Symmetric bases of locally convex spaces

by

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§ 1. Introduction. Let E be a Hausdorff locally convex space, with Schauder basis \( \{ e_n \} \), and let \( \{ f_n \} \) be the sequence of continuous linear functionals biorthogonal to \( \{ e_n \} \). In the case where E is a Banach space, Singer [9] introduced the following notion of symmetric basis: \( \{ e_n \} \) is a symmetric basis if

\[
\text{(SB}_1) \sup_{\sigma \in \mathcal{P}(N)} \sup_{|i| < n} \left\| \sum_{i=1}^{n} f_i(x) e_{\sigma(i)} \right\| < \infty \quad \text{for all } x \in E.
\]

\( \mathcal{P}(N) \) denotes the set of all permutations of \( N = \{ 1, 2, 3, \ldots \} \). As far as locally convex spaces are concerned, the condition \( \text{(SB}_1) \) has the following natural analogue:

\[
\text{(SB}_1) \left\{ \sum_{i=1}^{n} f_i(x) e_{\sigma(i)} : | \sigma | \leq 1, n \in \mathbb{N}, \sigma \in \mathcal{P}(N) \right\} \text{ is bounded in } E \text{ for each } x \in E.
\]

In [10], Singer investigated the relationship between \( \text{(SB}_1) \) and six other conditions \( (\text{SB}_2) \) - \( (\text{SB}_3) \). In this paper we consider the relationship between \( \text{(SB}_1) \) and six other conditions \( (\text{SB}_4), (\text{SB}_5), (\text{C}_1), (\text{C}_2), \) and \( (\text{C}_3)) \). Of these \( \text{(SB}_4) \) is identical to Singer's \( \text{(SB}_4) \), \( \text{(SB}_5) \) and \( \text{(SB}_1) \) are analogous to Singer's \( \text{(SB}_4) \) and \( \text{(SB}_5) \), and \( (\text{C}_1), (\text{C}_2), (\text{C}_3)) \) are new. In detail, these conditions are:

(\text{SB}_2) Every permutation \( \{ e_{\sigma(n)} \} \) of the basis \( \{ e_n \} \) is a basis of the space \( E \) equivalent to the basis \( \{ e_n \} \).

(\text{If } \{ e_n \} \text{ is a basis of a space } E, \text{ the sequence space associated with } \{ e_n \} \text{ is defined to be the linear space of all sequences } a = (a_n) \text{ for which } \sum_{n=1}^{\infty} a_n e_n \text{ is convergent. A basis } \{ e_n \} \text{ of a space } E \text{ is equivalent to a basis } \{ y_n \} \text{ of a space } F \text{ if the sequence space associated with } \{ e_n \} \text{ is the same as the sequence space associated with } \{ y_n \}.)

\[
\text{(SB}_1) \left\{ \sum_{i=1}^{n} f_i(x) e_{\sigma(i)} : n \in \mathbb{N}, \sigma \in \mathcal{P}(N) \right\} \text{ is bounded in } E \text{ for each } x \in E.
\]

\[
\text{(SB}_1) \left\{ \sum_{i=1}^{n} f_i(x) e_{\sigma(i)} : n \in \mathbb{N} \right\} \text{ is bounded in } E \text{ for each } x \in E \text{ and each } \sigma \in \mathcal{P}(N).
\]
(C) The sequence space associated with \( (a_n) \) is symmetric.

(A sequence space \( \mu \) is symmetric if the sequence \( a_\mu \) whenever \( a_\mu P(N) \), where \( a_\mu \) is defined by \( (a_\mu)_n = a_{\sigma(n)} \).

\[ (C) \quad \sum_{\sigma \in \Pi} \sum_{n \in N} a_\sigma a_n \text{ is bounded in } E \text{ for each } \sigma \in E. \]

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A little thought shows that the following pattern of implications always holds:

\[ (SB_3) \Rightarrow (SE_3) \]

\[ (C) \quad \Rightarrow (SB_3) \]

\[ (C) \quad \Rightarrow (C) \quad \Rightarrow (SB_3). \]

Notice also that \((SB_3)\) implies that \((a_n)\) is an unconditional basis, and if \((a_n)\) is an unconditional basis, then \((C)\) implies \((SB_3)\). In §3 we consider the circumstances under which \((C)\) implies \((SB_3)\). It will be seen that this depends to a large extent upon the size of the sequence space associated with \((a_n)\). In §4 we consider the other conditions mentioned above.

In [10], Singer asserted that \((SB_1)-(SB_3)\) are all equivalent. In §5, however, we give an example of a Banach space \( E \) with a Schauder basis \((a_n)\) which does not satisfy \((SB_3)\), but which satisfies \((SB_2)\).

\((SB_2)\) \((a_n)\) is an unconditional basis, and for every increasing sequence of positive integers \( \{n_i\} \) the basis \((a_{n_i})\) of the space \([a_{n_i}]\) is equivalent to the basis \((a_n)\).

\((a_{n_i})\) is the closed linear subspace of \( E \) spanned by the sequence \((a_{n_i})\).

§ 2. Preliminaries. We denote by \( \lambda \) the sequence space associated with \((x_n)\). \( \lambda \) is naturally algebraically isomorphic to \( E \), and we give \( \lambda \) the topology induced by this isomorphism from the topology of \( E \). It then follows that properties of \( E \) are reflected in properties of \( \lambda \), and conversely, and many of the results established belong properly to the theory of topological sequence spaces. \( \lambda \), with the induced topology, is an \( AK \)-space (1) and \((a_\lambda)\) is a Schauder basis for \( \lambda \) (where \((a_\lambda)_n = a_{\sigma(n)} \)). We can,

and shall, identify \( \lambda' \), the topological dual of \( \lambda \), with a linear subspace of \( \mathbb{R}^\omega \)

\[ \lambda' = \{ y: y \in \mathbb{R}^\omega, \sum_{i=1}^\omega y_i a_i \text{ is convergent for each } a \in \lambda \} \]

\((\omega \) denotes the linear space of all sequences, \( \varphi \) the linear subspace of sequences with finitely many non-zero terms, \( \varepsilon \) the sequence \((1, 1, \ldots) \); otherwise we use the terminology of [3]).

We shall also use the following terminology. If \( m \) and \( n \) are positive integers, with \( m \leq n \), \[ \ihm = \{ i: i \in N, m \leq i \leq n \} \] and \( \imn = \{ i: i \in N, i \geq m \} \). \( \Sigma \) denotes the collection of all finite subsets of \( N \), and \( \Delta \) the collection of all finite subsets of \( \{ r, \infty \} \). By a dyadic complex number we mean a complex number of the form \((p+q)/2^k\), where \( p, q \)

and \( k \) are integers. Finally we use \"weaker than\" and \"finer than\" in the broad sense (weaker than or equal to, finer than or equal to).

§ 3. The conditions \((SB_1)\) and \((C)\). Suppose that \((C)\) is satisfied. If \( x = \sum_{i=1}^\omega f_i(a_i) a_i \in E \)

and \( \sigma \in P(N) \), let

\[ T_\sigma(x) = \sum_{i=1}^\omega f_i(a_i) a_i. \]

\( T_\sigma \) is clearly a linear map of \( E \) onto itself.

Proposition 1. If \((C)\) is satisfied and each map \( T_\sigma \) is continuous, then \((SB_3)\) holds.

Suppose that \( x \in E \) and that \( \sigma \in P(N) \). Let \( \tau = \sigma^{-1} \). Then \( T_\tau(x) = x_\tau \), so that

\[ T_\tau \left( \sum_{i=1}^\omega f_i(a_i) a_i \right) = \sum_{i=1}^\omega f_i(a_i) x_\tau a_i. \]

But

\[ \sum_{i=1}^\omega f_i(a_i) x_\tau a_i \rightarrow x \text{ in } E, \]

so that the continuity of \( T_\tau \) implies that \( \sum_{i=1}^\omega f_i(a_i) x_\tau a_i \) is convergent; from this, \((SB_3)\) follows easily.

Corollary. If \( E \) is barrelled and fully complete, then \((C)\) implies \((SB_3)\).

For each map \( T_\sigma \), \( E \) has a closed graph. Note that this corollary applies in particular to Banach spaces and Fréchet spaces. In fact, the full completeness condition is redundant (Theorem 5).
PROPOSITION 2. If \((C_1)\) is satisfied and \(\lambda \not\equiv \bar{\Omega}\), then the topology on \(\lambda\) is the product topology.

Suppose that there exists a continuous semi-norm \(p\) on \(\lambda\) such that \(K = \{i | p(e_i) = 0\}\) is infinite. Let \(J\) be an infinite subset of \(K\) with an infinite complement, and let \(\alpha \not\equiv \bar{\Omega}\). Then there exists \(\epsilon\) in \(\mathcal{F}(\mathbb{N})\) such that \(|a_{e_j}| \geq p(e_j)\) for \(j\) in \(J\), so that \(\sum_{i \in \mathcal{C}} a_{e_i} e_i\) is not convergent, giving a contradiction. Thus any continuous semi-norm on \(\lambda\) vanishes on all but finitely many \(e_i\) and the topology of \(\lambda\) is the product topology.

THEOREM 1. Suppose that \(\Lambda \subseteq \mathcal{C}\) and that \(\lambda \not\equiv \epsilon\). \(\{a_i\}\) is an unconditional basis if and only if the topology of \(\lambda\) is weaker than the topology of uniform convergence on the compact sets of \(P\).

If \(\sigma \not\equiv \mathcal{E}\), \(\sigma \not\equiv \mathcal{P}(\mathbb{N})\), and \(A\) is relatively compact in \(P\),

\[
\sup_{(a_i) \in A} \sum_{i=1}^{\infty} |f_{\alpha}(a_i) a_{\alpha(i)}| \leq \left\| f_{\alpha}(a) \right\| \sup_{(a_{\alpha(i)}) \in A} \sum_{i=1}^{\infty} |a_{\alpha(i)}|.
\]

Since the right-hand side of this inequality tends to 0 as \(n \to \infty\) ([3], p. 338), the condition is sufficient. Suppose conversely that \(A\) is an equicontinuous subset of \(J\) which is not relatively compact in \(P\), so that there exists \(\epsilon > 0\) such that

\[
\sup_{(a_i) \in A} \sup_{r \in N} \sum_{i=1}^{\infty} |a_i| \geq \epsilon \quad \text{for all} \quad r \in N.
\]

It easily follows from this that there exists \(\epsilon > 0\) such that, given \(\epsilon\) in \(N\), there exists a finite subset \(J\) of \([r, \infty)\) and an element \(\alpha\) of \(A\) for which

\[
\left| \sum_{i=1}^{\infty} a_i \right| \geq \frac{\epsilon}{2} \sum_{i=1}^{\infty} |a_i| > \epsilon.
\]

We can therefore find a sequence \((J_i)\) in \(\Sigma\) and a sequence \((a^{(i)})\) in \(A\) such that

\[
\left| \sum_{i=1}^{\infty} a^{(i)} \right| = \frac{3}{4} \left| \sum_{i=1}^{\infty} a^{(i)} \right| > \epsilon
\]

and

\[
\sup_{j \in J_i} (j) \leq \inf_{j \in J_{i+1}} (j) + 2 \quad \text{for} \quad i = 1, 2, \ldots
\]

Let \(k_i = \sum_{j \in J_i} |j|\). There exists \(\sigma\) in \(P(\mathbb{N})\) such that \(\sigma([1, k_i]) = J_i\), and \(\sigma([k_i, i + 1, k_{i+1}, i, k_i + 1, i + 1]) = J_i\) for \(i = 2, 3, \ldots\). It follows from the conditions on \(\lambda\) and the fact that \(\lambda \not\equiv \sigma\) that there exists \(\alpha \in A\) such that

\[
\lim_{t \to 0} a_{e_i} = \gamma \neq 0 \quad \text{and} \quad |a_i - \gamma| \leq \frac{1}{4} |\gamma| \quad \text{for} \quad i = 1, 2, \ldots
\]

Now

\[
\sum_{j=1}^{k_i+1} a_{\sigma(j)} a_{\sigma(j)} = \sum_{j=1}^{k_i} a_{\sigma(j)},
\]

so that

\[
\left| \sum_{j=1}^{k_i+1} a_{\sigma(j)} a_{\sigma(j)} \right| = \left| \sum_{j=1}^{k_i} a_{\sigma(j)} \right| + \left| \gamma \sum_{j=1}^{k_i} a_{\sigma(j)} \right| \geq \frac{\epsilon}{3} + \frac{\epsilon}{4} \sum_{j=1}^{k_i} |a_{\sigma(j)}| \geq \frac{\epsilon}{2} \sum_{j=1}^{k_i} |a_{\sigma(j)}|.
\]

Thus \(\sum_{j=1}^{k_i+1} a_{\sigma(j)} a_{\sigma(j)}\) is not convergent, and the basis is conditional.

THEOREM 2. If \(\Lambda \subseteq \mathcal{P}, \lambda \not\equiv \epsilon\), and \((C_1)\) is satisfied, then the topology of \(\lambda\) is weaker than the topology of uniform convergence on the compact sets of \(P\).

If \(A\) is not relatively compact in \(P\), there exists \(\epsilon > 0\) such that

\[
\sup_{(a_i) \in A} \sup_{r \in N} \sum_{i=1}^{\infty} |a_i| > \epsilon \quad \text{for all} \quad r \in N
\]

(cf. Theorem 1). Using this fact we can construct inductively sequences \((m_i), (n_i)\) of positive integers, sequences \((J_i), (K_i)\) in \(\Sigma\), and a sequence \((a^{(i)})\) in \(A\) for which

(i) \(m_i \leq m_i + 2\),

(ii) \(J_i \subseteq [m_i, n_i], K_i = [m_i, n_i] \setminus J_i\),

and

(iii) \(\sum_{j=1}^{k_i} a_{\sigma(j)} \geq \epsilon/2\)

for \(i = 1, 2, \ldots\). Suppose that \(\sigma \not\equiv \epsilon\). There exist distinct numbers \(\alpha\) and \(\beta\), and disjoint increasing sequences \((p_i), (q_i)\) of positive integers such that \(p_i \to \alpha\) and \(q_i \to \beta\).

Let \(P = (p_i)\) and \(Q = (q_i)\). Let \(\sigma\) be a permutation for which \(\sigma(p_i) = q_i\) and \(\sigma(q_i) = p_i\) for \(i = 1, 2, \ldots\). Let \(y_i = x_{\sigma(q_i)}\), and let \(y = y_i\). Then \(x = \alpha y + \beta y_i\), and \(x_{\sigma(q_i)} \to a^{(i)} \beta = \gamma\) (say) \(\neq 0\), and \(x_{\sigma(q_i)} \to 0\). There exists a permutation \(\tau\) for which

\[
\tau(\bigcup_{i=1}^{k_i} J_i) \subseteq P, \quad \tau(\bigcup_{i=1}^{k_i} K_i) \subseteq Q,
\]

\[
\left| a_{\sigma(q_i)} - \gamma \right| \leq \left| \gamma \right| \left| \sum_{k=1}^{n_i} a_{\sigma(k)} \right| \quad \text{for} \quad j \in J_i
\]

and

\[
\left| x_{\sigma(q_i)} \right| \leq \left| \gamma \right| \left| \sum_{k=1}^{n_i} a_{\sigma(k)} \right| \quad \text{for} \quad j \in K_i.
\]
Then
\[
\left| \sum_{\ell=1}^{n} \sum_{m=1}^{n} z_{\ell}^{(m)} q_{(m)}^{(\ell)} \right| = \left| \sum_{\ell=1}^{n} z_{\ell}^{(m)} q_{(m)}^{(\ell)} \right| = \left| \sum_{\ell=1}^{n} |z_{\ell}^{(m)}| q_{(m)}^{(\ell)} \right| = \left| \sum_{\ell=1}^{n} |z_{\ell}^{(m)}| \right| q_{(m)}^{(\ell)} \geq \left| \gamma / s - 1 / 2 - \gamma / s \right| q_{(m)}^{(\ell)} \geq \left| \gamma / s - 1 / 2 \right|
\]

But \( \sum_{\ell=1}^{n} z_{\ell}^{(m)} q_{(m)}^{(\ell)} \) is convergent, so that \( \lambda \) cannot be equicontinuous.

**Theorem 3.** Suppose that \((C_2)\) is satisfied. If \( \lambda \equiv c_{\theta} \) or \( \lambda \equiv c_{\theta} \) (SB₄) is satisfied. If \( \lambda \equiv c_{\theta} \) and \( \lambda \equiv c_{\theta} \) (SB₄) is satisfied if and only if the topology of \( \lambda \) is weaker than the topology of uniform convergence on the compact sets of \( \mathcal{U} \).

If \( \lambda \equiv c_{\theta} \) has the product topology (Proposition 2) for which (SB₄) is certainly satisfied. If \( \lambda \equiv c_{\theta} \) and \( \lambda \equiv c_{\theta} \), the topology of \( \lambda \) is weaker than the topology of uniform convergence on the compact sets of \( \mathcal{U} \) (Theorem 2); under this finer topology (SB₄) is satisfied, so that (SB₄) is satisfied for the original topology. If \( \lambda \equiv c_{\theta} \), the required result follows from Theorem 1.

Suppose finally that \( \lambda \equiv c_{\theta} \), and that \( \{x_\theta\} \) is not an unconditional basis. There exists \( x \in E \) and \( x \in P(N) \) for which \( \sum_{i=1}^{n} f_i(x) x_{\theta}^{(i)} \) is not convergent to \( x \). There therefore exists a continuous semi-norm \( p \) on \( E \) and an increasing sequence \( (n_k) \) of positive integers such that
\[
p\left( x - \sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} \right) \geq 1/k \]
for \( j = 1, 2, \ldots \)

Since \( x = \sum_{i=1}^{n_k} f_i(x) x_i \), there exists \( g_k \) such that
\[
p\left( x - \sum_{i=1}^{g_k} f_i(x) x_i \right) \geq 1/k \]
for \( n \geq g_k \).

We show inductively that there exist increasing sequences \( \{j_1\}, \{m_1\}, \{p_1\} \), and \( \{q_1\} \) of positive integers, sequences \( \{J_1\} \) and \( \{K_1\} \) in \( \Sigma \), and a sequence \( \{\theta_1\} \) of maps, each \( \theta_1 \) mapping \( K_1 \) into \( N \), for which

(i) \( J_1 = \{1, \ldots, m_1\} \supset \{1, \ldots, \ell_1\} \),
(ii) \( m_1 = \sup\{j_1 \mid j_1 \in \mathbb{N}\} \),
(iii) \( K_1 = \{g_1 \mid g_1 \in \mathbb{N}\} \setminus J_1 \).

We define \( j_1 \) as follows:

(iv) \( \sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} \leq 1/k \sum_{i=1}^{n_k} p(x_{\theta}^{(i)}) \) for \( k \geq p_1 \),
(v) \( p_1 \geq m_1 \),
(vi) \( \theta_1 \) is a 1-1 map of \( K_1 \) into \( [p_1, \infty) \),
and
(vii) \( q_1 = \sup_{i=1}^{n_k} (j_1) + 1 \)
for \( i = 1, 2, \ldots \). Suppose that all terms have been defined for \( i < r \). Since \( \sigma \) maps \( N \) onto \( N \), we can find \( f \) for which (i) holds. \( J_i, m_i \), and \( K_i \) are then defined immediately. Since \( \lambda \equiv c_{\theta} \), we can find \( p_i \) for which (iv) and (v) hold; since \( \kappa \) is non-empty, (vi) and (vii) follows easily. Now define \( \tau \) as follows:

(i) \( \tau(j) = \theta_1(j) \) if \( j \in K_1 \),
(ii) \( \tau(j) = \theta_1^{-1}(j) \) if \( j \notin K_1 \),
and
(iii) \( \tau(j) = j \), otherwise.

It is easy to see that \( \tau \) is a properly defined element of \( P(N) \). Now consider
\[
\sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} = \sum_{i=1}^{n_k} f_i(x) x_i + \sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} + \sum_{i=1}^{n_k} f_i(x) x_i - \sum_{i=1}^{n_k} f_i(x) x_i.
\]

Thus
\[
p\left( x - \sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} \right) \geq 1 - \left( \sum_{i=1}^{n_k} p(x_{\theta}^{(i)}) \right) \left( \sum_{i=1}^{n_k} p(x_i) \right) - 1/4 \geq 1 - 1/4 - 1/4 = 1/2.
\]

So \( \sum_{i=1}^{n_k} f_i(x) x_{\theta}^{(i)} \) is not convergent, contradicting \((C_1)\).

We now consider the effect of imposing simple topological conditions.

**Theorem 4.** If \( E \) is sequentially complete and \((C_1)\) is satisfied, either \( \lambda \equiv c_{\theta} \) or \( \lambda = c \) or \( \lambda = l^\infty \) or \( \lambda = c_0 \).
If $\lambda \not\in \mathcal{F}^\sigma$, $\lambda$ has the product topology (Proposition 2). Since $E$ is sequentially complete, $\lambda = \omega$. If $\lambda \subseteq \mathcal{F}^\sigma$ and $\lambda \not\in \mathcal{C}$, the topology of $\lambda$ is weaker than the topology of uniform convergence on the compact sets of $\mathcal{P}$ (Theorem 2). Since $\mathcal{P}$ is an $\mathcal{AE}$-space under the topology of uniform convergence on the compact subsets of $\mathcal{P}$, it follows from the sequential completeness of $\mathcal{E}$ that $\lambda = \mathcal{F}^\sigma$. There remains the case where $\lambda \subseteq \mathcal{C}$ and $\lambda \not\in \mathcal{C}$. We shall show that if $A$ is an equiconnec-
titious subset of $\mathcal{C}$, $A$ is bounded in $\mathcal{P}$. $A$, being equiconnected, is coordinatewise bounded. Let

$$E_n = \sup_{\mathcal{P}} \sum_{i=1}^n |a_i| \quad \text{for } n = 1, 2, \ldots$$

As in Theorem 2, there exists $a$ in $\lambda$ such that

$$\lim_{n \to \infty} a_n = y \neq 0 \quad \text{and} \quad |a_n - y| \leq \frac{1}{2} |y| \quad \text{for } n = 1, 2, \ldots$$

Suppose that $A$ is not bounded in $\mathcal{P}$. Then if $a$ is any integer and $M > 0$, there exists a finite subset $J$ of $[a_n, \infty)$ and an element $a$ of $A$ for which

$$\sum_{i \in J} |a_i| \geq \frac{1}{2} \sum_{i \in J} |a_i| \geq M.$$ 

Using this fact, it is possible to find inductively a sequence $(a_i^n)$ in $A$, an increasing sequence $(n_i)$ of positive integers and a sequence $(d_i)$ in $\Sigma$ satisfying

(i) $J_i \subseteq [1, n_i]$,

(ii) $\sum_{i \in J_i} |a_i^n| \leq \frac{1}{2} \sum_{i \in J_i} |a_i^n| \geq 1$,

(iii) $J_i \subseteq [2n_{i-1} + 1, n_i]$ for $i = 2, 3, \ldots$

and

(iv) $\sum_{i \in J_i} |a_i^n| \geq \frac{3}{4} \sum_{i \in J_i} |a_i^n| \geq 2^{i-1} M_{n_i}$ for $i = 2, 3, \ldots$

Let $K_i = \{1, n_i\} \setminus J_i$, and let $K_i = [2n_{i-1} + 1, n_i] \setminus J_i$ for $i = 2, 3, \ldots$

Now let

$$y = \sum_{i} a_i e_i + \sum_{i=2}^{\infty} 2^{i-1} B_{n_i}^{-1} \left(\sum_{\mathcal{P}} a_i e_i\right).$$

(Since $\sum a_i e_i$ is bounded in $i$ and $E$ is sequentially complete, this sum is convergent in $\lambda$. Since $(e_i)$ is a Schauder basis, the values of $y_i$ are what one would expect.)

Now construct inductively a permutation $\sigma$ for which

$$(v) \quad \sigma([1, 2n_i]) \subseteq [1, 2n_i] \quad \text{for } i = 1, 2, \ldots$$

(vi) if $J_i \subseteq [1, n_i]$, then $\sigma(j) = j$

and

(vii) $\sigma(K_i) \subseteq [n_i + 1, 2n_i] \quad \text{for } i = 1, 2, \ldots$

Now if $i \geq 2$,

$$\sum_{i \in J_i} |a_i^n| y_{\sigma(i)} = \sum_{i \in J_i} |a_i^n| y_i \quad \text{(by (v))}$$

$$= \sum_{i \in J_i} 2^{-i} B_{n_i}^{-1} \sum_{i \in J_i} |a_i^n| e_i \quad \text{(by the construction of } y)$$

$$\geq 2^{-i} B_{n_i}^{-1} \sum_{i \in J_i} |a_i^n| - \sum_{i \in J_i} |a_i^n| |y - y_i|$$

$$\geq 2^{-i} B_{n_i}^{-1} \left(\frac{3}{4} |y| \sum_{i \in J_i} |a_i^n| \right)$$

$$\geq \left(\frac{2}{3} |y| \right) \quad \text{by (iv).}$$

Also

$$\sum_{i \in J_i} |a_i^n| y_{\sigma(i)} = B_{n_i} \sup_{\mathcal{P}} |y_{\sigma(i)}| \leq 2^{i-1} |a_i^n| \quad \text{(by (vii))} \leq 2^{i-1} |y|,$$

so that

$$\sum_{i=2}^{\infty} 2^{-i} |y_{\sigma(i)}| \geq (2/3 - 2^{-i}) |y|.$$ 

But

$$\sum_{i=2}^{\infty} 2^{-i} |y_{\sigma(i)}| \geq \lambda,$$ 

so that $A$ is not equiconnected, giving the required contradiction.

If now $\beta \in (\sum a_i e_i)\mathcal{C}$ is a Cauchy sequence in $\mathcal{C}$ in the $\mathcal{F}^\sigma$-norm topology. From what has just been proved, this is finer than the original topology. Thus $(\sum a_i e_i)\mathcal{C}$ is a Cauchy sequence in $\mathcal{C}$ in the original topology; since $E$ is sequentially complete, it is convergent, and since $(e_i)$ is a Schauder basis, it must converge to $\beta$. Hence $\lambda \equiv a_i$, since $\lambda \not\in \mathcal{C}$, $\lambda = \omega$. 

Locally convex spaces
Corollary 1. If $E$ is sequentially complete and $(C_d)$ is satisfied, $(SB_d)$ is satisfied if and only if $\lambda = 0$.

If $(SB_d)$ is satisfied, $(x_d)$ is a conditional basis, so that $E$ is bounded multiplier convergent. Thus, if $\lambda$ is solid. Since $\sigma$ is not solid, $\lambda = 0$. The converse follows from Theorems 3 and 4. Interpreting this result in terms of sequence spaces, we obtain

Corollary 2. If $\mu$ is a sequentially complete symmetric $\mathcal{A}$-space, either $\mu = 0$ or $\mu$ is solid.

Let us now give some examples to show that all the possibilities mentioned in Theorem 4 can occur. There are clearly plenty of spaces with Schauder bases satisfying the hypotheses of the theorem, for which $\lambda \leq \mu$. If $\lambda = \mu$, then the product topology is complete, and $(x_d)$ is a Schauder basis for which $\lambda = \mu$. If $\lambda = \mu$, then the product topology is complete, and $(x_d)$ is a Schauder basis for which $\lambda = \mu$. Finally let $E$ be and let $D = \{D : D$ is bounded in $P$ and $\sup_{n \to \infty} \sum_{d \in D} d_n \to 0$, $n \to \infty\}$.

$E$ is an $\mathcal{A}$-space under the topology of $E$-convergence, so that $(x_d)$ is a Schauder basis for $E$, and $\lambda = \mu$. We show that $E$ is complete under this topology. We can give $F$ the topology of $E$-convergence; since there is a base of $F$-closed neighbourhoods of $0$ for this topology, the topology of $E$-convergence is finer than the product topology, and since $F$ is $F_0$-complete under the topology of $E$-convergence (11, p. 53, Proposition 8). It therefore suffices to show that $\sigma$ is closed in $F$ for this topology. Suppose that $y \in F$. There exist distinct numbers $a$ and $\beta$, and increasing sequences $(n_0, m_0, m_0)$ of positive integers such that $y_n \to a$ and $y_m \to \beta$. By taking subsequences if necessary, we may assume that $n_0 \leq m_0 \leq n_{i+1}$, that $|y_n - a| \leq \frac{1}{i} |a - \beta|$ for $i = 1, 2, \ldots$. Now let $a_\lambda = 2^{d_\lambda} \sum_{i=1}^{d_\lambda} x_{i_{(d_\lambda)}}$, and let $D = \{a_\lambda : \lambda \in \mathcal{D}\}$. Clearly $D \in F$.

Further if $x \in E$, there exists $g$ such that $|x_i - y_i| \leq \frac{1}{i} |a - \beta|$ for $i, j \geq g$. If, then, $n_{i-1} \geq g$.

Hence $\langle y - x, a_\lambda \rangle = 2^{d_\lambda} \sum_{i=1}^{d_\lambda} (y_i - y_{i+1} - a_{i+1} + a_{i})$

$= 2^{d_\lambda} \sum_{i=1}^{d_\lambda} (a - \beta) + \sum_{i=1}^{d_\lambda} ((y_i - a) + (\beta - y_i) + (a_{i+1} - a_i)) = \frac{1}{i} |a - \beta|$.

Hence $\sup_{i \to \infty} \langle y - x, a_\lambda \rangle \geq \frac{1}{i} |a - \beta|$ for any $x$ in $e$, $a_\lambda$.

so that $\langle y - x, a_\lambda \rangle \geq \frac{1}{i} |a - \beta|$ for any $x$ in $e$.

Hence $\sup_{i \to \infty} \langle y - x, a_\lambda \rangle \geq \frac{1}{i} |a - \beta|$ for any $x$ in $e$.

so that $y \neq 0$, and $e$ is closed in $F$.

Theorem 5. If $E$ is barrelled and $(C_d)$ is satisfied, either $\lambda \leq \mu$ or $\lambda = \mu$. In either case, $(SB_d)$ is satisfied.

If $\lambda \leq \mu$, then $\lambda^2 \geq (\lambda^2)^0 = \lambda^2$. But $\lambda^2 = \lambda^2$ (6, Theorem 2, (c)). So that any norm-bounded subset of $P$ is weakly bounded in $\lambda^2$, and is therefore equiuniform. The topology of $\lambda$ is thus finer than the $\mathcal{F}$-norm topology. Since $\lambda$ is an $\mathcal{A}$-space, it follows that $\lambda \leq \mu$. The final result follows from Theorem 3.

§ 4. Further conditions. In this section we consider the relationship between the conditions $(SB_d)$, $(SB_d')$, $(SB_d)$, $(SB_d)$, $(C_d)$, $(C_d)$ and $(C_d)$.

If $E'$ is given the weak topology $\mathcal{E}(E', E)$, $(y_n)$ is a Schauder basis for $E'$. The next proposition follows from the definitions, and the fact that a subset of a locally convex space is bounded if and only if it is weakly bounded.

Proposition 3. Consider $(E', \mathcal{E}(E', E))$, with Schauder basis $(y_n)$.

(i) $(C_d)$ is satisfied for $E$ if and only if $(SB_d)$ is satisfied for $E'$.

(ii) $(SB_d)$ is satisfied for $E$ if and only if $(C_d)$ is satisfied for $E'$.

(iii) $(C_d)$ is satisfied for $E$ if and only if $(SB_d)$ is satisfied for $E'$.

(iv) $(SB_d)$ is satisfied for $E$ if and only if $(C_d)$ is satisfied for $E'$.

(v) $(SB_d)$ is satisfied for $E$ if and only if $(C_d)$ is satisfied for $E'$.

Proposition 4. If $\lambda = \sigma$ and either $(C_d)$ or $(SB_d)$ is satisfied, $(x_n)$ is bounded in $E$.

Suppose that $\lambda = \sigma$, and that $(x_n)$ is unbounded. Let $\mathcal{A} \subseteq \mathcal{P}$, and let $p$ be a continuous semi-norm on $E$ which is unbounded on $(x_n)$.

Let $J = \{1, 2, \ldots\}$, with $j_1 < j_2 < \ldots$, be an infinite set with an infinite complement with the property that $a_k = 0$ for all $k$. Let $\sigma$ be a permutation for which

$p(\sigma(x_n)) = p(a_k \sigma(x_n)) + 1$

for $i = 1, 2, \ldots$. Then

$p(\sum_{k=1}^{i} a_k \sigma(x_n)) \geq p(a_k \sigma(x_n)) + p(\sum_{k=1}^{i} a_k \sigma(x_n)) \geq i$
for \( i = 1, 2, \ldots \), so that \((\text{SB}_3')\) is not satisfied. Also
\[
p(i, q(i)) = p\left(\sum_{k=i}^{q(i)} a_k(x) x_k - \sum_{k=q(i)+1}^{q(i)+2a(i)} a_k(x) x_k\right)
\leq p\left(\sum_{k=i}^{q(i)} a_k(x) x_k\right) + p\left(\sum_{k=q(i)+1}^{q(i)+2a(i)} a_k(x) x_k\right),
\]
so that \((C_3)\) is not satisfied.

**Theorem 6.** Suppose that \((x)\) is bounded in \(E\) and \(\lambda \leq \rho\). If \((C_3)\) is satisfied, so is \((C_4)\). If \((\text{SB}_3')\) is satisfied, there exists an element \(x\) of \(E\) and a continuous semi-norm \(p\) on \(E\) such that

\[
\sup_{\alpha \in [0, \lambda]} \sup_{x \in E} \left\{ \sum_{i=1}^{\lambda} f(x) x_i \right\} = \infty.
\]

We shall show that it is possible to find a sequence \((\alpha)\) in \(P(N)\) and an increasing sequence \((\alpha)\) of positive integers such that
\[
\begin{align*}
(i) & \quad p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) \geq i \quad \text{for } i = 1, 2, \ldots, \\
(ii) & \quad i \leq \alpha_1, \alpha_2, \ldots \quad \text{and for } i = 2, 3, \ldots,
\end{align*}
\]
and
\[
\alpha_1(j) = \alpha_{i-1}(j) \quad \text{for } 1 \leq j \leq n_{i-1} \quad \text{and for } i = 2, 3, \ldots.
\]

We can find \(\alpha\) in \(P(N)\) and a positive integer \(m\) such that

\[
p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) \geq 1 + \left\| f(x) \right\| \sup_{\alpha \in [0, \lambda]} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\}.
\]

If \(\sigma^{-1}(1) \leq m\), we can take \(\alpha_1 = \alpha\) and \(n_1 = m\). If \(\sigma^{-1}(1) > m\), we define a new permutation \(\tau\) by setting \(\tau(m+1) = 1\), \(\tau^{-1}(1) = \sigma(m+1)\) and \(\tau(i) = \sigma(i)\), otherwise. Then putting \(\alpha_1 = \tau\) and \(n_1 = m+1\), (ii) is satisfied and

\[
p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) = p\left(\sum_{i=1}^{\lambda} f(x) x_i + f(x) \alpha_{m+1}\right)
\geq p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) - p\left(f(x)\right) \sup_{\alpha \in [0, \lambda]} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\} \geq 1.
\]

Suppose now that \(\alpha_1, \ldots, \alpha_{n_1\cdots n_{i-1}}\) and \(n_1, \ldots, n_{i-1}\) have been defined to satisfy conditions (i), (ii) and (iii). We can find \(\alpha\) in \(P(N)\) and a positive integer \(k > n_{i-1}\) such that

\[
p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) \geq r + (4n_{i-1}+1) \left\| f(x) \right\| \sup_{\alpha \in [0, \lambda]} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\}.
\]

Let \(R = \rho^{-1}(\{1, n_{i-1}\} \setminus \{1, n_{i-1}\}\)), \(S = \rho([1, n_{i-1}] \setminus \sigma_{i-1}(1, n_{i-1})), T = (R \cap [1, k]) \cup [1, n_{i-1}]\).

Then \(|R| = |S|\), so that there exists a 1-1 map \(\gamma\) of \(R\) onto \(S\). Now let

\[
\begin{align*}
\pi(i) & \quad \text{for } 1 \leq i \leq n_{i-1}, \\
\pi(i) & \quad \text{for } i \in R, \\
\pi(i) & \quad \text{otherwise}.
\end{align*}
\]

A straightforward verification shows that \(\pi \in P(N)\). Further

\[
\sum_{i=1}^{\lambda} f(x) x_i = \sum_{i=1}^{\lambda} f(x) x_i - \sum_{i=1}^{\lambda} f(x) x_i + \sum_{i=1}^{\lambda} f(x) x_i
\]
so that

\[
p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) \geq p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) - p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) - p\left(\sum_{i=1}^{\lambda} f(x) x_i\right)
\geq r + (4n_{i-1}+1) \left\| f(x) \right\| \sup_{\alpha \in [0, \lambda]} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\}
\geq r + \left\| f(x) \right\| \sup_{\alpha \in [0, \lambda]} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\},
\]

since \(|T| \leq 2n_{i-1}\). If \(\pi^{-1}(r) \leq n_{i-1}\), we can take \(\alpha = \pi\) and \(n_i = k\). Otherwise, we take \(n_i = k+1\), and alter \(\pi\) on \(k+1\), and \(\pi^{-1}(r)\), as before. This completes the proof of the induction. Now let

\[
\mu(i) = \begin{cases} 
\alpha_1(i) & \text{if } 1 \leq i \leq n_i, \\
\alpha_1(i) & \text{if } n_i < i \leq w_i.
\end{cases}
\]

\(\mu\) is a 1-1 map of \(N\) into itself (by (iii)), and condition (ii) ensures that \(\mu\) maps \(N\) onto \(N\), so that \(\mu \in P(N)\). Since

\[
p\left(\sum_{i=1}^{\lambda} f(x) x_i\right) \geq i \quad \text{for } i = 1, 2, \ldots,
\]

\((C_3)\) is not satisfied.

The proof that \((\text{SB}_3')\) implies \((\text{SB}_3)\) is extremely similar, and the details are omitted.

**Theorem 7.** If \(\alpha \leq \alpha_0\) and \((C_3)\) is satisfied, then \((\text{SB}_3')\) is satisfied.

Let \(\pi \in P(E)\), and let \(p\) be a continuous semi-norm on \(E\). Let

\[
M = \sup_{\alpha \in [0, \lambda]} \sup_{x \in E} \left\{ \sum_{i=1}^{\lambda} f(x) x_i\right\}.
\]
Suppose that $\sigma \in \mathcal{P}(\mathbb{N})$ and that $J \in \Sigma$. Let $m = \max \{ \sigma(j) \}$, and let $L = [1, m] \setminus \sigma(J)$. Since $\lambda \subseteq c_0$, there exists $n_0$ such that

$$\left| f_i(x) \right| \leq \left( \sum_{k=1}^{n_0} \left| p_k(x) \right| + 1 \right)$$

for $i \geq n_0$.

There exists $\tau$ in $\mathcal{P}(\mathbb{N})$ such that $\tau(\sigma(j)) = j$ for $j$ in $J$ and $\tau(i) \geq n_0$ if $i \in L$. Then

$$\sum_{\tau(i)} f_i(x) x_{\sigma(i)} = \sum_{i \in \tau(j)} f_j(x) x_{\sigma(j)} \leq \sum_{i \in \tau(j)} f_j(x) x_{\sigma(j)} - \sum_{i \in \tau(j)} f_j(x) x_{\sigma(j)},$$

so that

$$p \left\{ \sum_{\tau(i)} f_i(x) x_{\sigma(i)} \right\} \leq M + 1.$$

Now suppose that $\phi \in \mathcal{P}(\mathbb{N})$, that $\delta \subseteq \mathbb{N}$ and that $\delta = \{ \delta_1, \ldots, \delta_n \}$ is an $n$-tuple of complex numbers, with $|\delta_i| < 1$ for $1 \leq i \leq n$. We can find an $n$-tuple $(\gamma_1, \ldots, \gamma_n)$ of dyadic complex numbers, with $|\gamma_i| \leq 1$ for $1 \leq i \leq n$, for which

$$p \left\{ \sum_{i=1}^{n} (\gamma_i - \delta_i) f_j(x) x_{\sigma(i)} \right\} \leq 1.$$

We can also write

$$\sum_{i=1}^{n} \gamma_i f_j(x) x_{\sigma(i)} \leq \sum_{i=1}^{n} 2^{-k} \left( \sum_{\tau(i)} f_j(x) x_{\sigma(i)} - \sum_{\tau(i)} f_j(x) x_{\sigma(i)} + i \sum_{\tau(i)} f_j(x) x_{\sigma(i)} - i \sum_{\tau(i)} f_j(x) x_{\sigma(i)} \right),$$

where the $A_k, B_k, C_k$ and $D_k$ are suitable subsets of $[1, n]$. Thus

$$p \left\{ \sum_{i=1}^{n} \gamma_i f_j(x) x_{\sigma(i)} \right\} \leq 8(M + 1),$$

so that

$$p \left\{ \sum_{i=1}^{n} \delta_j f_j(x) x_{\sigma(i)} \right\} \leq 8M + 9.$$

**Corollary.** If $\lambda \subseteq c_0$, $\lambda \not\subseteq c_0$, and $(\text{SB}_2')$ is satisfied, then $(\text{SB}_2')$ is satisfied.

It follows from the conditions on $\lambda$ that $\lambda \not\subseteq c_0$. The result follows from the theorem and Proposition 3.

**Theorem 8.** If $\lambda \subseteq c_0$ and $\lambda \not\subseteq c_0$, the following are equivalent:

(i) $(\text{SB}_2')$ is satisfied;

(ii) $(\text{SB}_4')$ is satisfied;

(iii) $(\text{C}_4)$ is satisfied;

(iv) $\lambda \not\subseteq c_0$.

It is clear that (iv) implies (ii), and (i) implies both (ii) and (iii). We show that (ii) implies (iv) and that (iii) implies (iv).

Suppose that $a = (a_j) \in \mathcal{F}$. Let $\delta = (\delta_j) \in \lambda \setminus c$; so that there exist distinct numbers $\alpha$ and $\beta$, and disjoint increasing sequences $(m_i)$ and $(m_i')$ of positive integers such that $b_{m_i} \to \alpha$ and $b_{m_i'} \to \beta$.

Suppose first that $(\text{C}_4)$ is satisfied. Since $a \in \mathcal{F}$, given $M > 0$ there exists a finite set $J$ for which

$$\left| \sum_{\tau(i)} a_j \right| \leq \frac{3}{4} \sum_{\tau(i)} |a_j| \leq \frac{3M}{\alpha - \beta}.$$

Let $m = \sup \{ j \}$, and let $K = [1, m] \setminus J$. There exist permutations $\sigma$ and $\tau$ such that

$$|b_{m_j} - a_j| \leq \frac{1}{4} |\alpha - \beta|,$$

for $j$ in $J$ and $\sigma(\tau(j)) = j$ for $j$ in $K$. Then

$$\left| \sum_{\tau(i)} b_{m_j} a_j - \sum_{\tau(i)} b_{m_j} a_j \right| \leq \left| \sum_{\tau(i)} (b_{m_j} - b_{m_j}) a_j \right| \geq \left| \sum_{\tau(i)} (\alpha - \beta) a_j \right| - \sum_{\tau(i)} |b_{m_j} - \alpha| |a_j| - \sum_{\tau(i)} |b_{m_j} - \beta| |a_j| \geq \frac{3}{4} |\alpha - \beta| \sum_{\tau(i)} |a_j| - \frac{1}{4} |\alpha - \beta| \sum_{\tau(i)} |a_j| \geq M.$$

It therefore follows that $a \not\in \lambda$.

Suppose next that $(\text{SB}_4)$ is satisfied. If $a \in \mathcal{F}$, there exist distinct numbers $\gamma$ and $\delta$ and disjoint increasing sequences $(p_i)$ and $(q_i)$ of positive integers such that $a_{p_i} \to \gamma$ and $a_{q_i} \to \delta$. We can suppose that $\mathcal{N} \setminus \{ m_0 \}$ and $\mathcal{N} \setminus \{ p_0 \}$ are both infinite; let $\sigma$ and $\tau$ be permutations such that

$$\sigma(m_i) = p_i, \quad \tau(m_i) = q_i, \quad \sigma(q_i) = q_i, \quad \tau(p_i) = p_i$$

for $i = 1, 2, \ldots$ and such that $\sigma(j) = \tau(j)$ otherwise. Then a straightforward calculation shows that

$$\sum_{n} b_{m_{\sigma(n)}} - \sum_{n} b_{m_{\tau(n)}} = \infty.$$
so that \( a \in X \). If, next, \( a, c \in c_0 \), \( \sum_{n=1}^{\infty} b_n a_n \) is not convergent, so that \( a \notin X \).
Thus \( X \subseteq c_0 \). But since \((SB)_1\) is satisfied, \((C)_1 \) is satisfied for \( (E', \sigma(E', E)) \) (Proposition 3), \((SB)_1\) is satisfied for \( (E', \sigma(E', E)) \) (Theorem 3) and \((SB)_1\) is satisfied for \( E \) (Proposition 3).

Let us now relate the various implications as the size of \( \lambda \) varies, and give some examples. In the diagrams which follows, arrows denote implications, and conditions included in a bracket are equivalent.

Case 1. \( \lambda = p \). In this case \((SB)_1\), \((SB)_1\), \((C)_1\) and \((C)_1\) are always satisfied. When \( \lambda \) is given the direct sum topology none of \((SB)_1\), \((SB)_1\) or \((C)_1\) is satisfied.

Case 2. \( \lambda \equiv \xi \) and \( \lambda \neq \xi \).
\[ ([SB]_1 \wedge [C]_1) \Rightarrow ([SB]_1 \wedge [C]_1) \Rightarrow ([SB]_1 \wedge [SB]_1). \]

Let \( E = c_0 \), let \( F = \sigma \oplus \text{span} (e) \), and give \( E \) the weak topology \( \sigma(E, F) \) (for the natural duality between \( E \) and \( F \)). \( (e) \) is a Schauder basis for \( E \) for which \( \lambda = c_0 \subseteq c_0 \), and for which \((SB)_1\) is satisfied, while \((C)_1\) is not satisfied.

Case 3. \( \lambda \equiv c \) and \( \lambda \neq c_0 \).
\[ ([SB]_1 \wedge [C]_1) \Rightarrow ([SB]_1 \wedge [SB]_1 \wedge [SB]_1). \]

Give the space \( F \) of the preceding example the weak topology \( \sigma(F, E) \). \( (e) \) is a Schauder basis for \( F \) which satisfies \((C)_1\) and not \((SB)_1\), and for which the conditions on \( \lambda \) are satisfied.

Consider next \( E = c_0 \), with the topology of \( \xi \)-convergence, as described in \( \xi \). Under the Schauder basis \( (e) \), \((C)_1\) and \((SB)_1\) are satisfied, whereas \((SB)_1\) is not (Theorem 4, Corollary 2). Thus \((C)_1\) and \((SB)_1\) do not together imply \((SB)_1\).

Case 4. \( \lambda = p = \infty \) and \( \lambda 
eq c_0 \).
\[ ([SB]_1 \wedge [C]_1) \Rightarrow ([SB]_1 \wedge [SB]_1 \wedge [SB]_1). \]

Case 5. \( \lambda \neq p = \infty \).
\[ ([SB]_1 \wedge [C]_1) \Rightarrow ([SB]_1 \wedge [C]_1). \]

In this case, if \((SB)_1\) or \((C)_1\) is satisfied, the topology on \( \lambda \) must be the product topology (arguing as in Proposition 2); in particular \((SB)_1\) or \((C)_1\) can never be satisfied.

We end this section with a theorem in which topological conditions are imposed.

**Theorem 9.** If \( E \) is sequentially complete and barreled, then \((SB)_1\) is satisfied if either \((SB)_1\) or \((C)_1\) is satisfied.

The proof follows the proof of the Corollary to the Theorem of [10]. Suppose that \((SB)_1\) is satisfied. Let \( P \) be a collection of continuous semi-norms on \( E \) which defines the topology of \( E \). For each \( p \) in \( P \), let
\[ q(x) = \sup_{n=1}^{\infty} \sup_{a \in X} p \left( \sum_{i=1}^{n} f_i(x) a_i \right). \]

Each such \( q \) is a semi-norm on \( E \) (since \((SB)_1\) is satisfied) which is lower semi-continuous, and therefore continuous (since \( E \) is barreled — cf. [5], \( \xi \)). Further
\[ q(x) \geq \sup_{n=1}^{\infty} \sum_{i=1}^{n} f_i(x) a_i \geq p(x), \]
so that the collection \( Q \) of semi-norms defined in this way defines the topology of \( E \). Now if \( x \in E \), \( r \in P \) and \( q \in Q \),
\[ q \left( \sum_{i=1}^{n} f_i(x) a_i \right) = q \left( \sum_{i=1}^{n} f_i(x) a_i \right), \]
so that \( \sum_{i=1}^{n} f_i(x) a_i \) is convergent (since \( E \) is sequentially complete).
The fact that \((SB)_1\) is satisfied now follows easily. The proof is very similar when \((C)_1\) is satisfied, and is omitted.

Notice also that when the hypotheses of the theorem are satisfied, \( (P, \sigma_{P, \wedge}) \) is an equicontinuous group of linear operators from \( E \) into itself (cf. [5], Theorem 2).

The condition that \( E \) is barreled cannot be relaxed in this theorem (consider \( c_0 \) with the topology of \( \xi \)-convergence). Nor can the condition that \( E \) is sequentially complete be relaxed, as the following example shows. Let \( A \) be the set
\[ \{ a : a \in c, a = 0 \text{ or } 1 \} \text{ and } \left( \sum_{i=1}^{n} a_i \right)/n \to 0 \text{ as } n \to \infty \]
and let \( U = \{ x : x = a \cdot y \text{, where } a \in A \text{ and } y \in F \} \). \( U \) is a linear subspace of \( F \) which is barreled under the \( \ell^1 \)-norm topology ([7], p. 372). \( (e) \) is a Schauder basis for \( U \), for which \((SB)_1\) is satisfied, whereas \((C)_1\) is not.

**§ 5. A counterexample.** We give an example of a Banach space with a Schauder basis which satisfies \((SB)_1\), but not \((SB)_1\), contradicting an assertion of Singer [10].
Let $O$ be the collection of all increasing sequences of elements of $N$, For each $m = (m_i) \in O$, let $a_m$ be the sequence defined by $(a_m)_i = j^{-1/2}$ if $i = n_j$, and $(a_m)_i = 0$ otherwise, and let $A = (a_m : m \in O)$. Let

$$\lambda_d = \{x \in \mathbb{C}^\infty, \sup_{m \in A} \sum_i |x_i a_i| < \infty\},$$

and let

$$\|x\|_d = \sup_{m \in A} \sum_i |x_i a_i|.$$ 

$\|\cdot\|_d$ is a norm on $\lambda_d$, under which $\lambda_d$ is a Banach space (cf. [8], or [4], Proposition 3). Let $E$ be the closure of $\varphi$ in $\lambda_d$. It is clear that $\varphi$ is an unconditional basis for $E$, and that $(SB_d)$ is satisfied. If $(SB_d)$ were satisfied, $\{T_{\sigma} : \sigma \in P(N)\}$ would be equicontinuous (see the remark after Theorem 9) that is there would exist $M$ such that $\|T_{\sigma}\| \leq M$ for all $\sigma$ in $P(N)$. We shall show that this is not so.

Let $a''_i$ be the sequence defined by $a''_i = (r + 1 - i)^{-1/2}$, for $1 \leq i \leq r$, and $a''_i = 0$ otherwise. We shall show first that the sequence $(a''_i : r = 1, 2, \ldots)$ is bounded in $E$. For fixed $r$, there exists a finite set $n_1 < \ldots < n_s \leq r$ of integers such that

$$\|a''_i\|_d = \sum_{i=1}^r |(r + 1 - n_i)^{-1/2}|.$$

Let $n_i = n_i$ for $1 \leq i < s$, and let $n_s = r$. Then

$$0 \leq |a''_i| - \sum_{i=1}^r |(r + 1 - n_i)^{-1/2}| = \|a''_i\|_d - \sum_{i=1}^s |(r + 1 - n_i)^{-1/2}| = |(r + 1 - n_s)^{-1/2}| - \sum_{i=1}^s |(r + 1 - n_i)^{-1/2}|.$$

This implies that $n_s = r$. A similar argument then shows that $n_i = r - i - s$, for $1 \leq i < s$, so that

$$\|a''_i\|_d = \sum_{i=1}^s |(r + 1 - s)^{-1/2}| \leq \frac{2}{s+1} \left( \frac{s+1}{s+1} \right)^{1/2} = \frac{s+1}{s+1} \leq 1 + \pi.$$ 

Now let $\varphi_\sigma$ be defined by $\varphi_\sigma(i) = r + 1 - i$ for $1 \leq i \leq r$, and $\varphi_\sigma(i) = i$ otherwise. Then

$$T_{\varphi_\sigma}(a''_i) = (1, 2^{-1/2}, \ldots, r^{-1/2}, 0, 0, \ldots)$$

and

$$\|T_{\varphi_\sigma}(a''_i)\|_d = \sum_{i=1}^r (j^{-1/2}).$$

Since $\sum_{i=1}^r (j^{-1})$ is divergent, it follows that $\{T_{\sigma} : \sigma \in P(N)\}$ is not uniformly bounded.

The error in Singer's argument lies in asserting that his inequality (21) follows from $(SB_d)$ and his Lemma 2.

References