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# On a problem of moments

by

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Let $X$ and $Y$ be two real Banach spaces. Let $A$ be a continuous linear operator mapping $X$ into $Y$. Let us consider the equation

$$A(u) = c. \tag{1}$$

We are looking for the minimum norm solution of equation (1), i.e. for such a solution $u_0$ of equation (1) that

$$u_0 = \inf\{\|u\|: A(u) = c\}. \tag{2}$$

The problem considered in question often occurs in the theory of control. It has been considered for finite-dimensional $Y$ by many authors (see for example [3] and [4]). Obviously, in this case the operator $A(u)$ can be considered as a finite system of linear functionals

$$A(u) = \{A_1(u), \ldots, A_n(u)\}$$

and $c$ as a system of numbers

$$c = \{c_1, \ldots, c_n\}.$$

The main theorem in this case is

$$\inf\{\|u\|: A(u) = c\} = \sup_{\lambda_1,\ldots,\lambda_n} \inf\Big\{\|u\|: \sum_{i=1}^{n} \lambda_i A_i(u) = \sum_{i=1}^{n} \lambda_i c_i\Big\}. \tag{3}$$

The case $X = L^p, 1 < p \leqslant +\infty$, and $Y = l^p$ has been considered by Butkowski [2]. The method he used is the following: since $c \in l^p$, $c = \{c_1, \ldots, c_n, \ldots\}$ and the operator $A$ can be described as a sequence of functionals. The idea is to approximate equation (1) by a finite system of functional equations and then, using weak compactness or weak* compactness, to prove the existence of a minimum norm solution of equation (1).

In this note a generalization of formula (3) for infinite-dimensional Banach spaces will be given. The paper contains also applications of this generalization to the theory of control.

THEOREM. *Let $X$ and $Y$ be two real Banach spaces. Let $A$ be a continuous linear operator mapping $X$ into $Y$. Let the image $\Gamma$ of the unit ball*

$$K = \{x \in X\colon \|x\| \leqslant 1\}$$

*by the operator $A$ be a closed set. Let $c$ be a fixed element in $Y$. Let there exist a solution of equation $A(u) = c$. Then*

$$\inf\{\|u\|\colon A(u) = c,\ u \in X\} = \sup_{F \in Y^*} \inf\{\|u\|\colon F(A(u)) = F(c)\}, \tag{4}$$

*where as usual $Y^*$ denotes the conjugate space to the space $Y$, i.e. the space of all continuous linear functionals defined on $Y$.*

Proof. Obviously, each solution of equation (1) is simultaneously a solution of the equation

$$F(A(u)) = F(c). \tag{5}$$

Therefore

$$a \geqslant b, \tag{6}$$

where

$$a = \inf\{\|u\|\colon A(u) = c\}, \qquad b = \sup_{F \in Y^*} \inf\{\|u\|\colon F(A(u)) = F(c)\}. \tag{7}$$

Now we shall show that conversely also

$$a \leqslant b. \tag{8}$$

In fact, let us suppose that (8) does not hold, i.e. that

$$a > b. \tag{9}$$

Formula (9) implies that $c$ does not belong to $b\Gamma$. But under the assumption the set $b\Gamma$ is closed. Therefore the theorem on the separation implies that there are a positive number $\varepsilon$ and a linear continuous functional $F_1$ such that

$$F_1(c) = 1 \tag{10}$$

and

$$F_1(x) < 1 - \varepsilon \quad \text{for } x \in b\Gamma. \tag{11}$$

Hence

$$\inf\{\|u\|\colon F(A(u)) = F_1(c) = 1\} \geqslant \frac{b}{1-\varepsilon} \tag{12}$$

and this leads to a contradiction of the definition of $b$, q.e.d.

Remark 1. Let us remark that for finite-dimensional $Y$ the assumption that the set $\Gamma$ is closed is not necessary, since in this case formula (9) implies that $c \notin b\overline{\Gamma}$.



Remark 2. The theorem is also true if we replace Banach norms by Minkowski norms, i.e. norms which are not homogeneous but only positive homogeneous.

COROLLARY. *Let us assume that the assumptions of the theorem are satisfied and $\dim Y < +\infty$. Let us assume that there is a functional $F_0$ such that*

$$\inf\{\|u\|\colon A(u) = c\} = \inf\{\|u\|\colon F_0(A(u)) = F_0(c)\}. \tag{13}$$

*Let us assume that the equation*

$$F_0(A(u)) = F_0(c) \tag{14}$$

*has a unique minimal norm solution $u_0$. Then $u_0$ is the minimum norm solution of equation* (1) *also.*

Proof. Since $\Gamma$ is finite-dimensional and closed, it is compact. Since $\inf\{\|u\|\colon A(u) = c\}$ is finite, there is a minimum norm solution $u_1$ of equation (1). Obviously,

$$F_0(A(u_1)) = F_0(c). \tag{15}$$

Then formula (13) implies that $u_1$ is the minimum norm solution of equation (14). But equation (14) has a unique solution $u_0$. Therefore $u_1 = u_0$, q.e.d.

If the set $\Gamma$ is not closed, the corollary does not hold even if $Y$ is a two-dimensional space, as is shown by the following

Example. Let $X$ be a subspace of the space $L^1[0, 1]$ consisting of all functions constant on the interval $(\frac{1}{4}, \frac{3}{4})$. Let $Y$ be a two-dimensional space. Let

$$A(u) = \{F_1(u), F_2(u)\}, \tag{16}$$

where the functionals $F_1$ and $F_2$ are of type

$$F_i(u) = \int_0^1 f_i(t)u(t)\,dt, \qquad i = 1, 2, \tag{17}$$

and

$$f_1(t) = \begin{cases} 4t & \text{for } 0 \leqslant t < \frac{1}{4}, \\ \frac{1}{2} & \text{for } \frac{1}{4} \leqslant t < \frac{3}{4} \\ 0 & \text{for } \frac{3}{4} \leqslant t \leqslant 1, \end{cases} \tag{18}$$

$$f_2(t) = f_1(1-t).$$

By a simple calculation we obtain

$$\Gamma = \{(x, y)\colon |x| + |y| < 1\} \cup (\pm\tfrac{1}{2}, \pm\tfrac{1}{2}). \tag{19}$$

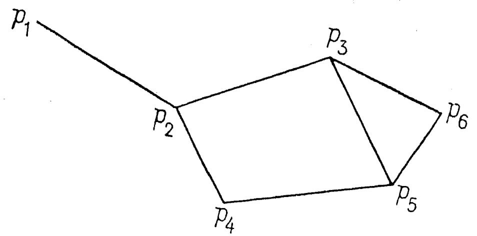
Let $c = (\frac{1}{3}, \frac{2}{3})$. Then there is a functional $F_0$ satisfying (13) and it is given by the formula

(20) $$F_0((x, y)) = x+y.$$

Then equation (14) has a unique solution $(\frac{1}{2}, \frac{1}{2}) \neq (\frac{1}{3}, \frac{2}{3})$.

Now we shall apply the theorem to a problem of the theory of control.

Let us consider the following system of strings:




Let us consider only perpendicular vibrations. Let the state of the system be described by the perpendicular deviation $Q(x, t)$ at the point $x$ and at the moment $t$. Let us control our system by perpendicular deviations $(u_1(t), \ldots, u_n(t))$ at the ends of the strings $p_1, \ldots, p_n$.

We shall consider the following minimum time problem for the system. Suppose we are given an initial state, i.e. the initial position $Q(x, 0) = Q_0(x)$ and the initial speed

$$\frac{\partial Q(x,t)}{\partial t}\bigg|_{t=0} = Q_0(x).$$

We are looking for a minimum time $T$ and control $u(t)$ bringing the system to rest, i.e. for such a control $u(t) = (u_1(t), \ldots, u_n(t))$ that

(21) $$Q(x, T) = 0$$

and

(22) $$\frac{\partial Q(x,t)}{\partial t}\bigg|_{t=T} = 0.$$

This problem has been considered for one string controlled by one end by Butkowski [2]. Butkowski has reduced the problem to the solving the infinite system of integral equations

(23) $$\int_0^T \sin k\pi \frac{a}{S} tu(t)\,dt = a_k, \qquad \int_0^T \cos k\pi \frac{a}{S} tu(t)\,dt = b_k, \qquad k = 1, 2, \ldots$$



where $S$ is the length of the string, $a$ is the coefficient of wave equation, the numbers $a_k$ and $b_k$ are defined by $Q_0(x)$ and $\dot{Q}(x)$ and

(24) $$\sum_{k=1}^{\infty} a_k^2 + b_k^2 < +\infty.$$

Using the same method we find that the minimum time problem for the system in question also can be reduced to the solving the system

(25) $$\begin{aligned} &\int_0^T \left[\sin\pi \frac{a_{i,j}}{S_{i,j}} ktu_i(t) + \sin\pi \left(\frac{a_{i,j}}{S_{i,j}} - 1\right) ktu_j(t)\right] dt = a_k^{i,j}, \\ &\int_0^T \left[\cos\pi \frac{a_{i,j}}{S_{i,j}} ktu_i(t) + \cos\pi \left(\frac{a_{i,j}}{S_{i,j}} - 1\right) ktu_j(t)\right] dt = b_k^{i,j} \\ &\qquad (k = 1, 2, \ldots, (i,j) \in A), \end{aligned}$$

where $S_{i,j}$ is the length of the string connecting points $p_i$ and $p_j$, $a_{i,j}$ is the coefficient of the wave equation of this string, $A$ is the set of pairs of positive integers such that there is a string connecting $p_i$ with $p_j$, the coefficients $a_k^{i,j}$ and $b_k^{i,j}$ are determined by $Q_0(x)$ and $\dot{Q}(x)$ and

(26) $$\sum_{(i,j)\in A} \sum_{k=1}^{\infty} (a_k^{i,j})^2 + (b_k^{i,j})^2 < +\infty.$$

Let us assume that

(27) $$a_{i,j}/S_{i,j} = \pi.$$

Then the system of equations (25) gives the system

(25′) $$\begin{aligned} &\int_0^T [\sin kt\cdot u_i(t) + \sin k(t-\pi) u_j(t)]\,dt = a_k^{i,j}, \\ &\int_0^T [\cos kt\cdot u_i(t) + \cos k(t-\pi) u_j(t)]\,dt = b_k^{i,j} \\ &\qquad (k = 1, 2, \ldots, (i,j) \in A). \end{aligned}$$

Situations in engineering imply constraints of different types for $u(u) = (u_1(t), \ldots, u_n(t))$. Let us assume that those constraints are of the following type. Given a function $H(z_1, \ldots, z_n)$ defined for non-negative $z_1, \ldots, z_n$. Let $H$ be a continuous function. Let $H(0, \ldots, 0) = 0$ and let

$$H(z_1+z_1', \ldots, z_n+z_n') \leqslant H(z_1, \ldots, z_n) + H(z_1', \ldots, z_n').$$

Let us suppose that $u_i(t) \in L^{p_i}[0, T], 1 < p_i \leqslant +\infty$. Let the constraints be of type

(27′) $$|u| \leqslant M,$$

where

$$|u| = H(\|u_1\|, \ldots, \|u_n\|). \tag{28}$$

Let us consider the product $X$ of the spaces $L^{p_i}[0, T]$, $i = 1, \ldots, n$:

$$X = L^{p_1}[0, T] \times \ldots \times L^{p_n}[0, T]. \tag{29}$$

Obviously, $X$ is a Banach space with respect to the norm $|u|$ defined by formula (28).

Using the classical technique of the theory of control we can reduce the minimum time problem to the minimum norm problem (see [2], [3], [4] and [6]), i.e. we are looking for an element with the minimal norm satisfying (25′).

Obviously equations (25′) can be considered as an operator equation

$$A(u) = c, \tag{1}$$

where $A$ is an operator mapping $X$ into $l^2$.

Simultaneously with the space $X$ we can consider a subspace $X_p \subset X$ constituted by periodic functions of the period $2\pi$ belonging to $X$. Let us consider the minimum problem in the space $X_p$. Obviously

$$\inf\{|u|: u \in X, A(u) = c\} \leqslant \inf\{|u|: u \in X_p, A(u) = c\}. \tag{30}$$

We shall show that in fact we have equality there.

Our further considerations are based on the following lemmas.

LEMMA 1. *Let $F$ be a functional defined on $X$ by a periodic function $f(t) = (f_1(t), \ldots, f_n(t))$ with the period $2\pi$ by the formula*

$$F(u) = \int_0^T [f_1(t)u_1(t) + \ldots + f_n(t)u_n(t)]\,dt. \tag{31}$$

*Then*

$$\sup_{\substack{u \in X \\ \|u\| \leqslant 1}} |F(u)| = \|F\|. \tag{32}$$

Proof. Let $u(t) = (u_1(t), \ldots, u_n(t)) \in X$. Since $u_i(t) \in L^{p_i}[0, T]$, there is a periodic function $u_i^p(t)$ such that $\|u_i\| = \|u_i^p\|$ and

$$\int_0^T f_i(t)u_i(t)\,dt \leqslant \int_0^T f_i(t)u_p^i(t)\,dt \tag{33}$$

(in fact we can put

$$u_i^p(t) = \frac{\|u_i\|}{\|f_i\|^{q_i}} |f_i(t)|^{1-q_i} \operatorname{sign} f_i(t),$$

where $q_i = p_i/(p_i - 1)$).

Let us write $u^p(t) = (u_1^p(t), \ldots, u_n^p(t))$. Then formula (33) implies

$$F(u^p) \geqslant F(u). \tag{34}$$

Formula (32) trivially follows from formula (34) and the definition of the norm of a functional.

LEMMA 2. *The set $\Gamma = A(K)$, where $K$ is a unit ball in the space $X$ and $A$ is the operator induced by formula (25′), is closed.*

Proof. The space $X$ can be considered as a space conjugate to the space

$$X_0 = L^{q_1}[0, T] \times \ldots \times L^{q_n}[0, T], \tag{35}$$

where $q_i = p_i/(1 - p_i)$ if $p_i < +\infty$ and $q_i = 1$ if $p_i = +\infty$, with the respective norm. The space $X_0$ is separable. Hence the unit ball $K$ in the space $X$ is weak* compact. The operator $A$ is weak* continuous. Therefore the set $\Gamma$ is weak* compact, whence closed.

LEMMA 3. *The set $\Gamma_p = A(K_p)$, where $K_p$ is a unit ball in the space $X_p$, is closed.*

The proof is similar to the proof of Lemma 2.

PROPOSITION 1. *We have*

$$\inf_{u \in X_p}\{\|u\|: A(u) = c\} = \inf_{u \in X}\{\|u\|: A(u) = c\}. \tag{36}$$

Proof. For an arbitrary functional $F \in Y^* = l^2$ the functional $AF(u) = F(A(u))$ is of type (31), where

$$f_i(t) = \sum_{k=1}^{\infty} a_k^i \sin kt + \beta_k^i \cos kt + \gamma_k^i \sin k(t-\pi) + \delta_k^i \cos k(t-\pi) \tag{37}$$

are periodic functions of the period $2\pi$. Therefore Lemma 1 implies

$$\inf_{u \in X_p}\{\|u\|: F(A(u)) = F(c)\} = \inf_{u \in X}\{\|u\|: F(A(u)) = F(c)\}. \tag{38}$$

Lemmas 2 and 3 imply that the theorem is applicable to both spaces, $X$ and $X_p$. Hence formula (36) follows from formula (38), q.e.d.

PROPOSITION 2. *If there is a solution of the equation $A(u) = c$, i.e. if the system is controllable with respect to the point $c$, then there is also a periodic minimum norm solution belonging to $X_p$ of the equation in the space $X$.*

Proof. Basing ourselves on Proposition 1 we can find a sequence $u_n \in X_p$ such that $A(u_n) = c$ and

$$\lim |u_n| = \inf\{|u|: u \in X, A(u) = c\}. \tag{39}$$

The set $X_p$ is weak* closed. The set $A^{-1}(c)$, as inverse image of a point by a weak* continuous operator, is also weak* closed. Therefore for all $r > a$ the set

(40) $$K_r^0 = \{u \in X_p \cap A^{-1}(c): |u| < r\}$$

is weak* compact. This implies that there is a weak* limit $u_0$ of the sequence $u_n$ and that

(41) $$|u_0| = \inf\{|u|: A(u) = c\}.$$

Since $A$ is weak* continuous, $A(u_0) = c$, q.e.d.

Propositions 1 and 2 can easily be extended to all systems for which the control problem is reduced to the solving a countable number of equations

(42) $$\int_0^T \sum_{j=1}^{n_i} f_{i,j}(t) u_j(t) = a_i,$$

where all functions $f_{i,j}(t)$ are periodic with the same period, $u \in X$, $(a_i) \in l^p$.

For example, Propositions 1 and 2 hold also for a system similar to those described above but controlled by the second derivatives of deviations (cf. paper [5] for one string).

The space $X$ can also be replaced by more general space namely by the space

(43) $$X_1 = L_{E_1}^{p_1}[0, T] \times \ldots \times L_{E_n}^{p_n}[0, T],$$

where $E_i$ are $k_i$-dimensional Minkowski spaces with the norm $\| \ \|_i$ and

(44) $$L_{E_i}^{p_i}[0, T] = \{u(t) = (u_1(t), \ldots, u_n(t)): |u|_i < +\infty\},$$

where the norm $|u|_i$ is defined by the formula

(45) $$|u|_i = \Big(\int_0^T \|u(t)\|_i^p \, dt\Big)^{1/p}$$

(see [6] and [7]).

The norm in $X_1$ is defined in the similar way as in $X$.

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