Literaturnachweis

[1] N. Bourbaki, *Intégration*, chap. 1-4, Paris 1965.
[2] — *Intégration*, chap. 5, Paris 1956.
[3] I. M. Gelfand und N. J. Wilenkin, *Verallgemeinerte Funktionen IV*, Berlin 1964.
[4] A. N. Kolmogorov, *A note on the papers of R. A. Minlos and V. Sazonov*, Theory of Prob. 4 (1959), S. 221-223.
[5] E. Nelson, *Regular probability measures on function space*, Ann. Math. 69 (1959), S. 630-643.
[6] V. Sazonov, *A remark on characteristic functionals*, Theory of Prob. 3 (1958), S. 188-192.
[7] W. v. Waldenfels, *Zur mathematischen Theorie der Druckverbreiterung von Spektrallinien*, Z. Wahrscheinlichkeitstheorie 6 (1966), S. 65-112.

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Symmetric bases of locally convex spaces

by

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§ 1. Introduction. Let E be a Hausdorff locally convex space, with Schauder basis $\{x_n\}$, and let $\{f_n\}$ be the sequence of continuous linear functionals biorthogonal to $\{x_n\}$. In the case where E is a Banach space, Singer [9] introduced the following notion of *symmetric basis*: $\{x_n\}$ is a symmetric basis if

$$(\mathrm{SB}_1)\ \sup_{\sigma\in P(N)}\ \sup_{\substack{|\delta_i|\leqslant 1\\ 1\leqslant n<\infty}} \Big\|\sum_{i=1}^{n}\delta_i f_i(x)x_{\sigma(i)}\Big\| < \infty \quad \text{for all } x \text{ in } E.$$

$P(N)$ denotes the set of all permutations of $N = \{1, 2, 3, \ldots\}$. As far as locally convex spaces are concerned, the condition (SB_1) has the following natural analogue:

$$(\mathrm{SB}_1')\ \Big\{\sum_{i=1}^{n}\delta_i f_i(x)x_{\sigma(i)}: |\delta_i|\leqslant 1,\ n\in N,\ \sigma\in P(N)\Big\} \text{ is bounded in } E \text{ for}$$

each x in E.

In [10], Singer investigated the relationship between (SB_1) and six other conditions $((\mathrm{SB}_2)$-$(\mathrm{SB}_7))$. In this paper we consider the relationship between (SB_1') and six other conditions $((\mathrm{SB}_3)$, (SB_4'), (SB_5'), (C_1), (C_2) and $(\mathrm{C}_3))$. Of these (SB_3) is identical to Singer's (SB_3), (SB_4') and (SB_5') are analogous to Singer's (SB_4) and (SB_5), and (C_1)-(C_3) are new. In detail, these conditions are:

(SB_3) Every permutation $\{x_{\sigma(n)}\}$ of the basis $\{x_n\}$ is a basis of the space E, equivalent to the basis $\{x_n\}$.

(If $\{x_n\}$ is a basis of a space E, the *sequence space associated with* $\{x_n\}$ is defined to be the linear space of all sequences $a = (a_i)$ for which $\sum_{i=1}^{\infty} a_i x_i$ is convergent. A basis $\{x_n\}$ of a space E is *equivalent* to a basis $\{y_n\}$ of a space F if the sequence space associated with $\{x_n\}$ is the same as the sequence space associated with $\{y_n\}$.)

$$(\mathrm{SB}_4')\quad \Big\{\sum_{i=1}^{n} f_i(x)x_{\sigma(i)} : n\in N,\ \sigma\in P(N)\Big\} \text{ is bounded in } E \text{ for each } x \text{ in } E.$$

$$(\mathrm{SB}_5')\ \Big\{\sum_{i=1}^{n} f_i(x)x_{\sigma(i)}: n\in N\Big\} \text{ is bounded in } E, \text{ for each } x \text{ in } E \text{ and}$$

each σ in $P(N)$.

(C_1) The sequence space associated with $\{x_n\}$ is symmetric.

(A sequence space μ is *symmetric* if the sequence $a_\sigma \in \mu$ whenever $a \in \mu$ and $\sigma \in P(N)$, where a_σ is defined by $(a_\sigma)_i = a_{\sigma(i)}$.)

(C_2) $\{\sum_{i=1}^{n} f_{\sigma(i)}(x)x_i \colon n \in N, \sigma \in P(N)\}$ is bounded in E for each x in E.

(C_3) $\{\sum_{i=1}^{n} f_{\sigma(i)}(x)x_i \colon n \in N\}$ is bounded in E for each x in E and each σ in $P(N)$.

A little thought shows that the following pattern of implications always holds:

$$\begin{array}{ccccc} & & (SB_3) & \Rightarrow & (SB_5') \\ & & \Downarrow & & \Uparrow \\ & & (C_1) & & (SB_4') \\ & & \Downarrow & & \Uparrow \\ (C_2) & \Rightarrow & (C_3) & \Leftarrow & (SB_1') \end{array}$$

Notice also that (SB_3) implies that $\{x_n\}$ is an unconditional basis, and that if $\{x_n\}$ is an unconditional basis, then (C_1) implies (SB_3). In § 3 we consider the circumstances under which (C_1) implies (SB_3). It will be seen that this depends to a large extent upon the size of the sequence space associated with $\{x_n\}$. In § 4 we consider the other conditions mentioned above.

In [10], Singer asserted that (SB_1)-(SB_7) are all equivalent. In § 5, however, we give an example of a Banach space E with a Schauder basis $\{x_n\}$ which does not satisfy (SB_3), but which does satisfy.

(SB_2) $\{x_n\}$ is an unconditional basis, and for every increasing sequence of positive integers (n_i) the basis $\{x_{n_i}\}$ of the space $[x_{n_i}]$ is equivalent to the basis $\{x_n\}$.

($[x_{n_i}]$ is the closed linear subspace of E spanned by the sequence (x_{n_i}).)

§ 2. Preliminaries. We denote by λ the sequence space associated with $\{x_i\}$. λ is naturally algebraically isomorphic to E, and we give λ the topology induced by this isomorphism from the topology of E. It then follows that properties of E are reflected in properties of λ, and conversely, and many of the results established belong properly to the theory of topological sequence spaces. λ, with the induced topology, is an AK-space (1) and $\{e_i\}$ is a Schauder basis for λ (where $(e_i)_j = \delta_{ij}$). We can,

(1) A sequence space E with a locally convex topology τ is called an *AK-space* if the inclusion map $(E, \tau) \to \omega$ is continuous and for each x in E, $P_n(x) \to x$, where $P_n(x)$ has the first n coordinates the same as x and the others equal to zero (cf. [5] and [11]).

and shall, identify λ', the topological dual of λ, with a linear subspace of

$$\lambda^\beta = \Big\{y \colon y \in \omega,\ \sum_{i=1}^{\infty} x_i y_i \text{ is convergent for each } x \text{ in } \lambda\Big\}$$

(ω denotes the linear space of all sequences, φ the linear subspace of sequences with finitely many non-zero terms, e the sequence $(1, 1, \ldots)$; otherwise we use the terminology of [3]).

We shall also use the following terminology. If m and n are positive integers, with $m \leqslant n$, $[m, n] = \{i \colon i \in N, m \leqslant i \leqslant n\}$, and $[m, \infty) = \{i \colon i \in N, i \geqslant m\}$. Σ denotes the collection of all finite subsets of N, and Σ_r the collection of all finite subsets of $[r, \infty)$. By a *dyadic* complex number we mean a complex number of the form $(p+iq)2^{-k}$, where p, q and k are integers. Finally we use "weaker than" and "finer than" in the broad sense (weaker than or equal to, finer than or equal to).

§ 3. The conditions (SB_3) **and** (C_1). Suppose that (C_1) is satisfied. If

$$x = \sum_{i=1}^{\infty} f_i(x) x_i \in E$$

and $\sigma \in P(N)$, let

$$T_\sigma(x) = \sum_{i=1}^{\infty} f_{\sigma(i)}(x) x_i .$$

T_σ is clearly a linear map of E onto itself.

PROPOSITION 1. *If* (C_1) *is satisfied and each map* T_σ *is continuous, then* (SB_3) *holds.*

Suppose that $x \in E$ and that $\sigma \in P(N)$. Let $\tau = \sigma^{-1}$. Then $T_\tau(x_j) = x_{\sigma(j)}$, so that

$$T_\tau\Big(\sum_{i=1}^{n} f_i(x) x_i\Big) = \sum_{i=1}^{n} f_i(x) x_{\sigma(i)} .$$

But

$$\sum_{i=1}^{n} f_i(x) x_i \to x \quad \text{in } E,$$

so that the continuity of T_τ implies that $\sum_{i=1}^{\infty} f_i(x) x_{\sigma(i)}$ is convergent; from this, (SB_3) follows easily.

COROLLARY. *If E is barrelled and fully complete, then* (C_1) *implies* (SB_3).

For each map T_σ has a closed graph. Note that this corollary applies in particular to Banach spaces and Fréchet spaces. In fact, the full completeness condition is redundant (Theorem 5).

PROPOSITION 2. *If* (C_1) *is satisfied and* $\lambda \nsubseteq l^\infty$, *then the topology on* λ *is the product topology.*

Suppose that there exists a continuous semi-norm p on λ such that $K = \{i: p(e_i) \neq 0\}$ is infinite. Let J be an infinite subset of K with an infinite complement, and let $a \in \lambda \setminus l^\infty$. Then there exists σ in $P(N)$ such that $|a_{\sigma(j)}| \geqslant (p(e_j))^{-1}$ for j in J, so that $\sum_{i=1}^{\infty} a_{\sigma(i)} e_i$ is not convergent, giving a contradiction. Thus any continuous semi-norm on λ vanishes on all but finitely many e_i, and the topology of λ is the product topology.

THEOREM 1. *Suppose that* $\lambda \subseteq c$ *and that* $\lambda \nsubseteq c_0$. $\{x_i\}$ *is an unconditional basis if and only if the topology of* λ *is weaker than the topology of uniform convergence on the compact sets of* l^1.

If $x \in E$, $\sigma \in P(N)$, and A is relatively compact in l^1,

$$\sup_{a \in A} \Big| \sum_{j=n}^{\infty} f_{\sigma(j)}(x) a_{\sigma(j)} \Big| \leqslant \|(f_i(x))\|_\infty \sup_{a \in A} \sum_{j=n}^{\infty} |a_{\sigma(j)}|.$$

Since the right-hand side of this inequality tends to 0 as $n \to \infty$ ([3], p. 338), the condition is sufficient. Suppose conversely that A is an equicontinuous subset of λ' which is not relatively compact in l^1, so that there exists $\varepsilon_1 > 0$ such that

$$\sup_{m \geqslant n \geqslant r} \sup_{a \in A} \sum_{j=n}^{\infty} |a_j| \geqslant \varepsilon_1 \quad \text{for all } r \text{ in } N.$$

It easily follows from this that there exists $\varepsilon > 0$ such that, given r in N, there exists a finite subset J of $[r, \infty)$ and an element a of A for which

$$\Big| \sum_{j \in J} a_j \Big| \geqslant \frac{3}{4} \sum_{j \in J} |a_j| \geqslant \varepsilon.$$

We can therefore find a sequence (J_i) in Σ and a sequence $(a^{(i)})$ in A such that

$$\Big| \sum_{j \in J_i} a_j^{(i)} \Big| \geqslant \frac{3}{4} \sum_{j \in J_i} |a_j^{(i)}| \geqslant \varepsilon$$

and

$$\sup_{j \in J_i}(j) \leqslant \inf_{j \in J_{i+1}}(j) + 2 \quad \text{for } i = 1, 2, \ldots$$

Let $k_i = \sum_{j=1}^{i} |J_j|$. There exists σ in $P(N)$ such that $\sigma([1, k_1]) = J_1$, and $\sigma([k_{i-1}+i, k_i+i-1]) = J_i$ for $i = 2, 3, \ldots$ It follows from the conditions on λ and the fact that $\lambda \supseteq \varphi$ that there exists a in λ such that

$$\lim_{i \to \infty} a_i = \gamma \neq 0 \quad \text{and} \quad |a_i - \gamma| \leqslant \frac{1}{4}|\gamma| \quad \text{for } i = 1, 2, \ldots$$

Now

$$\sum_{j=k_{i-1}+i}^{k_i+i-1} a_{\sigma(j)} e_{\sigma(j)} = \sum_{j \in J_i} a_j e_j,$$

so that

$$\Big|\Big\langle \sum_{j=k_{i-1}+i}^{k_i+i-1} a_{\sigma(j)} e_{\sigma(j)}, a^{(i)} \Big\rangle\Big| = \Big| \sum_{j \in J_i} a_j a_j^{(i)} \Big| = \Big| \gamma \Big(\sum_{j \in J_i} a_j^{(i)} \Big) + \sum_{j \in J_i} (a_j - \gamma) a_j^{(i)} \Big|$$
$$\geqslant \frac{3}{4}|\gamma| \sum_{j \in J_i} |a_j^{(i)}| - \frac{1}{4}|\gamma| \sum_{j \in J_i} |a_j^{(i)}| \geqslant \frac{1}{2}|\gamma|\varepsilon.$$

Thus $\sum_{j=1}^{\infty} a_{\sigma(j)} e_{\sigma(j)}$ is not convergent, and the basis is conditional.

THEOREM 2. *If* $\lambda \subseteq l^\infty$, $\lambda \nsubseteq c$, *and* (C_1) *is satisfied, then the topology of* λ *is weaker than the topology of uniform convergence on the compact sets of* l^1.

If A is not relatively compact in l^1, there exists $\varepsilon > 0$ such that

$$\sup_{J \in \Sigma_r} \sup_{a \in A} \Big| \sum_{j \in J} a_j \Big| \geqslant \varepsilon \quad \text{for all } r \text{ in } N$$

(cf. Theorem 1). Using this fact we can construct inductively sequences (m_i), (n_i) of positive integers, sequences (J_i), (K_i) in Σ, and a sequence $(a^{(i)})$ in A for which

(i) $m_i \leqslant n_i \leqslant m_i + 2$,

(ii) $J_i \subseteq [m_i, n_i]$, $K_i = [m_i, n_i] \setminus J_i$,

and

(iii) $\Big| \sum_{j \in J_i} a_j^{(i)} \Big| \geqslant \varepsilon/2$

for $i = 1, 2, \ldots$ Suppose that $x \in \lambda \setminus c$. There exist distinct numbers α and β, and disjoint increasing sequences (p_i) and (q_i) of positive integers such that $x_{p_i} \to \alpha$ and $x_{q_i} \to \beta$.

Let $P = \{p_i\}$ and let $Q = \{q_i\}$. Let σ be a permutation for which $\sigma(p_i) = q_i$ and $\sigma(q_i) = p_i$ for $i = 1, 2, \ldots$ Let $y_i = x_{\sigma(i)}$, and let $y = (y_i)$. Then $z = \alpha x - \beta y \in \lambda$, and $z_{p_i} \to \alpha^2 - \beta^2 = \gamma$ (say) $\neq 0$, and $z_{q_i} \to 0$. There exists a permutation τ for which

$$\tau\Big(\bigcup_{i=1}^{\infty} (J_i)\Big) \subseteq P, \quad \tau\Big(\bigcup_{i=1}^{\infty} (K_i)\Big) \subseteq Q,$$

$$|z_{\tau(j)} - \gamma| \leqslant |\gamma|\varepsilon \Big/ 8 \sum_{k=m_i}^{n_i} |a_k^{(i)}| \quad \text{for } j \text{ in } J_i$$

and

$$|z_{\tau(j)}| \leqslant |\gamma|\varepsilon \Big/ 8 \sum_{k=m_i}^{n_i} |a_k^{(i)}| \quad \text{for } j \text{ in } K_i.$$

Then

$$\Big|\Big\langle \sum_{j=m_i}^{n_i} z_{\tau(j)} e_j,\, a^{(i)}\Big\rangle\Big| = \Big|\sum_{j\in J_i} z_{\tau(j)} a_j^{(i)} + \sum_{j\in K_i} z_{\tau(j)} a_j^{(i)}\Big|$$

$$\geqslant \Big|\gamma \sum_{j\in J_i} a_j^{(i)}\Big| - \sum_{j\in J_i} |z_{\tau(j)}-\gamma|\,|a_j^{(i)}| - \sum_{j\in K_i} |z_{\tau(j)} a_j^{(i)}|$$

$$\geqslant |\gamma|\,\varepsilon/2 - |\gamma|\,\varepsilon/8 - |\gamma|\,\varepsilon/8 = |\gamma|\,\varepsilon/4.$$

But $\sum_{j=1}^{\infty} z_{\tau(j)} e_j$ is convergent, so that A cannot be equicontinuous.

THEOREM 3. *Suppose that* (C_1) *is satisfied. If* $\lambda \subseteq c_0$ *or if* $\lambda \nsubseteq c$, (SB_3) *is satisfied. If* $\lambda \subseteq c$ *and* $\lambda \nsubseteq c_0$, (SB_3) *is satisfied if and only if the topology of* λ *is weaker than the topology of uniform convergence on the compact sets of* l^1.

If $\lambda \subseteq l^\infty$, λ has the product topology (Proposition 2) for which (SB_3) is certainly satisfied. If $\lambda \subseteq l^\infty$ and $\lambda \nsubseteq c$, the topology of λ is weaker than the topology of uniform convergence on the compact sets of l^1 (Theorem 2); under this finer topology (SB_3) is satisfied, so that (SB_3) is satisfied for the original topology. If $\lambda \subseteq c$ and $\lambda \nsubseteq c_0$, the required result follows from Theorem 1.

Suppose finally that $\lambda \subseteq c_0$, and that $\{x_i\}$ is not an unconditional basis. There exists x in E and σ in $P(N)$ for which $\sum_{i=1}^{\infty} f_{\sigma(i)}(x) x_{\sigma(i)}$ is not convergent to x. There therefore exists a continuous semi-norm p on E and an increasing sequence (n_j) of positive integers such that

$$p\Big(x - \sum_{i=1}^{n_j} f_{\sigma(i)}(x) x_{\sigma(i)}\Big) \geqslant 1 \quad \text{for } j = 1, 2, \ldots$$

Since $x = \sum_{i=1}^{\infty} f_i(x) x_i$, there exists q_0 such that

$$p\Big(x - \sum_{i=1}^{n} f_i(x) x_i\Big) \leqslant 1/4 \quad \text{for } n \geqslant q_0.$$

We show inductively that there exist increasing sequences (j_i), (m_i), (p_i) and (q_i) of positive integers, sequences (J_i) and (K_i) in Σ, and a sequence (θ_i) of maps, each θ_i mapping K_i into N, for which

(i) $$J_i = \sigma[1, n_{j_i}] \supseteq [1, q_{i-1}],$$

(ii) $$m_i = \sup_{j\in J_i}(j) + 1,$$

(iii) $$K_i = [q_{i-1}, m_i] \setminus J_i,$$

(iv) $$|f_k(x)| \leqslant \frac{1}{4}\Big(\sum_{j=1}^{m_i} p(x_j)\Big) \quad \text{for } k \geqslant p_i,$$

(v) $$p_i > m_i,$$

(vi) θ_i is a 1-1 map of K_i into $[p_i, \infty)$,

and

(vii) $$q_i = \sup_{j\in\theta_i(K_i)}(j) + 1$$

for $i = 1, 2, \ldots$ Suppose that all terms have been defined for $i < r$. Since σ maps N onto N, we can find j_r for which (i) holds. J_r, m_r and K_r are then defined immediately. Since $\lambda \subseteq c_0$, we can find p_r for which (iv) and (v) hold; since K_r is non-empty, (vi) and (vii) follows easily. Now define τ as follows:

(i) $$\tau(j) = \theta_i(j) \quad \text{if } j \in K_i,$$

(ii) $$\tau(j) = \theta_i^{-1}(j) \quad \text{if } j \in \theta_i(K_i)$$

and

(iii) $$\tau(j) = j, \text{ otherwise}.$$

It is easy to see that τ is a properly defined element of $P(N)$. Now consider

$$\sum_{j=q_{j-1}+1}^{m_i} f_{\tau(j)}(x) x_j$$

$$= \sum_{j\in J_i} f_j(x) x_j + \sum_{j\in K_i} f_{\tau(j)}(x) x_j - \sum_{j=1}^{q_{i-1}} f_j(x) x_j$$

$$= \Big(\Big(\sum_{j=1}^{n_{j_i}} f_{\sigma(j)}(x) x_{\sigma(j)}\Big) - x\Big) + \sum_{j\in K_i} f_{\tau(j)}(x) x_j + \Big(x - \sum_{j=1}^{q_{i-1}} f_j(x) x_j\Big).$$

Thus

$$p\Big(\sum_{j=q_{i-1}+1}^{m_i} f_{\tau(j)} x_j\Big) \geqslant 1 - \Big(\sup_{j\in K_i} |f_{\tau(j)}(x)|\Big)\Big(\sum_{j\in K_i} p(x_j)\Big) - 1/4 \geqslant 1 - 1/4 - 1/4 = 1/2.$$

So $\sum_{j=1}^{\infty} f_{\tau(j)} x_j$ is not convergent, contradicting (C_1).

We now consider the effect of imposing simple topological conditions.

THEOREM 4. *If* E *is sequentially complete and* (C_1) *is satisfied, either* $\lambda \subseteq c_0$ *or* $\lambda = c$ *or* $\lambda = l^\infty$ *or* $\lambda = \omega$.

If $\lambda \nsubseteq l^\infty$, λ has the product topology (Proposition 2). Since E is sequentially complete, $\lambda = \omega$. If $\lambda \subseteq l^\infty$ and $\lambda \nsubseteq c$, the topology of λ is weaker than the topology of uniform convergence on the compact sets of l^1 (Theorem 2). Since $\lambda \supseteq \varphi$, and since l^∞ is an AK-space under the topology of uniform convergence on the compact subsets of l^1, it follows from the sequential completeness of E that $\lambda = l^\infty$. There remains the case where $\lambda \subseteq c$ and $\lambda \nsubseteq c_0$. We shall show that if A is an equicontinuous subset of λ', A is bounded in l^1. A, being equicontinuous, is coordinatewise bounded. Let

$$B_n = \sup \sum_{i=1}^{n} |a_i| \quad \text{for } n = 1, 2, \ldots$$

As in Theorem 2, there exists α in λ such that

$$\lim_{i\to\infty} \alpha_i = \gamma \neq 0 \quad \text{and} \quad |\alpha_i - \gamma| \leqslant \tfrac{1}{4}|\gamma| \quad \text{for } i = 1, 2, \ldots$$

Suppose that A is not bounded in l^1. Then if n is any integer and $M > 0$, there exists a finite subset J of $[n, \infty)$ and an element a of A for which

$$\Big|\sum_{j\in J} a_j\Big| \geqslant \tfrac{3}{4} \sum_{j\in J} |a_j| \geqslant M.$$

Using this fact, it is possible to find inductively a sequence $(a^{(i)})$ in A, an increasing sequence (n_i) of positive integers and a sequence (J_i) in Σ satisfying

(i) $$J_1 \subseteq [1, n_1],$$

(ii) $$\Big|\sum_{j\in J_1} a_j^{(1)}\Big| \geqslant \tfrac{3}{4} \sum_{j\in J_1} |a_j^{(1)}| \geqslant 1,$$

(iii) $$J_i \subseteq [2n_{i-1}+1, n_i] \quad \text{for } i = 2, 3, \ldots$$

and

(iv) $$\Big|\sum_{j\in J_i} a_j^{(i)}\Big| \geqslant 3/4 \sum_{j\in J_i} |a_j^{(i)}| \geqslant 2^i B_{n_{i-1}} \quad \text{for } i = 2, 3, \ldots$$

Let $K_1 = [1, n_1]\setminus J_1$, and let $K_i = [2n_{i-1}+1, n_i]\setminus J_i$ for $i = 2, 3, \ldots$ Now let

$$y = \sum_{j=1}^{n_1} \alpha_j e_j + \sum_{i=2}^{\infty} 2^{-i} B_{n_{i-1}}^{-1} \Big(\sum_{j=n_{i-1}+1}^{n_i} \alpha_j e_j\Big).$$

(Since $\{\sum_{j=1}^{n} \alpha_j e_j\}$ is bounded in λ, and E is sequentially complete, this sum is convergent in λ. Since $\{x_j\}$ is a Schauder basis, the values of y_j are what one would expect.)

Now construct inductively a permutation σ for which

(v) $$\sigma([1, 2n_i]) \subseteq [1, 2n_i] \quad \text{for } i = 1, 2, \ldots,$$

(vi) $$\text{if } j \in \bigcup_{i=1}^{\infty} (J_i), \text{ then } \sigma(j) = j$$

and

(vii) $$\sigma(K_i) \subseteq [n_i+1, 2n_i] \quad \text{for } i = 1, 2, \ldots$$

Now if $i \geqslant 2$,

$$\begin{aligned}
\Big|\sum_{j\in J_i} a_j^{(i)} y_{\sigma(j)}\Big| &= \Big|\sum_{j\in J_i} a_j^{(i)} y_j\Big| \quad \text{(by (vi))}\\
&= 2^{-i} B_{n_{i-1}}^{-1} \Big|\sum_{j\in J_i} a_j^{(i)} \alpha_j\Big| \quad \text{(by the construction of } y)\\
&\geqslant 2^{-i} B_{n_{i-1}}^{-1} \Big|\gamma \sum_{j\in J_i} a_j^{(i)}\Big| - \sum_{j\in J_i} |a_j^{(i)}||\gamma - \alpha_j|\\
&\geqslant 2^{-i} B_{n_{i-1}}^{-1} \Big(\frac{3}{4}|\gamma| \sum_{j\in J_i} |a_j^{(i)}| - \frac{1}{4}|\gamma| \sum_{j\in J_i} |a_j^{(i)}|\Big)\\
&\geqslant \frac{2}{3}|\gamma| \quad \text{(by (iv))}.
\end{aligned}$$

Also

$$\Big|\sum_{j\in K_i} a_j^{(i)} y_{\sigma(j)}\Big| \leqslant B_{n_i} \sup_{j\in K_i} |y_{\sigma(j)}| \leqslant 2^{-i-1} \|\alpha\|_\infty \quad \text{(by (vii))} \leqslant 2^{-i+1}|\gamma|,$$

so that

$$\Big|\sum_{j=2n_{i-1}+1}^{n_i} a_j^{(i)} y_{\sigma(j)}\Big| \geqslant (2/3 - 2^{-i-1})|\gamma|.$$

But

$$\sum_{j=1}^{\infty} y_{\sigma(j)} e_j \in \lambda,$$

so that A is not equicontinuous, giving the required contradiction.

If now $\beta \in c_0$, $(\sum_{i=1}^{n} \beta_i e_i)$ is a Cauchy sequence in λ in the l^∞-norm topology. From what has just been proved, this is finer than the original topology. Thus $(\sum_{i=1}^{n} \beta_i e_i)$ is a Cauchy sequence in λ in the original topology; since E is sequentially complete, it is convergent, and since $\{x_j\}$ is a Schauder basis, it must converge to β. Hence $\lambda \supseteq c_0$; since $\lambda \nsubseteq c_0$, $\lambda = c$.

COROLLARY 1. *If E is sequentially complete and* (C_1) *is satisfied,* (SB_3) *is satisfied if and only if $\lambda \neq c$.*

If (SB_3) is satisfied, $\{x_i\}$ is an unconditional basis, so that E is bounded-multiplier convergent ([2], p. 59); that is, λ is solid. Since c is not solid, $\lambda \neq c$. The converse follows from Theorems 3 and 4. Interpreting this result in terms of sequence spaces, we obtain

COROLLARY 2. *If μ is a sequentially complete symmetric AK-space, either $\mu = c$ or μ is solid.*

Let us now give some examples to show that all the possibilities mentioned in Theorem 4 can occur. There are clearly plenty of spaces with Schauder bases satisfying the hypotheses of the theorem, for which $\lambda \subseteq c_0$. If $E = l^\infty$ with any locally convex topology between the weak topology $\sigma(l^\infty, l^1)$ and the Mackey topology $\tau(l^\infty, l^1)$, E is sequentially complete, and $\{e_i\}$ is a Schauder basis for which $\lambda = l^\infty$. If $E = \omega$ with the product topology, E is complete; again $\{e_i\}$ is a Schauder basis for which $\lambda = \omega$. Finally let $E = c$ and let

$$\mathscr{D} = \Big\{D\colon D \text{ is bounded in } l^1 \text{ and } \sup_{d\in D}\Big|\sum_{i=n}^{\infty} d_i\Big| \to 0 \text{ as } n\to\infty\Big\}.$$

E is an AK-space under the topology of $\mathscr{D}$-convergence, so that $\{e_i\}$ is a Schauder basis for E, and $\lambda = c$. We show that E is complete under this topology. We can give l^∞ the topology of $\mathscr{D}$-convergence; since there is a base of $\tau(l^\infty, l^1)$-closed neighbourhoods of 0 for this topology, since the topology of $\mathscr{D}$-convergence is finer than $\tau(l^\infty, l^1)$, and since l^∞ is $\tau(l^\infty, l^1)$-complete, l^∞ is complete under the topology of $\mathscr{D}$-convergence ([1], p. 11, Proposition 8). It therefore suffices to show that c is closed in l^∞ in this topology. Suppose that $y \in l^\infty \setminus c$. There exist distinct numbers α and β, and increasing sequences (n_i), (m_i) of positive integers such that $y_{n_i} \to \alpha$ and $y_{m_i} \to \beta$. By taking subsequences if necessary, we may suppose that $n_i \leqslant m_i \leqslant n_{i+1}$, that $|y_{n_i} - \alpha| \leqslant \frac{1}{4}|\alpha-\beta|$ and that $|y_{m_i} - \beta| \leqslant \frac{1}{4}|\alpha-\beta|$ for $i = 1, 2, \ldots$ Now let

$$a^{(r)} = 2^{-r}\sum_{i=2^{r-1}+1}^{2^r}(e_{n_i} - e_{m_i}),$$

and let $D = \{a^{(r)}\colon r\in N\}$. Clearly $D\in\mathscr{D}$.

Further if $x\in c$, there exists q such that $|x_i - x_j| \leqslant \frac{1}{4}|\alpha-\beta|$ for $i, j \geqslant q$. If, then, $n_{2^{r-1}} \geqslant q$,

$$\langle y-x, a^{(r)}\rangle = 2^{-r}\sum_{i=2^{r-1}+1}^{2^r}(y_{n_i} - y_{m_i} - x_{n_i} + x_{m_i})$$

$$= 2^{-r}\Big(\sum_{i=2^{r-1}+1}^{2^r}(\alpha-\beta) + \sum_{i=2^{r-1}+1}^{2^r}\big((y_{n_i}-\alpha)+(\beta-y_{m_i})+(x_{m_i}-x_{n_i})\big)\Big)$$

so that

$$|\langle y-x, a^{(r)}\rangle| \geqslant \tfrac{1}{2}|\alpha-\beta| - \tfrac{1}{8}|\alpha-\beta| - \tfrac{1}{8}|\alpha-\beta| - \tfrac{1}{8}|\alpha-\beta| = \tfrac{1}{8}|\alpha-\beta|.$$

Hence

$$\sup_{d\in D}|\langle y-x, d\rangle| \geqslant \tfrac{1}{8}|\alpha-\beta| \quad \text{for any } x \text{ in } c,$$

so that $y \notin \bar{c}$, and c is closed in l^∞.

THEOREM 5. *If E is barrelled and* (C_1) *is satisfied, either $\lambda \subseteq c_0$ or $\lambda \nsubseteq l^\infty$. In either case,* ($SB_3$) *is satisfied.*

If $\lambda \subseteq l^\infty$, then $\lambda^\beta \supseteq (l^\infty)^\beta = l^1$. But $\lambda' = \lambda^\beta$ ([6], Theorem 2, Corollary), so that any norm-bounded subset of l^1 is weakly bounded in λ', and is therefore equicontinuous. The topology of λ is thus finer than the l^∞-norm topology. Since λ is an AK-space, it follows that $\lambda \subseteq c_0$. The final result follows from Theorem 3.

§ 4. Further conditions. In this section we consider the relationship between the conditions (SB_1'), (SB_3), (SB_4'), (SB_5') and (C_1), (C_2) and (C_3).

If E' is given the weak topology $\sigma(E', E)$, $\{f_i\}$ is a Schauder basis for E'. The next proposition follows from the definitions, and the fact that a subset of a locally convex space is bounded if and only if it is weakly bounded.

PROPOSITION 3. *Consider $(E', \sigma(E', E))$, with Schauder basis $\{f_i\}$.*

(i) (C_2) *is satisfied for E if and only if* (SB_4') *is satisfied for E'.*

(ii) (SB_4') *is satisfied for E if and only if* (C_2) *is satisfied for E'.*

(iii) (C_3) *is satisfied for E if and only if* (SB_5') *is satisfied for E'.*

(iv) (SB_5') *is satisfied for E if and only if* (C_3) *is satisfied for E'.*

(v) (SB_1') *is satisfied for E if any only if it is satisfied for E'.*

PROPOSITION 4. *If $\lambda \neq \varphi$ and either* (C_3) *or* (SB_5') *is satisfied, $\{x_i\}$ is bounded in E.*

Suppose that $\lambda \neq \varphi$, and that $\{x_i\}$ is unbounded. Let $\alpha \in \lambda\setminus\varphi$, and let p be a continuous semi-norm on E which is unbounded on $\{x_i\}$.

Let $J = \{j_1, j_2, \ldots\}$, with $j_1 < j_2 < \ldots$, be an infinite set with an infinite complement with the property that $\alpha_{j_i} \neq 0$ for any i. Let σ be a permutation for which

$$p(x_{\sigma(j_i)}) \geqslant |\alpha_{j_i}|\Big(p\Big(\sum_{k=1}^{j_i-1}\alpha_k x_{\sigma(k)}\Big) + i\Big),$$

for $i = 1, 2, \ldots$, and let $\tau = \sigma^{-1}$. Then

$$p\Big(\sum_{k=1}^{j_i}\alpha_k x_{\sigma(k)}\Big) \geqslant p(\alpha_{j_i}x_{\sigma(j_i)}) - p\Big(\sum_{k=1}^{j_i-1}\alpha_k x_{\sigma(k)}\Big) \geqslant i$$

for $i = 1, 2, \ldots$, so that (SB_5') is not satisfied. Also

$$i \leqslant p(a_{j_i} x_{\sigma(j_i)}) = p\Big(\sum_{k=1}^{\sigma(j_i)} a_{\tau}(k) x_k - \sum_{k=1}^{\sigma(j_i)-1} a_{\tau(k)} x_k\Big)$$
$$\leqslant p\Big(\sum_{k=1}^{\sigma(j_i)} a_{\tau(k)} x_k\Big) + p\Big(\sum_{k=1}^{\sigma(j_i)-1} a_{\tau(k)} x_k\Big),$$

so that (C_2) is not satisfied.

THEOREM 6. *Suppose that $\{x_i\}$ is bounded in E and $\lambda \subseteq l^\infty$. If (C_3) is satisfied, so is (C_2). If (SB_5') is satisfied, so is (SB_4').*

Suppose that (C_2) is not satisfied. There exists an element x of E and a continuous semi-norm p on E such that

$$\sup_{\sigma \in P(N)} \sup_{j \in N} p\Big(\sum_{i=1}^{j} f_{\sigma(i)}(x) x_i\Big) = \infty.$$

We shall show that it is possible to find a sequence (σ_j) in $P(N)$ and an increasing sequence (n_j) of positive integers such that

(i) $$p\Big(\sum_{j=1}^{n_i} f_{\sigma(j)}(x) x_j\Big) \geqslant i \quad \text{for } i = 1, 2, \ldots,$$

(ii) $$i \in \sigma_i[1, n_i] \quad \text{for } i = 1, 2, \ldots$$

and

(iii) $$\sigma_i(j) = \sigma_{i-1}(j) \quad \text{for } 1 \leqslant j \leqslant n_{i-1} \text{ and for } i = 2, 3, \ldots$$

We can find σ in $P(N)$ and a positive integer m such that

$$p\Big(\sum_{j=1}^{m} f_{\sigma(j)}(x) x_j\Big) \geqslant 1 + \|(f_i(x))\|_\infty \sup_i (p(x_i)).$$

If $\sigma^{-1}(1) \leqslant m$, we can take $\sigma_1 = \sigma$ and $n_1 = m$. If $\sigma^{-1}(1) > m$, we define a new permutation τ by setting $\tau(m+1) = 1$, $\tau\sigma^{-1}(1) = \sigma(m+1)$ and $\tau(i) = \sigma(i)$, otherwise. Then putting $\sigma_1 = \tau$, and $n_1 = m+1$, (ii) is satisfied and

$$p\Big(\sum_{j=1}^{n_1} f_{\sigma_1(j)}(x) x_j\Big) = p\Big(\sum_{j=1}^{m} f_{\sigma(j)}(x) x_j + f_1(x) x_{m+1}\Big)$$
$$\geqslant p\Big(\sum_{j=1}^{m} f_{\sigma(j)}(x) x_j\Big) - |f_1(x)|\, p(x_{m+1}) \geqslant 1.$$

Suppose now that $\sigma_1, \ldots, \sigma_{r-1}$ and $n_1, \ldots, n_{r-1}$ have been defined to satisfy conditions (i), (ii) and (iii). We can find ϱ in $P(N)$ and a positive integer $k > n_{r-1}$ such that

$$p\Big(\sum_{j=1}^{k} f_{\varrho(j)}(x) x_j\Big) \geqslant r + (4n_{r-1}+1)\|(f_i(x))\|_\infty \sup_i p(x_i).$$

Let

$$R = \varrho^{-1}\sigma_{r-1}([1, n_{r-1}]) \setminus [1, n_{r-1}],$$
$$S = \varrho([1, n_{r-1}]) \setminus \sigma_{r-1}([1, n_{r-1}]),$$
$$T = (R \cap [1, k]) \cup [1, n_{r-1}].$$

Then $|R| = |S|$, so that there exists a 1-1 map θ of R onto S. Now let

$$\pi(i) = \begin{cases} \sigma_{r-1}(i) & \text{for } 1 \leqslant i \leqslant n_{r-1}, \\ \theta(i) & \text{for } i \text{ in } R, \\ \varrho(i) & \text{otherwise.} \end{cases}$$

A straightforward verification shows that $\pi \in P(N)$. Further

$$\sum_{j=1}^{k} f_{\pi(j)}(x) x_j = \sum_{j=1}^{k} f_{\varrho(j)}(x) x_j - \sum_{j \in T} f_{\varrho(j)}(x) x_j + \sum_{j \in T} f_{\pi(j)}(x) x_j$$

so that

$$p\Big(\sum_{j=1}^{k} f_{\pi(j)}(x) x_j\Big) \geqslant p\Big(\sum_{j=1}^{k} f_{\varrho(j)}(x) x_j\Big) - p\Big(\sum_{j \in T} f_{\varrho(j)}(x) x_j\Big) - p\Big(\sum_{j \in T} f_{\pi(j)}(x) x_j\Big)$$
$$\geqslant r + (4n_{r-1}+1)\|(f_i(x))\|_\infty \sup_i p(x_i) - 2|T|\,\|(f_i(x))\|_\infty \sup_i p(x_i)$$
$$\geqslant r + \|(f_i(x))\|_\infty \sup_i p(x_i),$$

since $|T| \leqslant 2n_{r-1}$. If $\pi^{-1}(r) \leqslant k$, we can take $\sigma_r = \pi$ and $n_r = k$. Otherwise we take $n_r = k+1$, and alter π on $k+1$ and $\pi^{-1}(r)$, as before. This completes the proof of the induction. Now let

$$\mu(i) = \begin{cases} \sigma_1(i) & \text{if } i \leqslant n_r, \\ \sigma_j(i) & \text{if } n_{j-1} < i \leqslant n_j. \end{cases}$$

μ is a 1-1 map of N into itself (by (iii)), and condition (ii) ensures that μ maps N onto N, so that $\mu \in P(N)$. Since

$$p\Big(\sum_{j=1}^{n_i} f_{\mu(j)} x_j\Big) \geqslant i \quad \text{for } i = 1, 2, \ldots,$$

(C_3) is not satisfied.

The proof that (SB_5') implies (SB_4') is extremely similar, and the details are omitted.

THEOREM 7. *If $\lambda \subseteq c_0$ and (C_2) is satisfied, then (SB_1') is satisfied.*

Let $x \in E$, and let p be a continuous semi-norm on E. Let

$$M = \sup_{n \in N} \sup_{\sigma \in P(N)} p\Big(\sum_{j=1}^{n} f_{\sigma(j)}(x) x_j\Big).$$

Suppose that $\sigma \in P(N)$ and that $J \in \Sigma$. Let $m = \max_{j \in J} \sigma(j)$, and let $L = [1, m] \setminus \sigma(J)$. Since $\lambda \subseteq c_0$, there exists n_0 such that

$$|f_i(x)| \leqslant \Big(\sum_{j=1}^{m} p(x_j) + 1\Big)^{-1} \quad \text{for } i \geqslant n_0.$$

There exists τ in $P(N)$ such that $\tau\sigma(j) = j$ for j in J and $\tau(i) \geqslant n_0$ if $i \in L$. Then

$$\sum_{j \in J} f_i(x) x_{\sigma(i)} = \sum_{k \in \sigma(J)} f_{\tau(k)}(x) x_k = \sum_{k=1}^{m} f_{\tau(k)}(x) x_k - \sum_{k \in L} f_{\tau(k)}(x) x_k,$$

so that

$$p\Big(\sum_{i \in J} f_i(x) x_{\sigma(i)}\Big) \leqslant M + 1.$$

Now suppose that $\varrho \in P(N)$, that $n \in N$ and that $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tuple of complex numbers, with $|\delta_i| < 1$ for $1 \leqslant i \leqslant n$. We can find an n-tuple $(\gamma_1, \ldots, \gamma_n)$ of dyadic complex numbers, with $|\gamma_i| \leqslant 1$ for $1 \leqslant i \leqslant n$, for which

$$p\Big(\sum_{j=1}^{n} (\gamma_j - \delta_j) f_j(x) x_{\sigma(j)}\Big) \leqslant 1.$$

We can also write

$$\sum_{j=1}^{n} \gamma_j f_j(x) x_{\sigma(j)} = \sum_{k=0}^{r} 2^{-k}\Big(\sum_{j \in A_k} f_j(x) x_{\sigma(j)} - \sum_{j \in B_k} f_j(x) x_{\sigma(j)} + i \sum_{j \in C_k} f_j(x) x_{\sigma(j)} - i \sum_{j \in D_k} f_j(x) x_{\sigma(j)}\Big),$$

where the A_k, B_k, C_k and D_k are suitable subsets of $[1, n]$. Thus

$$p\Big(\sum_{j=1}^{n} \gamma_j f_j(x) x_{\sigma(j)}\Big) \leqslant 8(M+1),$$

so that

$$p\Big(\sum_{j=1}^{n} \delta_j f_j(x) x_{\sigma(j)}\Big) \leqslant 8M + 9.$$

COROLLARY. *If $\lambda \subseteq c$, $\lambda \nsubseteq c_0$, and* (SB_4') *is satisfied, then* (SB_1') *is satisfied.*

It follows from the conditions on λ that $\lambda' \subseteq c_0$. The result follows from the theorem and Proposition 3.

THEOREM 8. *If $\lambda \subseteq l^\infty$ and $\lambda \nsubseteq c$, the following are equivalent:*
(i) (SB_1') *is satisfied;*
(ii) (SB_4') *is satisfied;*
(iii) (C_2) *is satisfied;*
(iv) $\lambda' \subseteq l^1$.

It is clear that (iv) implies (ii), and (i) implies both (ii) and (iii). We show that (ii) implies (iv) and that (iii) implies (iv).

Suppose that $a = (a_i) \notin l^1$. Let $b = (b_i) \in \lambda \setminus c$; so that there exist distinct numbers α and β, and disjoint increasing sequences (n_i) and (m_i) of positive integers such that $b_{n_i} \to \alpha$ and $b_{m_i} \to \beta$.

Suppose first that (C_2) is satisfied. Since $a \notin l^1$, given $M > 0$ there exists a finite set J for which

$$\Big|\sum_{j \in J} a_j\Big| \geqslant \frac{3}{4} \sum_{j \in J} |a_j| \geqslant \frac{3M}{|\alpha - \beta|}.$$

Let $m = \sup_{j \in J}(j)$, and let $K = [1, m] \setminus J$. There exist permutations σ and τ such that

$$|b_{\sigma(j)} - \alpha| \leqslant \frac{1}{4}|\alpha - \beta|, \quad |b_{\tau(j)} - \beta| \leqslant \frac{1}{4}|\alpha - \beta| \quad \text{for } j \text{ in } J$$

and $\sigma(j) = \tau(j)$ for j in K. Then

$$\Big|\sum_{j=1}^{m} b_{\sigma(j)} a_j - \sum_{j=1}^{m} b_{\tau(j)} a_j\Big| = \Big|\sum_{j \in J} (b_{\sigma(j)} - b_{\tau(j)}) a_j\Big|$$

$$\geqslant \Big|\sum_{j \in J} (\alpha - \beta) a_j\Big| - \sum_{j \in J} |b_{\sigma(j)} - \alpha|\,|a_j| - \sum_{j \in J} |b_{\tau(j)} - \beta|\,|a_j|$$

$$\geqslant \frac{3}{4}|\alpha - \beta| \sum_{j \in J} |a_j| - \frac{1}{4}|\alpha - \beta| \sum_{j \in J} |a_j| - \frac{1}{4}|\alpha - \beta| \sum_{j \in J} |a_j| \geqslant M.$$

It therefore follows that $a \notin \lambda'$.

Suppose next that (SB_4') is satisfied. If $a \notin c$, there exist distinct numbers γ and δ and disjoint increasing sequences (p_i) and (q_i) of positive integers such that $a_{p_i} \to \gamma$ and $a_{p_i} \to \delta$. We can suppose that $N \setminus (\{m_i\} \cup \{n_i\})$ and $N \setminus (\{p_i\} \cup \{q_i\})$ are both infinite; let σ and τ be permutations such that

$$\sigma(m_i) = p_i, \quad \tau(m_i) = q_i, \quad \sigma(n_i) = q_i, \quad \tau(n_i) = p_i \quad \text{for } i = 1, 2, \ldots$$

and such that $\sigma(j) = \tau(j)$ otherwise. Then a straightforward calculation shows that

$$\sup_n \Big|\sum_{j=1}^{n} b_j a_{\sigma(j)} - \sum_{j=1}^{n} b_j a_{\tau(j)}\Big| = \infty,$$

so that $a \notin \lambda'$. If, next, $a \in c \setminus c_0$, $\sum_{i=1}^{\infty} b_i a_i$ is not convergent, so that $a \notin \lambda'$. Thus $\lambda' \subseteq c_0$. But since (SB_4') is satisfied, (C_2) is satisfied for $(E', \sigma(E', E))$ (Proposition 3), (SB_1') is satisfied for $(E', \sigma(E', E))$ (Theorem 7) and (SB_1') is satisfied for E (Proposition 3).

Let us now relate the various implications as the size of λ varies, and give some examples. In the diagrams which follows, arrows denote implications, and conditions included in a bracket are equivalent.

Case 1. $\lambda = \varphi$. In this case (SB_3), (SB_5'), (C_1) and (C_3) are always satisfied. When λ is given the direct sum topology none of (SB_1'), (SB_4') or (C_2) is satisfied.

Case 2. $\lambda \subseteq c_0$ and $\lambda \neq \varphi$.

$$[(SB_3), (C_1)] \Rightarrow [(SB_1'), (C_2), (C_3)] \Rightarrow [(SB_4'), (SB_5')].$$

Let $E = cs$, let $F = \varphi \oplus \text{span}\,(e)$, and give E the weak topology $\sigma(E, F)$ (for the natural duality between E and F). $\{e_i\}$ is a Schauder basis for E for which $\lambda = cs \subseteq c_0$, and for which (SB_4') is satisfied, while (C_2) is not satisfied.

Case 3. $\lambda \subseteq c$ and $\lambda \nsubseteq c_0$.

$$\begin{array}{ccc} (SB_3) & \Rightarrow & [(SB_1'),\ (SB_4'),\ (SB_5')] \\ \Downarrow & & \Downarrow \\ (C_1) & \Rightarrow & [(C_2),\ (C_3)] \end{array}$$

Give the space F of the preceding example the weak topology $\sigma(F, E)$. $\{e_i\}$ is a Schauder basis for F which satisfies (C_1) and not (SB_4'), and for which the conditions on λ are satisfied.

Consider next $E = c$, with the topology of $\mathcal{D}$-convergence, as described in § 3. Under the Schauder basis $\{e_i\}$, (C_1) and (SB_4') are satisfied, whereas (SB_3) is not (Theorem 4, Corollary 1). Thus (C_1) and (SB_4') do not together imply (SB_3).

Case 4. $\lambda \subseteq l^\infty$ and $\lambda \nsubseteq c$.

$$[(SB_3),\ (C_1)] \Rightarrow [(SB_1'),\ (SB_4'),\ (SB_5'),\ (C_2),\ (C_3)].$$

Case 5. $\lambda \nsubseteq l^\infty$.

$$[(SB_3),\ (C_1)] \Rightarrow [(SB_5'),\ (C_3)].$$

In this case, if (SB_5') or (C_3) is satisfied, the topology on λ must be the product topology (arguing as in Proposition 2); in particular (SB_4') or (C_2) can never be satisfied.

We end this section with a theorem in which topological conditions are imposed.

THEOREM 9. *If E is sequentially complete and barrelled, then* (SB_3) *is satisfied if either* (SB_4') *or* (C_2) *is satisfied.*

The proof follows the proof of the Corollary to the Theorem of [10]. Suppose that (SB_4') is satisfied. Let P be a collection of continuous semi-norms on E which defines the topology of E. For each p in P, let

$$q(x) = \sup_{n \in N} \sup_{\sigma \in P(N)} p\Big(\sum_{i=1}^{n} f_i(x) x_{\sigma(i)}\Big).$$

Each such q is a semi-norm on E (since (SB_4') is satisfied) which is lower semi-continuous, and therefore continuous (since E is barrelled — cf. [5], § 2).

Further

$$q(x) \geqslant \sup_{n \in N} p\Big(\sum_{i=1}^{n} f_i(x) x_i\Big) \geqslant p(x),$$

so that the collection Q of semi-norms defined in this way defines the topology of E. Now if $x \in E$, $\sigma \in P(N)$ and $q \in Q$,

$$q\Big(\sum_{i=m}^{n} f_i(x) x_{\sigma(i)}\Big) = q\Big(\sum_{i=m}^{n} f_i(x) x_i\Big),$$

so that $\sum_{i=1}^{\infty} f_i(x) x_{\sigma(i)}$ is convergent (since E is sequentially complete). The fact that (SB_3) is satisfied now follows easily. The proof is very similar when (C_2) is satisfied, and is omitted.

Notice also that when the hypotheses of the theorem are satisfied, $\{T_\sigma\}_{\sigma \in P(N)}$ is an equicontinuous group of linear operators from E into itself (cf. [5], Theorem 2).

The condition that E is barrelled cannot be relaxed in this theorem (consider c, with the topology of $\mathcal{D}$-convergence). Nor can the condition that E is sequentially complete be relaxed, as the following example shows. Let A be the set

$$\Big\{a\colon a \in \omega,\ a_i = 0 \text{ or } 1 \text{ and } \Big(\sum_{j=1}^{n} a_j\Big)/n \to 0 \text{ as } n \to \infty\Big\}$$

and let $l_A^1 = \{x\colon x = a \cdot y, \text{ where } a \in A \text{ and } y \in l^1\}$. l_A^1 is a linear subspace of l^1 which is barrelled under the l^1-norm topology ([7], p. 372). $\{e_i\}$ is a Schauder basis for l_A^1, for which (SB_4') is satisfied, whereas (C_1) is not.

§ 5. A counterexample. We give an example of a Banach space with a Schauder basis which satisfies (SB_2), but not (SB_3), contradicting an assertion of Singer [10].

Let O be the collection of all increasing sequences of elements of N. For each $m = (m_j)$ in O, let a_m be the sequence defined by $(a_m)_i = j^{-1/2}$ if $i = m_j$, and $(a_m)_i = 0$ otherwise, and let $A = \{a_m: m \in O\}$. Let

$$\lambda_A = \Big\{x: x \in \omega,\ \sup_{a \in A} \sum_{i=1}^{\infty} |x_i a_i| < \infty\Big\},$$

and let

$$\|x\|_A = \sup_{a \in A} \sum_{i=1}^{\infty} |x_i a_i|.$$

$\| \ \|_A$ is a norm on λ_A, under which λ_A is a Banach space (cf. [8], or [4], Proposition 3). Let E be the closure of φ in λ_A. It is clear that $\{e_i\}$ is an unconditional basis for E, and that (SB_2) is satisfied. If (SB_3) were satisfied, $\{T_\sigma: \sigma \in P(N)\}$ would be equicontinuous (see the remark after Theorem 9)-that is there would exist M such that $\|T_\sigma\| \leqslant M$ for all σ in $P(N)$. We shall show that this is not so.

Let $a^{(r)}$ be the sequence defined by $a_i^{(r)} = (r+1-i)^{-1/2}$, for $1 \leqslant i \leqslant r$, and $a_i^{(r)} = 0$ otherwise. We shall show first that the sequence $\{a^{(r)}: r = 1, 2, \ldots\}$ is bounded in E. For fixed r, there exists a finite set $n_1 < \ldots < n_s \leqslant r$ of integers such that

$$\|a^{(r)}\|_A = \sum_{i=1}^{s} \big(i(r+1-n_i)\big)^{-1/2}.$$

Let $m_i = n_i$ for $1 \leqslant i < s$, and let $m_s = r$. Then

$$0 \leqslant \|a^{(r)}\|_A - \sum_{i=1}^{s} \big(i(r+1-m_i)\big)^{-1/2} = \big(s(r+1-n_s)\big)^{-1/2} - s^{-1/2}$$

This implies that $n_s = r$. A similar argument then shows that $n_i = r+i-s$, for $1 \leqslant i < s$, so that

$$\|a^{(r)}\|_A = \sum_{i=1}^{s} \big(i(s+1-i)\big)^{-1/2}$$

$$\leqslant \frac{2}{s+1} + 2\int_1^{(s+1)/2} \frac{dx}{\sqrt{x(s+1-x)}} \leqslant 1 + 2\sin^{-1}\left(\frac{s-1}{s+1}\right) \leqslant 1+\pi.$$

Now let σ_r be defined by $\sigma_r(i) = r+1-i$ for $1 \leqslant i \leqslant r$, and $\sigma_r(i) = i$ otherwise. Then

$$T_{\sigma_r}(a^{(r)}) = (1, 2^{-1/2}, \ldots, r^{-1/2}, 0, 0, \ldots)$$

and

$$\|T_{\sigma_r}(a^{(r)})\|_A = \sum_{=1}^{r} (j^{-1}).$$

Since $\sum_{j=1}^{\infty} (j^{-1})$ is divergent, it follows that $\{T_\sigma: \sigma \in P(N)\}$ is not uniformly bounded.

The error in Singer's argument lies in asserting that his inequality (21) follows from (SB_2) and his Lemma 2.

References

[1] N. Bourbaki, *Espaces vectoriels topologiques*, Chapitres I, II, Paris 1953.

[2] M. M. Day, *Normed linear spaces*, Berlin 1958.

[3] N. Dunford and J. T. Schwartz, *Linear operators*, New York 1958.

[4] D. J. H. Garling, *On symmetric sequence spaces*, Proc. London Math. Soc. (3) 16 (1966), p. 85-106.

[5] — *On topological sequence spaces*, Proc. Cambridge Philos. Soc. 63 (1967), p. 997-1019.

[6] — *The β and γ-duality of sequence spaces*, ibidem 63 (1967), p. 963-981.

[7] G. Köthe, *Topologische lineare Räume*, Berlin 1960.

[8] G. G. Lorentz and D. G. Wertheim, *Representation of linear functionals on Köthe spaces*, Canadian J. Math. 5 (1953), p. 568-575.

[9] I. Singer, *On Banach spaces with symmetric basis* (Russian), Rev. Math. Pures Appl. 6 (1961), p. 159-166.

[10] — *Some characterizations of symmetric bases in Banach spaces*, Bull. Acad. Pol. Sci., Ser. sci. math., astr. et phys., 10 (1962), p. 185-192.

[11] K. Zeller, *Theorie der Limitierungsverfahren*, Berlin 1958.

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