we observe that $|a_n|\,\|x^n\|_i \leqslant 2M_i^n(\|x\|_{i+1}^*)^n$ for every $x \in X$, $n = 0, 1, \ldots$

From [3] it follows that X is an m-convex B_0-algebra. By remark 4 we infer that X is a Q-algebra, q.e.d.

References

[1] Н. И. Ахиезер, *Лекции по теории аппроксимации*, Москва 1965.

[2] E. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. 11 (1952).

[3] W. Żelazko, *Metric generalization of Banach algebras*, Rozprawy Matematyczne 47, Warszawa 1965.

Reçu par la Rédaction le 4. 7. 1967

STUDIA MATHEMATICA, T. XXX. (1968)

On the generation of tight measures

by

J. KISYŃSKI (Warszawa)

A non-negative measure μ defined on a σ-algebra $\mathscr{A}$ of subsets of a topological space is called *tight* if

$$\mu(A) = \sup\{\mu(C) : A \supset C \in \mathscr{A},\ C\text{—compact}\}$$

for every $A \in \mathscr{A}$. The main result of this paper is theorem 2.1 concerning extensions to tight measures of some set functions in arbitrary Hausdorff spaces. This theorem generalizes a theorem given by Bourbaki ([1], Chap. IV, § 4, N° 10, theorem 5) for locally compact spaces. The proof of theorem 2.1 is based on the idea of Halmos ([3], § 53 and 54) of extending to a measure a certain "semi-regular content" obtained from a given set function. However, the method of such extension presented here is different from that of Halmos.

Throughout this paper the Borel subsets of any topological space X are defined as elements of the smallest σ-algebra of subsets of X, containing all the closed subsets of X.

1. Extension of a content to a tight measure. We call a *content* any non-negative, finite, non-decreasing set function λ defined on the class of all compact subsets of topological space X, such that for every pair A, B of compact subsets of X we have

$$\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$$

and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B) \quad \text{if} \quad A \cap B = \emptyset.$$

We say that a content λ is *tight* if

$$\lambda(A) - \lambda(B) = \sup\{\lambda(C) : C \subset A \setminus B,\ C\text{—compact}\}$$

for every pair A, B of compacts such that $B \subset A$.

We say that a content λ is *semi-regular*, if for every compact A and every $\varepsilon > 0$ there is an open set U such that $A \subset U$ and $\lambda(B) < \lambda(A) + \varepsilon$ for every compact $B \subset U$.

THEOREM 1.1. *Every semi-regular content in a topological Hausdorff space is tight.*

Proof. Let λ be a semi-regular content in a Hausdorff space X and let A and B be compact subsets of X such that $B \subset A$. Since $\lambda(C) \leqslant \lambda(A)-\lambda(B)$ for any compact $C \subset A \setminus B$, in order to prove that λ is tight it is sufficient to show that, for every $\varepsilon > 0$, there is a compact $C \subset A \setminus B$ such that $\lambda(C) > \lambda(A)-\lambda(B)-\varepsilon$. By the semi-regularity of λ, for any $\varepsilon > 0$, there is an open set U such that $B \subset U$ and $\lambda(D) < \lambda(B)+\varepsilon$ for every compact $D \subset U$. Since the compact set A is contained in the union of open sets U and $X \setminus B$, by a theorem of Halmos ([3], § 50, theorem A, p. 216), there are compact sets C and D such that $C \cup D = A$, $C \subset X \setminus B$ and $D \subset U$. We then have $\lambda(C) \geqslant \lambda(A)-\lambda(D) > \lambda(A)-\lambda(B)-\varepsilon$.

THEOREM 1.2. *Every tight content in a topological space may be extended to a tight measure defined on the σ-algebra of all Borel subsets of this space.*

Proof. Let λ be a tight content defined on the class $\mathscr{C}$ of all compact subsets of a space X. For any $E \subset X$ put

$$\mu(E) = \sup\{\lambda(C) : E \supset C \in \mathscr{C}\},$$

$$\mathscr{M}_E = \{F : F \subset X,\ \mu(E) \leqslant \mu(E \cap F)+\mu(E \setminus F)\}$$

and let

$$\mathscr{M} = \bigcap_{C \in \mathscr{C}} \mathscr{M}_C .$$

The theorem will be proved if we show that $\mathscr{M}$ is a σ-algebra of subsets of X containing all the Borel subsets of X and the set function μ restricted to $\mathscr{M}$ is a measure.

Clearly $\mu(\emptyset) = 0$ and, by the monotonity and additivity of λ,

$$\mu(E)+\mu(F) \leqslant \mu(E \cup F)$$

for any pair E, F of disjoint subsets of X. Further, it is obvious that, since λ is tight, $\mathscr{M}$ contains all the closed subsets of X and that if $E \in \mathscr{M}$, then $X \setminus E \in \mathscr{M}$. Hence, in order to complete the proof, we need only to show that, if $E_1, E_2, \ldots$ is a sequence of sets in $\mathscr{M}$, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathscr{M} \quad \text{and} \quad \mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) \leqslant \sum_{n=1}^{\infty} \mu(E_n).$$

To prove this, let $E_1, E_2, \ldots$ be a sequence of sets in $\mathscr{M}$ and let $C \in \mathscr{C}$ and $\varepsilon > 0$ be arbitrarily fixed. Then, by the definition of $\mathscr{M}$, for every $n = 1, 2, \ldots$ there are $A_n \in \mathscr{C}$ and $B_n \in \mathscr{C}$ such that

$$A_n \subset C \cap E_n, \quad B_n \subset C \setminus E_n \quad \text{and} \quad \lambda(A_n)+\lambda(B_n) > \lambda(C)-\frac{\varepsilon}{2^n}.$$

Since, for every $n = 1, 2, \ldots$, $(A_1 \cup \ldots \cup A_{n-1}) \cap A_n$ and $(B_1 \cap \ldots \cap B_{n-1}) \cup B_n$ are disjoint subsets of C and since λ is tight, we have

$$\begin{aligned}
-\frac{\varepsilon}{2^n} < \lambda(A_n)+\lambda(B_n)-\lambda(C) \leqslant \lambda(A_n)+\lambda(B_n)-\lambda\big((A_1 \cup \ldots \cup A_{n-1}) \cap A_n\big)-\\
-\lambda\big((B_1 \cap \ldots \cap B_{n-1}) \cup B_n\big)\\
= \mu\big(A_n \setminus (A_1 \cup \ldots \cup A_{n-1})\big)-\mu\big((B_1 \cap \ldots \cap B_{n-1}) \setminus B_n\big)\\
= \lambda(A_1 \cup \ldots \cup A_n)+\lambda(B_1 \cap \ldots \cap B_n)-\lambda(A_1 \cup \ldots \cup A_{n-1})-\\
-\lambda(B_1 \cap \ldots \cap B_{n-1}),
\end{aligned}$$

from which, by induction

$$\lambda(A_1 \cup \ldots \cup A_n)+\lambda(B_1 \cap \ldots \cap B_n) > \lambda(C)-\sum_{k=1}^{n}\frac{\varepsilon}{2^k} > \lambda(C)-\varepsilon .$$

Since λ is tight, there is a compact

$$D \subset B_1 \setminus \bigcap_{n=1}^{\infty} B_n$$

such that

$$\lambda(D) > \lambda(B_1)-\lambda\Big(\bigcap_{n=1}^{\infty} B_n\Big)-\varepsilon .$$

Since

$$D \cap \Big(\bigcap_{n=1}^{\infty} B_n\Big) = \emptyset$$

and D and B_n are compact, we have

$$D \cap \Big(\bigcap_{n=1}^{N} B_n\Big) = \emptyset$$

for sufficiently large N and thus

$$\lambda\Big(\bigcap_{n=1}^{N} B_n\Big) \leqslant \lambda(B_1)-\lambda(D) < \lambda\Big(\bigcap_{n=1}^{\infty} B_n\Big)+\varepsilon .$$

It follows that

$$\sum_{n=1}^{N} \lambda(A_n)+\lambda\Big(\bigcap_{n=1}^{\infty} B_n\Big) \geqslant \lambda\Big(\bigcup_{n=1}^{N} A_n\Big)+\lambda\Big(\bigcap_{n=1}^{\infty} B_n\Big) > \lambda(C)-2\varepsilon$$

for sufficiently large N. Hence, because

$$A_1 \cup \ldots \cup A_N \subset C \cap \Big(\bigcup_{n=1}^{\infty} E_n\Big), \qquad \bigcap_{n=1}^{\infty} B_n \subset C \setminus \Big(\bigcup_{n=1}^{\infty} E_n\Big)$$

and $\varepsilon > 0$ is arbitrary, we have

$$\mu\Big(C \cap \big(\bigcup_{n=1}^{\infty} E_n\big)\Big) + \mu\Big(C \setminus \big(\bigcup_{n=1}^{\infty} E_n\big)\Big) \geqslant \lambda(C)$$

and

$$\sum_{n=1}^{\infty} \mu(C \cap E_n) + \mu\Big(C \setminus \big(\bigcup_{n=1}^{\infty} E_n\big)\Big) \geqslant \lambda(C)$$

for every $C \in \mathscr{C}$. The first of these inequalities shows that $\bigcup_{n=1}^{\infty} E_n \in \mathscr{M}$ and the second — that

$$\mu\big(\bigcup_{n=1}^{\infty} E_n\big) = \sup\{\lambda(C): \bigcup_{n=1}^{\infty} E_n \supset C \in \mathscr{C}\} \leqslant \sum_{n=1}^{\infty} \mu(E_n).$$

Thus theorem 1.2 is proved.

The proof given above is closely related to the reasonings of Neveu [5], Chapter 1-5, p. 19-23. The analogy is more expressive in view of the remark that

$$\mathscr{M}_E = \bigcap_{E \supset C \in \mathscr{C}} \mathscr{M}_C$$

for every $E \subset X$ such that $\mu(E) < \infty$. To prove this equality, let $E \subset X$ and $\mu(E) < \infty$. If $F \in \mathscr{M}_E$, then, for any $\varepsilon > 0$, there are $A \in \mathscr{C}$ and $B \in \mathscr{C}$ such that $A \subset E \cap F$, $B \subset E \setminus F$ and $\lambda(A \cup B) = \lambda(A) + \lambda(B) > \mu(E) - \mathscr{C}$. If $E \supset C \in \mathscr{C}$, then

$$\begin{aligned}\mu(C \cap F) + \mu(C \setminus F) &\geqslant \lambda(C \cap A) + \lambda(C \cap B) = \lambda\big(C \cap (A \cup B)\big) \\ &= \lambda(C) - \mu\big(C \setminus (A \cup B)\big) = \lambda(C) + \lambda(A \cup B) - \lambda(A \cup B \cup C) \\ &\geqslant \lambda(C) + \lambda(A \cup B) - \mu(E) > \lambda(C) - \varepsilon,\end{aligned}$$

which shows that

$$F \in \bigcap_{E \supset C \in \mathscr{C}} \mathscr{M}_C.$$

On the other hand, if

$$F \in \bigcap_{E \supset C \in \mathscr{C}} \mathscr{M}_C,$$

then for $E \supset C \in \mathscr{C}$ we have

$$\lambda(C) \leqslant \mu(C \cap F) + \mu(C \setminus F) \leqslant \mu(E \cap F) + \mu(E \setminus F) \leqslant \mu(E),$$

so that $\mu(E) = \sup\{\lambda(C): E \supset C \in \mathscr{C}\} = \mu(E \cap F) + \mu(E \setminus F)$ and $F \in \mathscr{M}_E$.

2. Generalization of a theorem of Bourbaki.

THEOREM 2.1. *Let X be a topological Hausdorff space. Let $\mathcal{O}$ be a class of open subsets of X such that*

(i) *if $C \subset X$ is compact and $U \in \mathcal{O}$, then $U \setminus C \in \mathcal{O}$.*

Let $\mathscr{L}$ be a lattice of subsets of X such that

(ii) *if C is compact, $U \in \mathcal{O}$ and $C \subset U$, then there is an $A \in \mathscr{L}$ such that $C \subset A \subset U$;*

(iii) *for every pair C, D of disjoint compact subsets of X there are an $A \in \mathscr{L}$ and a $B \in \mathscr{L}$ such that $C \subset A$, $D \subset B$ and $A \cap B = \emptyset$.*

Let μ_0 be a non-negative, finite, non-decreasing set function on $\mathscr{L}$, such that for any $A \in \mathscr{L}$ and $B \in \mathscr{L}$ we have

(iv) $$\mu_0(A \cup B) \leqslant \mu_0(A) + \mu_0(B)$$

and

(v) $$\mu_0(A \cup B) = \mu_0(A) + \mu_0(B) \quad \text{if} \quad A \cap B = \emptyset.$$

Assume furthermore that

(vi) *for every $A \in \mathscr{L}$ and every $\varepsilon > 0$ there are C and U such that C is compact, $U \in \mathcal{O}$, $C \subset A \subset U$ and that $|\mu_0(A) - \mu_0(B)| < \varepsilon$ for every $B \in \mathscr{L}$ satisfying $C \subset B \subset U$.*

Under these assumptions there is a unique tight measure μ defined on the σ-algebra of all Borel subsets of X, whose completion is an extension of μ_0, i.e. such that for every $A \in \mathscr{L}$ there are Borel sets F and G such that $F \subset A \subset G$ and $\mu(F) = \mu_0(A) = \mu(G)$.

Proof. By (iii), by the finiteness and monotonity of μ_0 and by (iv) and (v), if for any compact $C \subset X$ we put

$$\lambda(C) = \inf\{\mu_0(A): C \subset A \in \mathscr{L}\},$$

then λ is a content. By (ii) and (vi), for every compact C we have

(α) $$\lambda(C) = \inf\{v(U): C \subset U \in \mathcal{O}\},$$

where

$$v(U) = \sup\{\mu_0(B): U \supset B \in \mathscr{L}\} = \sup\{\lambda(D): D \subset U,\ D\text{-compact}\}$$

for every $U \in \mathcal{O}$. This in particular shows that λ is a semi-regular content; and so, by theorems 1.1 and 1.2, it may be extended to a tight measure μ defined on the σ-algebra of all Borel subsets of X. If $A \in \mathscr{L}$ and $\varepsilon > 0$ then, by (vi), there are C and U such that C is compact, $U \in \mathcal{O}$, $C \subset A \subset U$, $\mu_0(A) - \varepsilon \leqslant \lambda(C)$ and $v(U) \leqslant \mu_0(A) + \varepsilon$. Since $\mu(C) = \lambda(C) \leqslant \mu_0(A) \leqslant v(U) = \mu(U)$, we have $\mu_0(A) - \varepsilon \leqslant \mu(C) \leqslant \mu_0(A) \leqslant \mu(U) \leqslant \mu_0(A) + \varepsilon$. This shows that the completion of μ is an extension of μ_0. Thus the existence of a measure μ satisfying the theorem is proved.

For the proof of uniqueness it is sufficient to show that, if a measure μ satisfies the theorem, then $\mu(C) = \lambda(C)$ for every compact $C \subset X$. So, suppose that μ is such a measure. Let $U \in \mathcal{O}$. If $C \subset U$ is compact, then, by (ii), there is $A \in \mathscr{L}$ such that $C \subset A \subset U$. Since the completion of μ

is an extension of μ_0, we have $\mu(C) \leqslant \mu_0(A) \leqslant \mu(U)$, which, μ being tight, implies that $\mu(U) = v(U)$. Hence, by (α), we have

$$(\beta) \qquad \lambda(C) = \inf\{\mu(U) : C \subset U \in \mathcal{O}\} \geqslant \mu(C)$$

for every compact $C \subset X$. It follows that for every compact C there is a $V \in \mathcal{O}$ such that $V \supset C$ and $\mu(V) < \infty$, so that, since μ is tight, by (i) and (β)

$$\begin{aligned}\mu(C) &= \mu(V) - \mu(V \setminus C) = \mu(V) - \sup\{\mu(D) : D \subset V \setminus C,\ D - \text{compact}\} \\ &= \inf\{\mu(V \setminus D) : D \subset V \setminus C,\ D - \text{compact}\} \\ &\geqslant \inf\{\mu(U) : C \subset U \in \mathcal{O}\} = \lambda(C).\end{aligned}$$

Thus $\mu(C) = \lambda(C)$ for every compact $C \subset X$, which completes the proof.

A *neighbourhood* of a subset A of a topological space X (or of a point $x \in X$) is any subset G of X such that $A \subset \operatorname{Int} G$ (or $x \in \operatorname{Int} G$). A non-negative, finite, non-decreasing set function μ defined on a class $\mathscr{L}$ of subsets of a topological space is called *regular* if for every $A \in \mathscr{L}$ and every $\varepsilon > 0$ there are $F \in \mathscr{L}$ and $G \in \mathscr{L}$ such that $\overline{F} \subset A \subset \operatorname{Int} G$, $\mu_0(F) > \mu_0(A) - \varepsilon$ and $\mu_0(G) < \mu_0(A) + \varepsilon$.

THEOREM 2.2. *Let $\mathscr{L}$ be a lattice of subsets of a Hausdorff space X such that every two distinct points of X have disjoint neighbourhoods belonging to $\mathscr{L}$. Let μ_0 be a non-negative, finite, non-decreasing, regular set function on $\mathscr{L}$, satisfying* (iv) *and* (v).

Then the following two statements are equivalent:

(a) *there is a unique non-negative, finite, tight measure defined on the σ-algebra of all Borel subsets of X, whose completion is an extension of μ_0;*

(b) *for every $\varepsilon > 0$ there is a compact subset K of X such $X \setminus K \supset A \in \mathscr{L}$ implies $\mu_0(A) < \varepsilon$.*

Proof. It is clear that (a) implies (b). To prove the opposite implication, observe that since every two distinct points of X have disjoint neighbourhoods in $\mathscr{L}$, it follows that every two disjoint compact subsets of X have disjoint neighbourhoods in $\mathscr{L}$. So, if we put $\mathcal{O} = \{X \setminus C : C \subset X,\ C - \text{compact}\}$, then conditions (i), (ii) and (iii) are satisfied.

Further, suppose that (b) holds and let $A \in \mathscr{L}$ and $\varepsilon > 0$. Let K be such a compact subset of X that $X \setminus K \supset E \in \mathscr{L}$ implies $\mu_0(E) < \varepsilon/2$. Let $F \in \mathscr{L}$ and $G \in \mathscr{L}$ be such that

$$\overline{F} \subset A \subset \operatorname{Int} G, \quad \mu_0(F) > \mu_0(A) - \frac{\varepsilon}{2} \quad \text{and} \quad \mu_0(G) < \mu_0(A) + \frac{\varepsilon}{2}.$$

Put $C = K \cap \overline{F}$ and $U = (X \setminus K) \cup \operatorname{Int} G$. Then C is compact, $U \in \mathcal{O}$ and $C \subset A \subset U$. If $B \in \mathscr{L}$ and $C \subset B \subset U$, then $F \setminus B \subset \overline{F} \setminus C \subset X \setminus K$ and $B \setminus G \subset U \setminus \operatorname{Int} G \subset X \setminus K$, so that

$$\begin{aligned}\mu_0(A) - \varepsilon < \mu_0(F) - \frac{\varepsilon}{2} &\leqslant \mu_0(F \setminus B) + \mu_0(B) - \frac{\varepsilon}{2} \leqslant \mu_0(B) \\ &\leqslant \mu_0(B \setminus G) + \mu_0(G) \leqslant \mu_0(G) + \frac{\varepsilon}{2} < \mu_0(A) + \varepsilon.\end{aligned}$$

Thus we see that condition (vi) is satisfied. Hence the implication (b) $\Rightarrow$ (a) is a corollary to theorem 2.1.

3. Application to projective systems of measure spaces.

ASSUMPTIONS 3.1. *Let $(I, \leqslant)$ be a directed set. For any $i \in I$ let X_i be a topological Hausdorff space and let p_i be a probability measure defined on the σ-algebra $\mathscr{B}_i$ of all the Borel subsets of X_i. Suppose that for every pair $(i, j) \in I^2$ such that $i \leqslant j$ a continuous mapping π_{ij} of X_j onto X_i is given, such that*

(3.1.1) $\quad \pi_{ii} = $ *identity for every* $i \in I$,

(3.1.2) $\quad \pi_{ij} \circ \pi_{jk} = \pi_{ik}$

for every triple $(i, j, k) \in I^3$ such that $i \leqslant j \leqslant k$ and

(3.1.3) $\quad p_i(E) = p_j(\pi_{ij}^{-1}(E))$

for every pair $(i, j) \in I^2$ such that $i \leqslant j$ and for every $E \in \mathscr{B}_i$.

Suppose further that X is a topological Hausdorff space and that for every $i \in I$ a continuous mapping π_i of X onto X_i is given, such that

(3.1.4) $\quad$ *if $(i, j) \in I^2$ and $i \leqslant j$, then $\pi_i = \pi_{ij} \circ \pi_j$.*

THEOREM 3.2. *Under assumption 3.1 suppose, moreover, that all the measures p_i, $i \in I$, are regular and that for every two distinct points x and y of X there exists an $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$. Then the following two statements are equivalent:*

(3.2.1) *there is a unique tight probability measure p defined on the σ-algebra of all the Borel subsets of X, such that $p_i(E) = p(\pi_i^{-1}(E))$ for every $i \in I$ and $E \in \mathscr{B}_i$;*

(3.2.2) *for every $\varepsilon > 0$ there is a compact subset K of X, such that $p_i(X_i \setminus \pi_i(K)) \leqslant \varepsilon$ for every $i \in I$.*

Proof. We shall show that theorem 3.2 is a corollary to theorem 2.2. Put

$$\mathscr{L} = \{\pi_i^{-1}(E) : i \in I,\ E \in \mathscr{B}_i\}.$$

Then, by (3.1.4), we see that $\mathscr{L}$ is an algebra of the Borel subsets of X. If $x \in X$, $y \in Y$ and $x \neq y$, then there is an $i \in I$ such that $\pi_i(x) \neq \pi_i(y)$. Since X_i is a Hausdorff space, there are disjoint open subsets U and V of X_i, such that $\pi_i(x) \in U$ and $\pi_i(y) \in V$. The mapping π_i being continuous, $\pi_i^{-1}(U)$ and $\pi_i^{-1}(V)$ are open subsets of X. Clearly $\pi_i^{-1}(U) \cap \pi_i^{-1}(V) = \emptyset$, $x \in \pi_i^{-1}(U)$ and $y \in \pi_i^{-1}(V)$. Thus we see that every two distinct points of X have disjoint open neighbourhoods belonging to $\mathscr{L}$.

If $E \in \mathscr{L}$ and $E = \pi_i^{-1}(E_i)$, where $i \in I$ and $E_i \in \mathscr{B}_i$, then, for any $j \in I$ such that $i \leqslant j$, since the mappings π_{ij} and π_j are surjective, we have $\pi_j(E) = \pi_{ij}^{-1}(E_i)$ and $E = \pi_j^{-1}(\pi_j(E))$, so that $\pi_j(E) \in \mathscr{B}_j$ and $p_j(\pi_j(E)) = p_i(E_i)$. Hence, if for any $E \in \mathscr{L}$ we put

$$\mu_0(E) = \lim_{i \in I} p_i(\pi_i(E)),$$

then μ_0 is a well-defined set function on $\mathscr{L}$ with values in $[0, 1]$. If $E \in \mathscr{L}$, $F \in \mathscr{L}$, and $E \cap F = \emptyset$, then, for every $i \in I$ greater than a certain $i_0 \in I$, we have $E = \pi_i^{-1}(\pi_i(E))$ and $F = \pi_i^{-1}(\pi_i(F))$, so that $\pi_i^{-1}(\pi_i(E) \cap \pi_i(F)) = E \cap F = \emptyset$, and therefore $\pi_i(E) \cap \pi_i(F) = \emptyset$, since π_i is surjective. Hence

$$\mu_0(E \cup F) = \lim_{i \in I} p_i(\pi_i(E \cup F)) = \lim_{i \in I} p_i(\pi_i(E) \cup \pi_i(F))$$
$$= \lim_{i \in I} \big(p_i(\pi_i(E)) + p_i(\pi_i(F))\big) = \mu_0(E) + \mu_0(F),$$

which shows that μ_0 is an additive set function on the algebra $\mathscr{L}$. Since all p_i, $i \in I$, are regular, it follows that μ_0 is a regular set function.

Thus, for $\mathscr{L}$ and μ_0 defined above, all the assumptions of theorem 2.2 are satisfied. Since, by the definition of μ_0, for any $i \in I$ and $E \in \mathscr{B}_i$, we have $p_i(E) = \mu_0(\pi_i^{-1}(E))$, the statement (3.2.1) is equivalent to (a). Further, for any $K \in X$, $i \in I$ and $E \subset X_i$ we have $\pi_i^{-1}(E) \cap K = \emptyset$ if and only if $E \cap \pi_i(K) = \emptyset$, so that the statement (3.2.2) is equivalent to (b). So, the equivalence of (3.2.1) and (3.2.2) follows from the equivalence of (a) and (b).

THEOREM 3.3. *Under the assumptions* 3.1 *suppose that all X_i, $i \in I$, are finite-dimensional linear spaces, X is a separable Banach space and all the mappings π_{ij} and π_i, $i \in I$, $j \in I$, $i \leqslant j$, are linear. Furthermore, suppose that I contains a countable subset I_0 such that, for every $x \in X$,*

(3.3.1) $$\|x\| = \sup_{i \in I_0} \inf\{\|y\|: y \in X,\ \pi_i(y) = \pi_i(x)\}.$$

Then the smallest σ-algebra $\mathscr{S}$ of subsets of X containing all the sets of the form $\pi_i^{-1}(E)$, $i \in I$, $E \in \mathscr{B}_i$, is the algebra of all the Borel subsets of X and the statement

(3.3.2) *there is a unique probability measure p defined on $\mathscr{S}$ such that $p_i(E) = p(\pi_i^{-1}(E))$ for every $i \in I$ and $E \in \mathscr{B}_i$*

is equivalent to (3.2.2).

Proof. For any $i \in I$ and $z \in X_i$ put

$$\|z\|_i = \inf\{\|y\|: y \in X,\ \pi_i(y) = z\}.$$

Then, for any $i \in I$, $\|\ \|_i$ is a pseudonorm on X_i, continuous since X is finite-dimensional. So, for any $x \in X$, $r > 0$ and $i \in I$,

$$B_i = \{z: z \in X_i,\ \|z - \pi_i(x)\|_i \leqslant r\}$$

is a closed subset of X_i. Since, by (3.3.1),

$$\{y: y \in X,\ \|y - x\| \leqslant r\} = \bigcap_{i \in I_0} \pi_i^{-1}(B_i),$$

it follows that every closed ball in the space X belongs to $\mathscr{S}$, and so, X being separable, $\mathscr{S}$ contains all the Borel subsets of X. On the other hand, it is obvious from the continuity of the mappings π_i that all the sets in $\mathscr{S}$ are Borel subsets of X. Thus $\mathscr{S}$ is the algebra of all the Borel subsets of X. Since by (3.3.1) for any $x \in X$, $x \neq 0$, there is an $i \in I_0$, such that $\pi_i(x) \neq 0$ and since every non-negative, finite measure defined on the σ-algebra of all the Borel subsets of a separable complete metric space is tight, the equivalence of (3.3.2) and (3.2.2) follows from theorem 3.2.

THEOREM 3.4. *Let I be the family of all finite non-void subsets of the interval $[0, T]$, $T > 0$, directed by inclusion. For any $i \in I$, let X_i be the space of all real functions defined on i and let π_i be the operator of restriction to i of functions defined on $[0, T]$. For any $i \in I$ and $j \in I$ such that $i \subset j$, let π_{ij} be the operator of restriction to i of functions defined on j. For any $i \in I$ let p_i be a probability measure defined on the σ-algebra $\mathscr{B}_i$ of all the Borel subsets of X_i and suppose that* (3.1.3) *holds. Under these assumptions the smallest σ-algebra $\mathscr{S}$ of subsets of the space $C[0, T]$ containing all the sets of the form $\pi_i^{-1}(E)$, $i \in I$, $E \in \mathscr{B}_i$, is the σ-algebra of all the Borel subsets of $C[0, T]$ and the statement* (3.3.2) *is equivalent to the following statement:*

(3.4.1) *for every $\varepsilon > 0$ and every $\eta > 0$ there is a $\delta > 0$ such that $p_i(A_i, \delta, \varepsilon) \leqslant \eta$ for every $i \in I$, where*

$$A_{i,\delta,\varepsilon} = \big\{x: x \in X_i,\ \sup\{|x(t) - x(s)|: t, s \in i,\ |t - s| \leqslant \delta\} > \varepsilon\big\}.$$

The equivalence of (3.4.1) and (3.3.2) may be expressed in other words by saying that (3.4.1) is a necessary and sufficient condition of the existence of a stochastic process $(X_t)_{t \in [0,T]}$ with all sample paths continuous, such that, for any finite sequence $(t_1, t_2, \ldots, t_n)$ of instants in $[0, T]$ and every Borel subset E of R_n, the probability of the event $\{(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in E\}$ is equal to

$$p_{\{t_1, t_2, \ldots, t_n\}}\Big(\big\{x: x \in X_{\{t_1, t_2, \ldots, t_n\}},\ (x(t_1), x(t_2), \ldots, x(t_n)) \in E\big\}\Big).$$

This result is known (see [5], proposition III — 5 — 1, or [6], §§ 2.1-2.2), but we shall give a proof by an application of theorem 3.3.

A similar direct proof of the existence of the Wiener measure in the space $C[0,T]$ is given in [4], p. 14-16.

Proof of theorem 3.4. Defining I_0 as the family of all non-void finite sets of rational numbers belonging to $[0,T]$, we are under the assumptions of theorem 3.3. Hence we need only to prove that for $X = C[0,T]$ conditions (3.4.1) and (3.2.2) are equivalent. This equivalence follows from the Arzelà-Ascoli theorem (see [2], p. 289).

Indeed, let (3.4.1) hold and let $\varepsilon > 0$ be arbitrarily fixed. For any $n = 1, 2 \ldots$ choose $\delta_n > 0$ such that

$$\sup_{i\in I} p_i(A_{i,\delta_n,1/n}) \leqslant \frac{\varepsilon}{2^{n+1}}$$

and let $C < \infty$ be so large that

$$p_{\{0\}}\big(\{x: x\in X_{\{0\}},\ |x(0)| > C\}\big) \leqslant \frac{\varepsilon}{2}.$$

Put

$$K_0 = \{x: x\in C[0,T],\ |x(0)| \leqslant C\},$$

$$K_n = \left\{x: x\in C[0,T],\ \sup\{|x(t)-x(s)|: s,t\in[0,T],\ |t-s| \leqslant \delta_n\} \leqslant \frac{1}{n}\right\}$$

for $n = 1, 2, \ldots$ and

$$K = \bigcap_{n=0}^{\infty} K_n.$$

The K is a bounded set of equicontinuous functions, closed in $C[0,T]$, and so, by the Arzelà-Ascoli theorem, a compact subset of $C[0,T]$. For any $i\in I$ we have

$$X_i \setminus \pi_i(K) = \begin{cases} \{x: x\in X_i,\ |x(0)| > C\} \cup \bigcup_{n=1}^{\infty} A_{i,\delta_n,1/n} & \text{if } 0\in i, \\ \bigcup_{n=1}^{\infty} A_{i,\delta_n,1/n} & \text{if } 0\notin i, \end{cases}$$

so that, since

$$p_i\big(\{x: x\in X_i,\ |x(0)| > C\}\big) = p_{\{0\}}\big(\{x: x\in X_{\{0\}},\ |x(0)| > C\}\big),$$

if $0\in i$, we have

$$p_i\big(X_i\setminus\pi_i(K)\big) \leqslant \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} p_i(A_{i,\delta_n,1/n}) \leqslant \frac{\varepsilon}{2} + \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n+1}} = \varepsilon.$$

Thus (3.4.1) implies (3.3.2).

On the other hand, for a certain $\eta > 0$, let K be such a compact subset of $C[0,T]$ that

$$\sup_{i\in I} p_i\big(X_i\setminus\pi_i(K)\big) \leqslant \eta.$$

Then, by the Arzelà-Ascoli theorem, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sup\{|x(t)-x(s)|: t, s\in[0,T],\ |t-s| \leqslant \delta\} \leqslant \varepsilon$$

for every $x\in K$, so that $A_{i,\delta,\varepsilon} \subset X_i\setminus\pi_i(K)$ for every $i\in I$ and hence

$$\sup_{i\in I} p_i(A_{i,\delta,\varepsilon}) \leqslant \eta.$$

Thus (3.3.2) implies (3.4.1)

Bibliography

[1] N. Bourbaki, *Éléments de Mathématique*, Livre VI, Intégration, Paris 1952.

[2] N. Dunford and J. T. Schwartz, *Linear operators*, Part I, General Theory, russian edition, Moscow 1962.

[3] P. R. Halmos, *Measure theory*, New York 1950.

[4] K. Itô and H. P. Mc Kean, Jr., *Diffusion processes and their sample paths*, Berlin 1965.

[5] J. Neveu, *Bases mathématiques du Calcul des Probabilités*, Paris 1964.

[6] Ю. В. Прохоров, *Сходимость случайных процессов и предельные теоремы теории вероятностей*, Теория вероятностей и ее применения 1, № 2 (1956), p. 177-237.

Reçu par la Rédaction le 1. 8. 1967