta)}{4\pi(\alpha-\beta)}\Big\}$$

and its limiting case

$$\int_{-\infty}^{\infty} \frac{\sin^4 \pi x}{\pi^4 x^4}\, dx = \frac{2}{3},$$

we have

$$\|h\|_4^4 \sim\leqslant \sum_n n^{-2}\frac{2}{3} + \frac{1}{2\pi^2} \sum_{1\leqslant m<n<\infty} n^{-1} m^{-1} (2^n - 2^m)^{-2} < \infty.$$

Thus $h \in L^4$.

$$\|f\|_4^4 \sim \int_{-\infty}^{\infty} \Big(\sum_n |n^{-1/2} f_n(x)|^2\Big)^2 dx = \sum_{m,n} n^{-1} m^{-1} \int_{-\infty}^{\infty} \frac{\sin^4 \pi x}{\pi^4 x^4}\, dx = \frac{2}{3}\Big(\sum_n n^{-1}\Big)^2.$$

Thus $f \notin L^4$ and g is not a multiplier on L^4.

6. Paley-Littlewood systems of operators. In this section, we abstract the operator-theoretic features of the Paley-Littlewood theorem. We say that a family of operators $\{A_n\}$ on L^p (or some subspace thereof) is a *Paley-Littlewood system* if there is a constant A for which

$$\int |f(x)|^p dx \lesseqgtr A^{\pm p} \Big(\sum_n |A_n f(x)|^2\Big)^{p/2} dx \quad \text{for all } f \in L^p.$$

In the cases considered in Section 2, the operators A_n are disjoint idempotents with $\sum_n A_n = I$, the identity operator; it then follows that $\big|\sum_{n\in\sigma} A_n\big| \leqslant A^2$ uniformly over all subsets σ of indices and thus $\{A_n\}$ is a bounded Boolean algebra of projections in the sense of Dunford [5]. For many of our applications, however, it is more convenient to deal with $A_n = M_{\varphi_n}$, the multiplier operator given by the function φ_n, where $\{\varphi_n\}$ form a certain C^∞-partition of unity. For this reason we establish the second principal theorem of this section (theorem 6.6) which gives a natural equivalent condition for such $\{A_n\}$ to be a Paley-Littlewood system.

Our first principal theorem (theorem 6.2) shows that if $\{A_n\}$ and $\{B_m\}$ are two arbitrary Paley-Littlewood systems of operators on L^p with the range of B_m contained in the domain of A_n for all m, n, then the doubly indexed family of operators $\{A_n B_m\}$ is again a Paley-Littlewood system. This general fact may be used to replace the induction step in theorem 2.2; in addition it is valid in the full range $0 < p < \infty$ and for L^p of an arbitrary measure space.

Preparatory to proving the theorems of this section we prove lemma 6.1 which is closely related to Khinchin's inequality and its elaboration by Marcinkiewicz (lemma 6); it is actually a special instance of a general group-theoretic or probability-theoretic inequality which was put to many good uses by Marcinkiewicz (see *Collected Works*, *passim*). For the range $p > 2$, lemma 6.1 may be found in McCarthy [14], propositions 1 and 2. We denote by T the infinite dimensional torus $\mathop{\mathsf{X}}_{n=1}^{\infty} \{\theta_n\colon 0 \leqslant \theta_n < 2\pi\}$ with measure $d\theta = \prod_{n=1}^{\infty} (d\theta_n/2\pi)$.

LEMMA 6.1. *Let N be the multi-index $N = (n_1, \ldots, n_S)$ and let a_N be elements of the Hilbert space H. Let T^S denote the Cartesian product of S copies, of T (denoted by $T^{(1)}, \ldots, T^{(S)}$),*

$$\Theta_N = \theta_{n_1}^{(1)} + \ldots + \theta_{n_S}^{(S)}, \qquad d\Theta = d\theta^{(1)} \ldots d\theta^{(S)}.$$

Then there exists a constant C_p $(0 < p < \infty)$ depending only upon p such that

$$\int_{T^S} |\Sigma_N e^{i\Theta N} a_N|^p d\Theta \lesseqgtr C_p^{\pm pS} \Big(\sum_N |a_N|^2\Big)^{p/2}.$$

Proof. Since the measure $d\Theta$ on T^S has mass 1, the Hölder inequality gives immediately

$$\int_{T^S} d\Theta \Big|\sum_N e^{i\Theta N} a_N\Big|^p \leqslant \Big(\int_{T^S} d\Theta\Big|\sum_N e^{i\Theta N} a_N\Big|^2\Big)^{p/2} = \Big(\sum_N |a_N|^2\Big)^{p/2} \quad \text{for} \quad p \leqslant 2.$$

For the inequality in the opposite sense, we use induction on S. For the case $S = 1$, we first consider $p \geqslant 2$, and for $p \geqslant 2$, first the case $p = 2P$, P integral. We then have

$$\int_T d\theta \Big|\sum_n e^{i\theta_n} a_n\Big|^{2P}$$

$$= \int_T d\theta \sum_{\substack{n_1,\dots,n_P \\ m_1,\dots,m_P}} \exp i(\theta_{n_1} + \dots + \theta_{n_P} - \theta_{m_1} - \dots - \theta_{m_P})(a_{n_1}, a_{m_1}) \dots (a_{n_P}, a_{m_P}).$$

For each choice of $n_1, \dots, n_P$ we obtain a non-zero contribution after integration over T only when $m_1, \dots, m_P$ is one of the at most $P!$ permutations of $n_1, \dots, n_P$, and this contribution has absolute value at most

$$|a_{n_1}|\,|a_{m_1}| \dots |a_{n_P}|\,|a_{m_P}| = |a_{n_1}|^2 \dots |a_{n_P}|^2.$$

Thus

$$\int_T d\theta \Big|\sum_n e^{i\theta_n} a_n\Big|^{2P} \leqslant P! \sum_{n_1,\dots,n_P} |a_{n_1}|^2 \dots |a_{n_P}|^2 = P!\Big(\sum_n |a_n|^2\Big)^P.$$

Thus $C_p^p = (p/2)!$ for p an even integer suffices and for $p > 2$, p not an even integer, use of the Hölder inequality again shows that $C_p^p = P!$ suffices for any integral P exceeding $p/2$.

For $p < 2$, $S = 1$, we let A be that set of $\theta \in T$ for which

$$\sum_n |a_n|^2 \leqslant 2\Big|\sum_n e^{i\theta_n} a_n\Big|^2,$$

A' the complementary set to A, and $\lambda = \int_A d\theta$. Using the fact that $C_4^4 = 2$ suffices, we have

$$\sum_n |a_n|^2 = \int_A \Big|\sum_n e^{i\theta_n} a_n\Big|\, d\theta + \int_{A'} \Big|\sum_n e^{i\theta_n} a_n\Big|^2 d\theta$$

$$\leqslant \Big(\int_A \Big|\sum_n e^{i\theta_n} a_n\Big|^4\Big)^{1/4} \cdot \Big(\int_A d\theta\Big)^{1/2} + \int_{A'} \frac{1}{2} \sum_n |a_n|^2 d\theta$$

$$\leqslant \Big(2\Big(\sum_n |a_n|^2\Big)^2\Big)^{1/2} \lambda^{1/2} + (1-\lambda) \cdot \frac{1}{2} \sum_n |a_n|^2.$$

Thus λ satisfies the inequality $1 \leqslant 2^{3/2}\lambda^{1/2} - \lambda$ and so we have $\lambda \geqslant 3 - 2\sqrt{2} > \frac{1}{7}$.

We then have

$$\int_T \Big|\sum_n e^{i\theta_n} a_n\Big|^p d\theta \geqslant \int_A \Big(\frac{1}{2}\sum_n |a_n|^2\Big)^{p/2} d\theta \geqslant \frac{1}{14}\Big(\sum_n |a_n|^2\Big)^{p/2},$$

and $C_p^p = 14$ suffices uniformly for $0 < p < 2$ which completes the demonstration for $S = 1$.

Now assuming the validity of lemma 6.1 for $S-1$, we pass to S by the same technique as that in the proof of theorem 2.2 above. Let $\mathfrak{H}$ be the Hilbert space of H-valued square-integrable functions on $T^{S-1} = T^{(2)} \times \dots \times T^{(S)}$. We denote by N' the multi-index $(n_2, \dots, n_S)$, $d\psi = d\theta^{(2)} \dots d\theta^{(S)}$, $\psi_{N'} = (\theta_{n_2}^{(2)} + \dots + \theta_{n_S}^{(S)})$, so that some elements of $\mathfrak{H}$ are of the form $\mathfrak{b}(\psi) = \sum_{N'} b_{N'} e^{i\psi_{N'}}$, $b_{N'} \in H$, with norm given by

$$|\mathfrak{b}|_{\mathfrak{H}}^2 = \int_{T^{S-1}} d\psi \Big|\sum_{N'} b_{N'} e^{i\psi_{N'}}\Big|^2 = \sum_{N'} |b_{N'}|^2.$$

Apply the case $S = 1$ to the sequence of elements $\mathfrak{a}_n(\psi) = \sum_{N'} a_N e^{i\psi_{N'}} \in \mathfrak{H}$ (where we have set $n = n_1$, $N = (n, N')$) to obtain

$$\int_T d\theta \Big|\sum_n e^{i\theta_n} \mathfrak{a}_n\Big|_{\mathfrak{H}}^p \lesseqgtr C_p^{\pm p}\Big(\sum_n |\mathfrak{a}_n|_{\mathfrak{H}}^2\Big)^{p/2}.$$

Expressing the norms in $\mathfrak{H}$ in terms of the H-norms of the coefficients of $\mathfrak{a}(\psi)$, we have

$$\int_T d\theta \Big(\sum_{N'} \Big|\sum_n e^{i\theta_n} a_N\Big|^2\Big)^{p/2} \gtreqless C_p^{\pm p}\Big(\sum_n \sum_{N'} |a_N|^2\Big)^{p/2} = C_p^{\pm p}\Big(\sum_N |a_N|^2\Big)^{p/2}.$$

But for each fixed θ the induction hypothesis shows that

$$\int_{T^{S-1}} d\psi \Big|\sum_{N'} e^{i\psi_{N'}} \sum_n e^{i\theta_n} a_{n,N'}\Big|^p \lesseqgtr C_p^{\pm p(S-1)} \Big(\sum_{N'}\Big|\sum_n e^{i\theta_n} a_{n,N'}\Big|^2\Big)^{p/2}.$$

Integrating this inequality in θ we have finally

$$\int_{T^S} d\Theta \Big|\sum_N e^{i\Theta_N} a_N\Big|^p = \int_T d\theta \int_{T^{S-1}} d\psi \Big|\sum_{N'} e^{i\psi_{N'}} \sum_n e^{i\theta_n} a_{n,N'}\Big|^p$$

$$\lesseqgtr C_p^{\pm p(S-1)} \int_T d\theta \Big(\sum_{N'}\Big|\sum_n e^{i\theta_n} a_{n,N'}\Big|^2\Big)^{p/2} \gtreqless C_p^{\pm pS}\Big(\sum_N |a_N|^2\Big)^{p/2}.$$

With lemma 6.1, the first principal theorem of this section becomes immediate.

THEOREM 6.2. *Let $\{A_n^{(1)}\}, \ldots, \{A_n^{(S)}\}$ be S Paley-Littlewood systems of operators on L^p such that*

$$\int |f(x)|^p\,dx \gtrless (A^{(s)})^{\pm p} \int \Big(\sum_n |A_n^{(s)} f(x)|^2\Big)^{p/2} dx \qquad (s = 1, \ldots, S).$$

For the S-multi-index $N = (n_1, \ldots, n_S)$, let A_N be the operator $A_{n_1}^{(1)} \ldots A_{n_S}^{(S)}$. Then the family of operators $\{A_N\}$ is a Paley-Littlewood system and we have

$$\int |f(x)|^p\,dx \gtrless C_p^{\pm 4pS} \Big(\prod_{s=1}^{S} A^{(s)}\Big)^{\pm p} \int \Big(\sum_N |A_N f(x)|^2\Big)^{p/2} dx.$$

Proof. To simplify the notation, we consider first the case $S = 2$, and set $A_n^{(1)} = A_n$, $A_m^{(2)} = B_m$; for $S > 2$, we simply iterate the case $S = 2$. Since $\{B_m\}$ is a Paley-Littlewood system, we have, using lemma 6.1,

$$\int |f(x)|^p\,dx \gtrless (A^{(2)})^{\pm p} \int \Big(\sum_m |B_m f(x)|^2\Big)^{p/2} dx$$
$$\gtrless (A^{(2)})^{\pm p} C_p^{\pm p} \int_T d\psi \int dx \Big| \sum_m e^{i\psi_m} B_m f(x)\Big|^p.$$

Since $\{A_n\}$ is a Paley-Littlewood system, we have for each fixed ψ

$$\int dx \Big| \sum_m e^{i\psi_m} B_m f(x)\Big|^p$$
$$\gtrless (A^{(1)})^{\pm p} C_p^{\pm p} \int_T d\theta \int dx \Big| \sum_n e^{i\theta_n} \sum_m e^{i\psi_m} A_n B_m f(x)\Big|^p,$$

so that

$$\int |f(x)|^p\,dx \gtrless C_p^{\pm 2p} (A^{(1)} A^{(2)})^{\pm p} \int dx \int_{T^2} d\theta\, d\psi \Big| \sum_{n,m} e^{i(\theta_n + \psi_m)} A_n B_m f(x)\Big|^p$$
$$\gtrless C^{\pm 2p} (A^{(1)} A^{(2)})^{\pm p} \int dx\, C_p^{\pm 2p} \Big(\sum_{n,m} |A_n B_m f(x)|^2\Big)^{p/2}.$$

Half of the occurrences of C_p^p above may be eliminated, the particular occurrences eliminated depending upon whether $p > 2$ or $p < 2$ and upon which inequality is being demonstrated, and we have

$$\int |f(x)|^p\,dx \gtrless C_p^{\pm 2p} (A^{(1)} A^{(2)})^{\pm p} \int \Big(\sum_N |A_N f(x)|^2\Big)^{p/2} dx.$$

For S of the form 2^τ, τ integral, we obtain in general by induction on τ

$$\int |f(x)|^p\,dx \gtrless C^{\pm 2p(S-1)} \Big(\prod_{s=1}^{S} A^{(s)}\Big)^{\pm p} \int \Big(\sum_N |A_N f(x)|^2\Big)^{p/2} dx$$

and for other S the factor $C_p^{2p(S-1)}$ may be taken as $C_p^{2p\tau} \leqslant C_p^{4pS}$, where τ is the least power of 2 exceeding S.

Before proceeding, we wish to make a few remarks. The facts that C_p may be taken uniformly bounded in the range $0 < p < 2$ but not as $p \to \infty$, and that for $1 \leqslant p < \infty$ the conjugate space $L^p(l^2)$ is $L^{p'}(l^2)$ with the pairing

$$(\{f_n\}, \{f_n^*\}) = \int \sum_n f_n(x) f_n^*(x)\,dx \qquad (p + p' = pp'),$$

might lead one to suspect that better bounds could be obtained in theorem 6.2 by considering only the case $p < 2$ and passing from there to $p > 2$ by an adjointness argument. Unfortunately, it does not seem at all clear whether $\{A_n\}$ a Paley-Littlewood system in L^p, is equivalent to $\{A_n^*\}$ being a Paley-Littlewood system in $L^{p'}$, although in all cases which have so far appeared to us in applications this is true; in particular, the equivalent condition for certain classes of Paley-Littlewood systems given by theorem 6.6 is in terms of operator norms and thus for these families $\{A_n\}$ is a Paley-Littlewood system if and only if $\{A_n^*\}$ is. We also note, for whatever use it may be, that no commutativity whatever among the operators $A_n^{(s)}$ is required for theorem 6.2.

The second principal theorem of this section deals with those special families of operators $\{A_n\}$ on $L^p (1 \leqslant p < \infty)$ which satisfy

(a) $A_n A_m = A_m A_n$ for all m, n;

(b) there exists a finite integer N such that for every m, the set of indices $\{n\colon A_n A_m \neq 0\}$ has at most N elements;

(c) $\sum_n A_n = I$, the identity operator on L^p. (It is sufficient to assume only that $\sum_n A_n = A$, a bicontinuous operator, by treating $\{A_n A^{-1}\}$.)

We shall show that such a family $\{A_n\}$ is a Paley-Littlewood system on L^p if and only if

(d) there is a finite number M such that for every set σ of indices $\big|\sum_{n\in\sigma} A_n\big| \leqslant M$.

The sense in which $\sum_n A_n = I$ is in the weak operator topology, the convergence being unconditional by (d); by modifications of the work of Badé [1, 2] and Foguel [7] we can see that (a) conversely shows that

$$\lim_{k\to\infty} \sum_{n\in\sigma_k} A_n = \sum_{n\in\sigma} A_n$$

in the strong operator topology if $\sigma_1 \subset \sigma_2 \subset \ldots$, $\sigma = \bigcup_{k=1} \sigma_k$. Further, assuming this sort of countable additivity of the family $\{A_n\}$, it may be shown that (d) is implied by the weaker assumption $\big|\sum_{n\in\sigma} A_n\big| < \infty$ for every σ, but not necessarily uniformly. The theorem of Hörmander that a uniformly bounded sequence of multipliers in $L^p(1<p<\infty)$ which converges in the $\mathscr{S}'$-topology must converge in the strong operator topology (Hörmander [8], lemma 1.5) may also be used to reassure us that it is not questions of convergence which need concern us here, but only the problem of obtaining estimates. To obtain these estimates, we first prove some preparatory lemmas which hypothesize only some of (a)-(d).

LEMMA 6.3. *Let $\{A_n\}$ satisfy* (a) *and* (b). *Then there exists a partition $\tau_1,\ldots,\tau_J$ $(J=N^2+N+1)$ of the set of indices $\{n\}$ such that for every m and every j the set $\{n\in\tau_j: A_mA_n\neq 0\}$ has at most one element.*

Proof. We build up the sets τ_j inductively. We may assume without loss of generality that the operators A_n are indexed in the following convenient manner: A_1 arbitrary; $A_2,\ldots,A_{n+1}$ to include all A_m for which $A_mA_1\neq 0$ (there are at most N such A_m by (b)); and $A_{(k-1)N+2},\ldots,A_{kN+1}(2\leqslant k\leqslant N+1)$ to include all A_m for which $A_mA_k\neq 0$. We begin the construction of the sets τ_j by requiring $j\in\tau_j(1\leqslant j\leqslant J=N^2+N+1)$. Obviously, for every m and every j, the set $\{n\in\tau_j, n\leqslant J: A_mA_n\neq 0\}$ has at most the one element j. Suppose then that we have placed every index $l<L$ into one of the sets τ_j in such a way that for every m and every j the set $\{n\in\tau_j, n<L: A_mA_n=0\}$ has at most one element. We show how to determine which set τ_j may asorb the index L without destroying this property. By (b) there are at most N indices $n_1,\ldots,n_N$ for which $A_{n_i}A_L\neq 0$ and at most N^2 more indices $n_{1,1},\ldots,n_{N,N}$ for which $A_{n_{i,k}}A_{n_i}\neq 0$. Of these at most N^2+N indices $n_i, n_{i,k}$, some, all, or none may be less than L, but in any case there is at least one of the N^2+N+1 sets τ_j which contains none of them. Put L into such a τ_j. To complete the induction, we need only show that for every m and every j the set $\{n\in\tau_j, n\leqslant L: A_mA_n\neq 0\}$ contains at most one element. If j is such that $L\notin\tau_j$ or if $L\in\tau_j$ but $A_mA_L=0$, then $\{n\in\tau_j, n\leqslant L: A_mA_n\neq 0\}=\{n\in\tau_j, n\leqslant L: A_mA_n\neq 0\}$ which contains at most one element. If $L\in\tau_j$ and $A_mA_L\neq 0$, then m must be one of the indices $n_1,\ldots,n_N$. Any other n for which $A_mA_n=0\,(=A_nA_m$ by (a)) must be one of $n_{1,1},\ldots,n_{N,N}$ which cannot be in the set τ_j chosen to asorb L.

LEMMA 6.4 (Dunford [5], theorem 7). *Let $\{A_n\}$ be a family of operators which satisfies* (d). *Then for any set $\{a_n\}$ of complex numbers we have*

$$\Big|\sum_n a_nA_n\Big|\leqslant 4\,M\sup_n|a_n|.$$

Proof. Let $f\in L^p$, $g\in L^{p'}$. Let $\sigma_1,\ldots,\sigma_4$ be the sets of indices

$$\sigma_1=\{n:\operatorname{Re}g(A_nf)\geqslant 0\},\qquad \sigma_2=\{n:\operatorname{Re}g(A_nf)<0\},$$
$$\sigma_3=\{n:\operatorname{Im}g(A_nf)\geqslant 0\},\qquad \sigma_4=\{n:\operatorname{Im}g(A_nf)<0\}.$$

Then we have

$$\Big|g\Big(\sum_n a_nA_nf\Big)\Big|\leqslant\sum_n|a_n|g(A_nf)|\leqslant\sup_n|a_n|\sum_n|g(A_nf)|$$
$$\leqslant\sup_n|a_n|\sum_n\big(|\operatorname{Re}g(A_nf)|+|\operatorname{Im}g(A_nf)|\big)$$
$$=\sup_n|a_n|\Big(\sum_{n\in\sigma_1}\operatorname{Re}g(A_nf)-\sum_{n\in\sigma_2}\operatorname{Re}g(A_nf)+\sum_{n\in\sigma_3}\operatorname{Im}g(A_nf)-\\-\sum_{n\in\sigma_4}\operatorname{Im}g(A_nf)\Big)$$
$$\leqslant\sup_n|a_n|\sum_{i=1}^4\Big|\sum_{n\in\sigma_i}g(A_nf)\Big|\leqslant\sup_n|a_n|\cdot 4M\,|f|\,|g|.$$

If the operator A_n were the atoms of a bounded Boolean algebra of projections, the next step would be to show that

$$\Big|\sum_n a_nA_nf\Big|\geqslant(4M)^{-1}\inf|a_n|\,|f|;$$

such an inequality is no longer true in our more general situation, but a substitute is given by lemma 6.3 which shows that for each j the subset of operators $\{A_n: n\in\tau_j\}$ do not interact and the obvious remark:

LEMMA 6.5. *Let $\{A_n\}$ satisfy* (a), (b), (c), *and let $\tau_1,\ldots,\tau_J$ be the J sets of indices given by Lemma 2.3. Then for each $f\in L^p$ there exists a j (depending upon f) such that*

$$\Big|\sum_{n\in\tau_j}A_nf\Big|\geqslant J^{-1}|f|.$$

Proof. $f=\sum\limits_n A_nf=\sum\limits_{j=1}^J\sum\limits_{n\in\tau_j}A_nf$, so

$$|f|\leqslant\sum_{j=1}^J\Big|\sum_{n\in\tau_j}A_nf\Big|\leqslant J\max_j\Big|\sum_{n\in\tau_j}A_nf\Big|.$$

We are now ready to prove the theorem which establishes the equivalence between (d) and the Paley-Littlewood nature of the family of operators $\{A_n\}$.

THEOREM 6.6. *Let $\{A_n\}$ be a family of operators on L^p $(1\leqslant p<\infty)$ which satisfies* (a), (b), (c), *and* (d). *Then*

$$\int|f(x)|^p\,dx\lessgtr(4MC_pJ)^{\pm p}\int\Big(\sum_n|A_nf(x)|^2\Big)^{p/2}dx.$$

Conversely, suppose that $\{A_n\}$ *satisfies* (a), (b), (c) *and*

$$\int |f(x)|^p dx \lessgtr A^{\pm p} \int \Big(\sum_n |A_n f(x)|^2\Big)^{p/2} dx.$$

Then $\{A_n\}$ *satisfies* (d) *with* $M = A^3 C_p J$.

Proof. Suppose $\{A_n\}$ satisfies (a), (b), (c) and

$$\int \Big|\sum_n e^{i\theta_n} A_n f(x)\Big|^p dx \leqslant (4M)^p \int |f(x)|^p dx.$$

Integrating over all $\theta \in T$ we have by lemma 6.1 ($S = 1$)

$$\int \Big(\sum_n |A_n f(x)|^2\Big)^{p/2} dx \leqslant (4MC_p)^p \int |f(x)|^p dx.$$

For the inequality in the opposite sense, set $Q_n = \sum_{\{m: A_m A_n \neq 0\}} A_m$ and note that $A_n = \big(\sum_n A_m\big) A_n = Q_n A_n$. Let $\tau_1, \ldots, \tau_J$ be the sets of indices guaranteed by lemma 6.3; then if n and l belong to the same τ_j, $n \neq l$, we have $\{m: A_m A_n \neq 0\}$ and $\{m: A_m A_l \neq 0\}$ are disjoint sets of indices and so $A_n Q_l = 0$; also an operator of the form $\sum_{n\in\tau_j} e^{-i\theta_n} Q_n$ is of the form $\sum_n a_n A_n$ with $|a_n| = 1$ or $a_n = 0$ and thus has norm at most $4M$ by lemma 6.4. For any choice of θ_n we have

$$\int \Big|\sum_{n\in\tau_j} A_n f(x)\Big|^p dx = \int \Big|\Big(\sum_{n\in\tau_j} e^{-i\theta_n} Q_n\Big)\Big(\sum_{n\in\tau_j} e^{i\theta_n} A_n\Big) f(x)\Big|^p dx$$
$$\leqslant (4M)^p \int \Big|\Big(\sum_{n\in\tau_j} e^{i\theta_n} A_n\Big) f(x)\Big|^p dx.$$

Integrating over all θ we have for any j

$$\int \Big|\sum_{n\in\tau_j} A_n f(x)\Big|^p \leqslant (4MC_p)^p \int \Big(\sum_{n\in\tau_j} |A_n f(x)|^2\Big)^{p/2} dx$$

and use of lemma 6.5 gives

$$\int |f(x)|^p dx \leqslant J^p \max_j \int \Big|\sum_{n\in\tau_j} A_n f(x)\Big|^p$$
$$\leqslant (4MC_p J)^p \max_j \int \Big(\sum_{n\in\tau_j} |A_n f(x)|^2\Big)^{p/2} dx$$
$$\leqslant (4MC_p J)^p \int \Big(\sum_n |A_n f(x)|^2\Big)^{p/2} dx.$$

Conversely, suppose that $\{A_n\}$ is a Paley-Littlewood system, which satisfies (a), (b), and (c); we thus have the sets $\tau_1, \ldots, \tau_J$ of lemma 6.3.

For any choice of θ, the fact that $\{A_n\}$ is a Paley-Littlewood system yields for each j

$$\int \Big|\sum_{n\in\tau_j} e^{i\theta_n} A_n f(x)\Big|^p dx \lessgtr A^{\pm p} \int \Big(\sum_m \Big|A_m \sum_{n\in\tau_j} e^{i\theta_n} A_n f(x)\Big|^2\Big)^{p/2} dx.$$

For each m, there is at most one $n \in \tau_j$ for which $A_m A_n \neq 0$. Denoting this n by $n(m)$,

$$\int \Big(\sum_m \Big|A_m \sum_{n\in\tau_j} e^{i\theta_n} A_n f(x)\Big|^2\Big)^{p/2} dx = \int \Big(\sum_m |A_m e^{i\theta_{n(m)}} A_{n(m)} f(x)|^2\Big)^{p/2} dx$$

which is independent of θ. Thus

$$\sup_\theta \int \Big|\sum_{n\in\tau_j} e^{i\theta_n} A_n f(x)\Big|^2 dx \leqslant A^{2p} \inf_\theta \int \Big|\sum_{n\in\tau_j} e^{i\theta_n} A_n f(x)\Big|^p dx$$
$$\leqslant A^{2p} C_p^p \int \Big(\sum_{n\in\tau_j} |A_n f(x)|^2\Big)^{p/2} \leqslant A^{3p} C_p^p \int |f(x)|^p dx.$$

Using lemma 6.5, it follows that for any choice of θ,

$$\Big|\sum_n e^{i\theta_n} A_n f\Big| \leqslant A^3 C_p J |f|.$$

Let σ now be an arbitrary set of indices. $\sum_{n\in\sigma} A_n$ is a convex combination of operators of the form $\sum_n e^{i\theta_n} A_n$ and thus $\big|\sum_{n\in\sigma} A_n\big| \leqslant A^3 C_p J$ uniformly in σ, which is (d).

We remark that the obstruction to proving theorem 6.6 in the range $0 < p < 1$ is lemma 6.4, and its dependence upon a conjugate space; in lemma 6.5, the use of the triangle inequality may be replaced by the inequality

$$\int \Big|\sum_{j=1}^{J} f_j(x)\Big|^p dx \leqslant \sum_{j=1}^{J} \int |f_j(x)|^p dx$$

valid for $0 < p < 1$, where $f_j(x) = \sum_{n\in\tau_j} A_n f(x)$. For want of applications, we have not here developed a substitute for lemma 6.4 in the range $0 < p < 1$, although theorem 6.6 does remain valid. We remark additionally that judicious use of adjoints allows the factors C_p in the estimates of theorem 6.6 to be replaced (in the range $2 < p < \infty$) by a factor involving J alone.

As an application of theorem 6.6, we obtain the classical Paley-Littlewood theorem in R^1 from the L^p multiplier theorem of Hörmander ([8], theorem 2.5) which shows that a uniformly bounded function $\varphi(\xi)$ ($\xi \in R^1$) is a multiplier in $L^p(R^1)$ ($1 < p < \infty$) if

$$2^m \int_{2^m < |\xi| < 2^{m+1}} |\varphi'(\xi)|^2 d\xi$$

is uniformly bounded in m, $-\infty < m < \infty$. The same sort of development may be carried out in R^d even using the multiplier theorems with mixed homogenity of Fabes and Rivière [6].

THEOREM 6.7 (The Classical Paley-Littlewood theorem). *The family of operators*

$$A_n: f \to (\chi_{2^n<|\xi|<2^{n+1}}\hat{f})^\vee \qquad (-\infty < m < \infty)$$

form a Paley-Littlewood system in $L^p(R^1)$, $1 < p < \infty$.

Proof. $\{A_n\}$ clearly satisfies (a), (b), (c), so that only (d) need be demonstrated. To this end, let $\varphi(\xi)$ be a C^∞-function with support in $\frac{1}{2} < |\xi| < 4$ such that

$$\sum_{n=-\infty}^{\infty} \varphi(2^{-n}\xi) = 1 \quad \text{a.e.};$$

we first show that the family of operators $B_n: f \to (\varphi(2^{-n}\xi)\hat{f})^\vee$ is a Paley-Littlewood system. Again it is obvious that $\{B_n\}$ satisfies (a), (b), and (c) with $N = 5$. To demonstrate (d) for the family $\{B_n\}$, let σ be any subset of indices. Then $\left|\sum_{n\epsilon\sigma}\varphi(2^{-n}\xi)\right| \leqslant 3|\varphi|_\infty$ and

$$2^m \int_{2^m}^{2^{m+1}} \left|\Big(\sum_{n\epsilon\sigma}\varphi(2^{-n}\xi)\Big)'\right|^2 d\xi$$

$$= \int_2^2 \left|\Big(\sum_{n\epsilon\sigma}\varphi(2^{m-n}\eta)\Big)'\right|^2 d\eta \quad (\eta = 2^{-m}\xi) = \int_1^2 \left|\Big(\sum_{\substack{n\epsilon\sigma\\ |n-m|\leqslant 2}}\varphi(2^{m-n}\eta)\Big)'\right|^2 d\eta$$

$$\leqslant \int_1^2 5 \sum_{|n-m|<2} |2^{m-n}\varphi'(2^{m-n}\eta)|^2 d\eta \qquad \text{(Schwarz inequality)}$$

$$\leqslant 5\cdot 16\cdot |\varphi'|_\infty^2.$$

This last estimate is uniform in m, which shows that, for every σ, $\sum_{n\epsilon\sigma} B_m$ is bounded. This last estimate is uniform in σ also, and by the estimates in the Hörmander multiplier theorem, $|\sum_{n\epsilon\sigma} B_n|$ has a bound uniform in σ, which is (d); thus $\{B_n\}$ is a Paley-Littlewood system and we have for some constant B

$$\int |f(x)|^p dx \gtrless B^{\pm p} \int \Big(\sum_n |B_n f(x)|^2\Big)^{p/2} dx.$$

Denote $\sum_{n\epsilon\sigma} A_n$ by A_σ. We have

$$\int |A_\sigma f(x)|^p dx \leqslant B^p \int \Big(\sum_n |B_n A_\sigma f(x)|^2\Big) dx.$$

Now $B_n A_\sigma f = M_{g_n} B_n f$, where g is zero off the support of $\varphi(2^{-n}\xi)$ that is $g_n = 0$ for $2^{n-1} < |\xi|$ and $|\xi| < 2^{n+2}$-and g_n is constantly 0 or 1

on each of the sets $2^{n-1} < |\xi| < 2^n$, $2^n < |\xi| < 2^{n+1}$, $2^{n+1} < |\xi| < 2^{n+2}$. It is clear that $V(g_n) \leqslant 8$ uniformly in n, so by theorem 2.2,

$$\int \Big(\sum |B_n A_\sigma f(x)|^2\Big)^{p/2} dx \leqslant 8^p \int \Big(\sum |B_n f(x)|\Big)^{p/2} dx \leqslant 8^p h_p^p B^p \int |f(x)|^p dx.$$

It follows that $\{A_n\}$ satisfies (d) with $|\sum_{n\epsilon\sigma} A_n| \leqslant 8h_p B^2$ which completes the proof of the theorem.

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